Linear Elasticity III - Conformal mapping and a simple composite material

This TA completes the 4 week series on linear elasticity. We believe that the problems we'll be solving today, together with the thermo-elastic problem of the previous TA, really capture a lot of the tools used to tackle real problems, and also provide an insight into the beauty and complexity of even such a simple linear theory. Next week we'll be moving forward to non-linear elasticity, so don't get too comfortable.

1 Complex representation of scalar elasticity

We study a case of scalar elasticity, where $\boldsymbol{u} = u_z(x,y)\boldsymbol{e}_z$. The strains are

$$\varepsilon_{yz} = \frac{1}{2}(\partial_y u_z + \partial_z u_y) = \frac{1}{2}\partial_y u_z \ , \tag{1}$$

$$\varepsilon_{xz} = \frac{1}{2}(\partial_x u_z + \partial_z u_x) = \frac{1}{2}\partial_x u_z \ , \tag{2}$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0 \ . \tag{3}$$

We have seen in class that $\nabla^2 u_z = 0$, that is, u_z is a harmonic function. This means that we can write u_z as

$$u_z = 2\Re(\phi) = \phi(z) + \overline{\phi(z)}, \qquad z = x + iy \ , \tag{4}$$

where ϕ is an analytic complex function. We will use the Cauchy-Riemman equations, that tell us that

$$\partial_x \phi = -i\partial_y \phi = \phi' \ , \tag{5}$$

$$\partial_x \overline{\phi} = \overline{\partial_x \phi} = i\partial_y \overline{\phi} = \overline{\phi}' \ , \tag{6}$$

and therefore the stresses are

$$\begin{aligned} \sigma_{xz} &= \mu\partial_x u_z = \mu\left(\partial_x\phi + \partial_x\overline{\phi}\right) = \mu\left(\phi' + \overline{\phi}'\right) = 2\mu\Re(\phi') \ , \\ \sigma_{yz} &= \mu\partial_y u_z = \mu\left(\partial_y\phi + \partial_y\overline{\phi}\right) = i\mu\left(\phi' - \overline{\phi}'\right) = -2\mu\Im(\phi') \ , \\ \Rightarrow \qquad & 2\mu\phi' = \sigma_{xz} - i\sigma_{yz} \ , \end{aligned} \tag{7}$$

and all other components vanish.

If our domain contains a free boundary, given by a curve that is parameterized by $x(s), y(s)$ with s being arc-length parametrization, then the normal to the boundary is given by $\boldsymbol{n} = (\partial_s y, -\partial_s x)$. On the boundary we thus have

$$\begin{aligned} 0 &= \sigma_{zx} n_x + \sigma_{zy} n_y \\ &= \mu\left[\left(\partial_x\phi + \partial_x\overline{\phi}\right)\partial_s y - \left(\partial_y\phi + \partial_y\overline{\phi}\right)\partial_s x\right] \\ &= \mu\left[\left(-i\partial_y\phi + i\partial_y\overline{\phi}\right)\partial_s y - \left(i\partial_x\phi - i\partial_x\overline{\phi}\right)\partial_s x\right] \\ &= -i\mu\left[\left(\frac{\partial\phi}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial\phi}{\partial x}\frac{\partial x}{\partial s}\right) - \left(\frac{\partial\overline{\phi}}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial\overline{\phi}}{\partial x}\frac{\partial x}{\partial s}\right)\right] \\ &= -\mu\left(\frac{\partial\phi}{\partial s} - \frac{\partial\overline{\phi}}{\partial s}\right) = 2\mu\frac{\partial\Im(\phi)}{\partial s} \ , \end{aligned} \tag{8}$$

so on the boundary $\Im(\phi)$ is constant. Since ϕ is only given up to an additive constant, we can choose $\Im(\phi) = 0$, or, in other words, $\phi = \overline{\phi}$ on the boundary. We see that solving for the displacement field is equivalent to finding an analytic function whose imaginary part is constant on the boundary.

2 Conformal mapping: Inglis crack

(Refernce: Marder & Fineberg, Physics Reports 1999)

Complex treatment of 2D elasticity is very useful because Laplace's equation is conformally invariant, so one can use conformal mappings to deform the region over which we need to solve the equation into a more convenient geometry. Here we'll see an application of this method, which is called the Inglis (mode III) problem. In 1913 Charles Inglis solved the general problem of an elliptic hole in an infinite plate subject to distant loading. His solution turned out to be one of the cornerstones of fracture mechanics, and was later used and generalized by the works of Griffith, Irwin, and others.

So let's look at an infinite plane with an elliptic hole, subject to antiplane shear $\sigma_{yz} = \sigma_\infty$ at $y \to \pm\infty$. As working with ellipses is unpleasant, we want to find a conformal mapping that maps the region outside the ellipse to a region outside a circle. Luckily, such a mapping is well known, and is given by

$$z = f(\omega) = R\left(\omega + \frac{\rho}{\omega}\right) , \tag{9}$$

$$\omega = f^{-1}(z) = \frac{z}{2R} + \sqrt{\left(\frac{z}{2R}\right)^2 - \rho} . \tag{10}$$

f maps the unit circle in the ω-plane to an ellipse with axes $R(1 \pm \rho)$ in the z-plane. $0 \leq \rho \leq 1$ is a parameter that measures the ellipse's eccentricity[1] - when $\rho = 0$ the ellipse is a circle, while for $\rho = 1$ it is a 1D crack of length $4R$. The conformal mapping is shown in Fig. (1).

The crux of the conformal mapping technique is that while in the real coordinates the geometry is elliptic (and thus complicated), in the ω-plane the domain is a circle (simple!), and therefore we want to reformulate the problem in terms of ω. That is, we want to describe ϕ as a function of ω, by the mapping $\phi(\omega) = \phi(\omega(z))$.

On the hole's boundary, which is the unit circle in ω-plane, we have

$$\phi(\omega) = \overline{\phi(\omega)} = \overline{\phi}(\overline{\omega}) = \overline{\phi}(1/\omega) , \tag{11}$$

because on the unit circle $\overline{\omega} = 1/\omega$. The property (11) can be analytically extended to all the ω-plane.

What are the singularities of ϕ? Outside the hole, it must be completely regular, except at infinity where it diverges as $\phi \sim z$. This is because Eq. (7) tells us that far from the hole we have $\partial_z\phi \propto \sigma/\mu$, and therefore we conclude that

$$\phi \approx -i\frac{\sigma_\infty}{\mu}z \approx -i\frac{\sigma_\infty}{\mu}R\omega, \qquad \text{for } \omega, z \to \infty . \tag{12}$$

[1] Note that ρ isn't the eccentricity as usually defined in geometry, which is $e = \frac{2\sqrt{\rho}}{\rho+1}$.

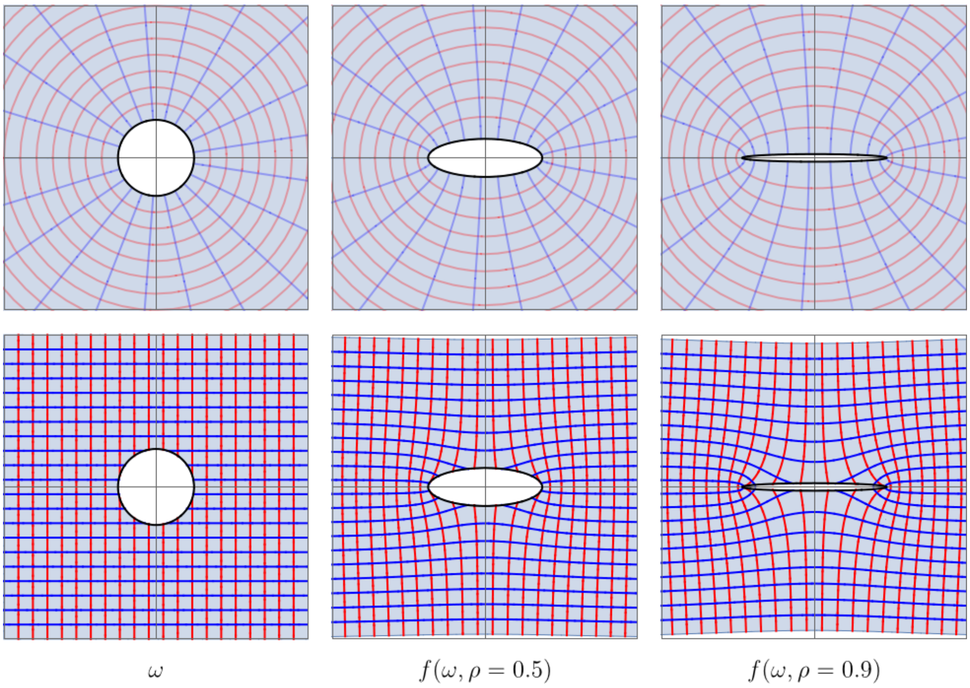


Figure 1: The conformal mapping. Polar lines (top row) and the Cartesian lines (bottom row) are shown. Note that, after the mapping, the lines remain perpendicular.

Using the analytical continuation of Eq. (11), we get that

$$\overline{\phi}(1/\omega) \approx -i\frac{\sigma_\infty}{\mu}R\omega, \qquad \text{for } \omega \to \infty \ , \tag{13}$$

or equivalently,

$$\phi(\omega) \approx i\frac{\sigma_\infty R}{\mu\omega}, \qquad \text{for } \omega \to 0 \ , \tag{14}$$

and there are no other singularities inside the unit circle. Having determined all the possible singularities of ϕ, it is determined up to an additive constant. It must be

$$\phi(\omega) = i\frac{\sigma_\infty R}{\mu}\left(\frac{1}{\omega} - \omega\right) \ . \tag{15}$$

As discussed above, another way of finding ϕ is to find a function whose imaginary part vanishes on the boundary on the hole, i.e. on the unit circle. The function $i\,(1/\omega - \omega)$ fits this requirement, therefore, it is exactly the function we're looking for, up to a multiplicative factor which we have obtained from the external BC.

We can now calculate the displacement field in the "real" coordinate z by joining Eqs. (15) and (10):

$$u_z = 2\Re\left\{-i\frac{\sigma_\infty R}{\mu}\left(\zeta + \sqrt{\zeta^2 - \rho} - \frac{1}{\zeta + \sqrt{\zeta^2 - \rho}}\right)\right\} \ , \qquad \text{where } \zeta \equiv \tfrac{z}{2R} \ . \tag{16}$$

What is the stress at the tip of the ellipse? We can differentiate $u_z(z)$ of Eq. (16) explicitly, but this gives a nasty expression that is very difficult to understand. It is simpler to use the conformal property of the mapping:

$$
\begin{aligned}
\partial_z \phi(z) &= \partial_z \phi(\omega(z)) = \phi'(\omega)\frac{\partial \omega}{\partial z} \ , \\
\phi'(\omega) &= -i\frac{\sigma_\infty R}{\mu}\left(1+\frac{1}{\omega^2}\right) \ , \\
\frac{\partial \omega}{\partial z} &= \left(\frac{\partial z}{\partial w}\right)^{-1} = \frac{1}{f'(\omega)} \ , \\
f'(\omega) &= R(1-\frac{\rho}{\omega^2}) \ , \\
\phi'(z) &= \frac{-i\frac{\sigma_\infty R}{\mu}\left(1+\frac{1}{\omega^2}\right)}{R(1-\frac{\rho}{\omega^2})} = -\frac{i\sigma_\infty}{\mu}\,\frac{\omega(z)^2+1}{\omega(z)^2-\rho} \ .
\end{aligned}
\tag{17}
$$

Note that in the last equation ω is a function of z.

Now let's examine the solution. One thing we would like to know is where in space is the stress maximal. Clearly, ϕ' diverges for $w = \pm\sqrt{\rho}$, but remember that $\rho < 1$ and our domain is outside the unit circle, so this point is inside the hole. Some trivial algebra shows that the ϕ' is maximal for $\omega = \pm 1$, which are, not surprisingly, the closest points outside the unit circle to $\pm\rho$. When $\omega = \pm 1$ we have $z = \pm R(1+\rho)$ - these are the horizontal tips of the ellipse. The stresses there are

$$
\begin{aligned}
\sigma_{xz} - i\sigma_{yz} = \mu\phi' = -\sigma_\infty\frac{2i}{1-\rho} \Rightarrow \\
\sigma_{xz} = 0 \qquad , \qquad \sigma_{yz} = \frac{2\sigma_\infty}{1-\rho} \ .
\end{aligned}
\tag{18}
$$

The case $\rho = 0$ gives $\sigma_{yz} = 2\sigma_\infty$, in accordance with what was done in class. In the opposite extremity, $\rho \to 1$, the stress field diverges (but the displacement doesn't). We see that the stress at the tip decreases with the radius there. An interesting consequence of this is that in order to arrest a crack from propagating, one can drill a hole at its tip (!). This will reduce the radius of curvature at the tip and weaken the singularity.

The limiting case $\rho \to 1$ is of particular interest, as it describes a 1-dimensional cut in the material. It is known in the literature as Mode III crack. The power with which σ_{zy} diverges in the case $\rho = 1$ can be easily obtained. In this case we have

$$
\phi = -\frac{iR\sigma_\infty}{\mu}\sqrt{\frac{z^2}{R^2}-4} \ . \tag{19}
$$

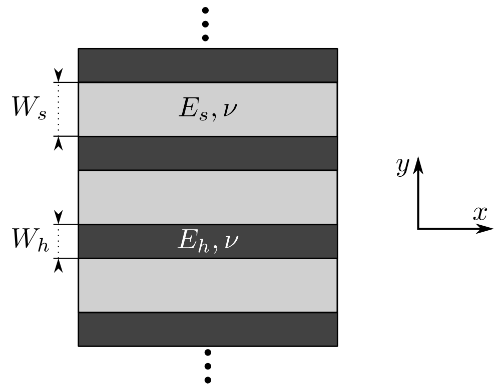
Plugging in $z = 2R\left(1+\delta\right)$ (where $\delta \in \mathbb{C}$) and keeping the leading order in δ gives

$$
\begin{aligned}
\phi = -i\frac{2\sqrt{2}R\sigma_\infty}{\mu}\sqrt{\delta} + O\left(\delta^{3/2}\right) \Rightarrow \\
\sigma_{yz} \sim \frac{\sigma_\infty}{\sqrt{2}\sqrt{\delta}} \ .
\end{aligned}
\tag{20}
$$

The fact that near the crack tip the stress field diverges as the square root of the distance from the crack tip, and that the displacement field is regular, is of general applicability, and is true for static cracks in all loading configurations. The square-root divergence is a consequence of the branch-cut at the crack surface.

3 A simple model for a simple composite material

Composite materials are materials that have a microscopic structure. They are abundant in nature, and examples include bone, wood, dentin (the material your teeth are made of), and many more (graphic examples will be shown in class). In the last few decades there are also many man-made composite materials. The vast advancement in composite material technology is one of the most influential revolutions in modern technology, and it allows manufacturing materials that have desirable characteristics – light-weight, high strength, shape memory, etc. – which are orders of magnitude better than homogeneous materials. This is a fascinating topic which is the subject of huge and very active ongoing research. In this short exercise we'll examine some simple outcomes of a simple model of a simple composite material. This model was given as a question in the final exam of the 2012 course.



Consider a material that is composed of layers of two linear-elastic isotropic materials, one is hard and the other soft. The hard material has a Young's modulus E_h and the soft material has a Young's modulus of E_s. For simplicity, we'll assume that both materials have the same Poisson's ratio ν. The width of each layer is denoted by W_i, and the layers are glued perfectly to each other. We assume the material is infinite in all directions, has a periodic structure in the y direction and is translationally invariant in the x direction. We also assume plane-stress conditions in the z direction. We define the volume ratio of the hard material by

$$\phi = \frac{W_h}{W_h + W_s} \ . \tag{21}$$

Our goal is to calculate the coarse-grained Hooke's law of the composite system, that is, it's (linear) response to loading on length-scales much larger than the scale of the microscopic structure (in our case - W_s and W_h). To this end, we consider a small segment of the material over which the stress and strain field are constant *in each material.* We denote these fields by $\sigma^{(i)}$, $\varepsilon^{(i)}$ where $i = h, s$.

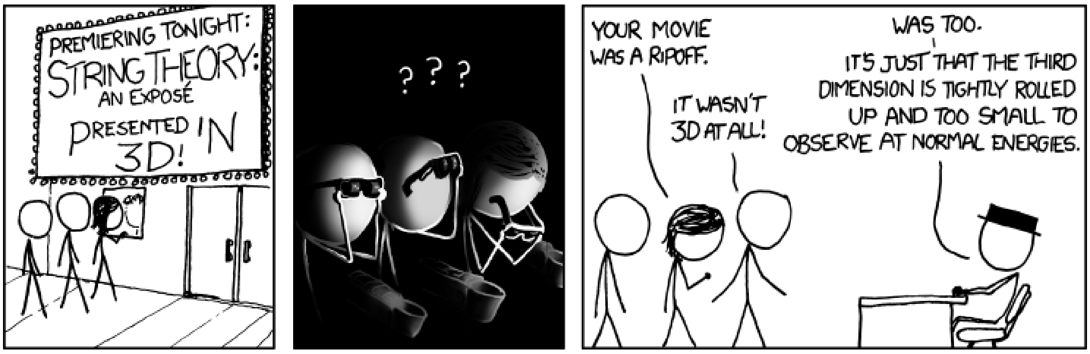
Each of these fields follows Hooke's law with the relevant elastic constants:

$$\begin{pmatrix} {\varepsilon_{xx}}^{(h,s)} \\ {\varepsilon_{yy}}^{(h,s)} \\ {\varepsilon_{xy}}^{(h,s)} \end{pmatrix} = \frac{1}{E^{(h,s)}} \begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{pmatrix} \begin{pmatrix} {\sigma_{xx}}^{(h,s)} \\ {\sigma_{yy}}^{(h,s)} \\ {\sigma_{xy}}^{(h,s)} \end{pmatrix} , \tag{22}$$

$$\begin{pmatrix} {\sigma_{xx}}^{(h,s)} \\ {\sigma_{yy}}^{(h,s)} \\ {\sigma_{xy}}^{(h,s)} \end{pmatrix} = \frac{E^{(h,s)}}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \begin{pmatrix} {\varepsilon_{xx}}^{(h,s)} \\ {\varepsilon_{yy}}^{(h,s)} \\ {\varepsilon_{xy}}^{(h,s)} \end{pmatrix} . \tag{23}$$

3.1 Boundary conditions at the interfaces

We now need to determine are the boundary conditions at the interfaces between the different layers. Clearly, the displacement field $\boldsymbol{u}$ must be continuous across the interface, because we assumed perfect bonding between the materials, which means no relative displacement. But what does that tell us about $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$? Which components are continuous across the interface and which experience a jump? How is the jump determined? If you are reading this before the class, I strongly recommend that you stop here for a minute and try to answer this question on your own. If you need a pause, here's an XKCD strip about string theory:



Now that you've answered the question here's the correct answer. Let's say that some of the interfaces lies on $y = 0$. As said before, $\boldsymbol{u}$ is continuous across the interface and therefore

$$u_i^{(s)}(x, y = 0) = u_i^{(h)}(x, y = 0) , \tag{24}$$

for all x. Specifically, this means that $\partial_x u_i$ is continuous across the interface. From this we conclude immediately that $\varepsilon_{xx}^{(s)} = \varepsilon_{xx}^{(h)}$.

Next, consider force balance. Think of a small volume element of length L and infinitesimal height, which is half in the soft region and half in the hard region. The vertical forces applied to it (per unit thickness in the z direction) sum up to

$$L\left(\sigma_{yy}^{(h)} - \sigma_{yy}^{(s)}\right)\hat{y} + L\left(\sigma_{xy}^{(h)} - \sigma_{xy}^{(s)}\right)\hat{x} , \tag{25}$$

and since the situation is static, we conclude that σ_{iy} is continuous across the interface. To summarize:

- $\boldsymbol{u}$, σ_{yy}, σ_{xy}, ε_{xx} are continuous across the interface.
- σ_{xx}, ε_{yy}, ε_{xy} experience a jump across the interface. The jump can be calculated from Hooke's law.

3.2 Coarse graining

Now comes a crucial part of the intellectual path that we try to follow. We want to describe the large-scale/macroscopic/coarse-grained fields $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ in terms of the small-scale/microscopic elastic fields of the constituent materials $\sigma^{(i)}, \varepsilon^{(i)}$. The final goal is to find the macroscopic Hooke's law, i.e. the linear relation

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \boldsymbol{C} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} . \tag{26}$$

In our case this step fairly easy: consider a large square which contains a lot of layers. The total force exerted on the x side is equal to the total force exerted by all the different layers:

$$\sigma_{xx} = \frac{W_h \sigma_{xx}^{(h)} + W_s \sigma_{xx}^{(s)}}{W_h + W_s} = \phi\, \sigma_{xx}^{(h)} + (1-\phi)\sigma_{xx}^{(s)} . \tag{27}$$

The macroscopic strain in the x direction is $\varepsilon_{xx} = \varepsilon_{xx}^{(h)} = \varepsilon_{xx}^{(s)}$, which translates through the microscopic Hooke's law Eq. (22) to

$$\varepsilon_{xx} = \varepsilon_{xx}^{(h)} = \varepsilon_{xx}^{(s)} = \frac{\sigma_{xx}^{(h)} - \nu \sigma_{yy}^{(h)}}{E_h} = \frac{\sigma_{xx}^{(s)} - \nu \sigma_{yy}^{(s)}}{E_s} . \tag{28}$$

In the y direction things are exactly the other way around:

$$\sigma_{yy} = \sigma_{yy}^{(h)} = \sigma_{yy}^{(s)} = \frac{E^{(s)}}{1-\nu^2}\left(\varepsilon_{yy}^{(s)} + \nu \varepsilon_{xx}^{(s)}\right) = \frac{E^{(h)}}{1-\nu^2}\left(\varepsilon_{yy}^{(h)} + \nu \varepsilon_{xx}^{(h)}\right) , \tag{29}$$

$$\varepsilon_{yy} = \phi\, \varepsilon_{yy}^{(h)} + (1-\phi)\varepsilon_{yy}^{(s)} . \tag{30}$$

We now want to write the macroscopic Hooke's law and we'll begin with the diagonal part. Before we start, let's count to see that we already have all the ingredients. We use the fact that $\varepsilon_{xx} = \varepsilon_{xx}^{(h,s)}$ and $\sigma_{yy} = \sigma_{yy}^{(h,s)}$ to summarize our equations as

$$\sigma_{xx} = \phi\, \sigma_{xx}^{(h)} + (1-\phi)\sigma_{xx}^{(s)} , \tag{31}$$

$$\sigma_{yy} = \frac{E^{(s)}}{1-\nu^2}\left(\varepsilon_{yy}^{(s)} + \nu \varepsilon_{xx}\right) = \frac{E^{(h)}}{1-\nu^2}\left(\varepsilon_{yy}^{(h)} + \nu \varepsilon_{xx}\right) , \tag{32}$$

$$\varepsilon_{yy} = \phi\, \varepsilon_{yy}^{(h)} + (1-\phi)\varepsilon_{yy}^{(s)} , \tag{33}$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}^{(h)} - \nu \sigma_{yy}}{E_h} = \frac{\sigma_{xx}^{(s)} - \nu \sigma_{yy}}{E_s} . \tag{34}$$

These are linear 6 equations, and we want to eliminate 4 variables: $\sigma_{xx}^{(h,s)}$ and $\varepsilon_{yy}^{(h,s)}$. Thus, we expect to finish with two linear equations, which we will be able to put in the desired form

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \end{pmatrix} . \tag{35}$$

The full solution is therefore obtained by simple algebra, which we will not do because it's not very interesting.

For the shear part of Hooke's law we have

$$\partial_x u_y = \partial_x u_y^{(s)} = \partial_x u_y^{(h)} \ . \tag{36}$$

Therefore,

$$\frac{E_s\left(\partial_x u_y + \partial_y u_x^{(s)}\right)}{2(1+\nu)} = \frac{E_h\left(\partial_x u_y + \partial_y u_x^{(h)}\right)}{2(1+\nu)} = \sigma_{xy} \ . \tag{37}$$

Inverting, we get

$$\partial_y u_x^{(i)} = \frac{2(1-\nu)}{E^{(i)}}\sigma_{xy} - \partial_x u_y \ . \tag{38}$$

We now use

$$\partial_y u_x = (1-\phi)\partial_y u_x^{(s)} + \phi\, \partial_y u_x^{(h)} \ , \tag{39}$$

to obtain

$$\varepsilon_{xy} = \frac{1}{2}(\partial_x u_y + \partial_y u_x) = (1+\nu)\left[\frac{(1-\phi)}{E_s} + \frac{\phi}{E_h}\right]\sigma_{xy} \equiv \frac{\sigma_{xy}}{\mu_{\text{eff}}} \ . \tag{40}$$

3.3 Discussion

As you see we did not write the full expressions for the coarse-grained Hooke's law (as it is a bit ugly). The point was to show the structure of the solution. Nonetheless, some interesting cases can be easily obtained without having to write the full expression.

First of all, we note that the resulting coarse-grained material is, unsurprisingly, anisotropic. We see that the symmetry $x \leftrightarrow y$ is clearly broken. However, plugging in $E_s = E_h$, the isotropic case is immediately recovered. Also, the cases $\phi \to 1$ or $\phi \to 0$ easily recover a homogeneous isotropic material.

Second, let's consider the (very realistic) case of $E_h \gg E_s$. We are only interested in approximate trends, so we'll assume ϕ is not to close to 0 or to 1, and omit all pre-factors of the form $\phi, \phi/(1-\phi)$ and so on. We rewrite Eqs. (31) and (34) as

$$\sigma_{xx}^{(h)} = \frac{\sigma_{xx} - (1-\phi)\sigma_{xx}^{(s)}}{\phi} \ , \tag{41}$$

$$\sigma_{xx}^{(s)} = \frac{\sigma_{xx} + \phi\,\nu\left(\frac{E_h}{E_s} - 1\right)\sigma_{yy}}{\phi\,\frac{E_h}{E_s} + (1-\phi)} \ . \tag{42}$$

Expanding these to order in E_s/E_h, we get

$$\sigma_{xx}^{(s)} \approx \frac{E_s}{E_h}\sigma_{xx} + \sigma_{yy} \ , \tag{43}$$

$$\sigma_{xx}^{(h)} \approx \sigma_{xx} + \sigma_{yy} \ . \tag{44}$$

Plugging this into Hooke's law we get

$$\varepsilon_{xx} \approx \frac{\frac{E_s}{E_h}\sigma_{xx} + \sigma_{yy} - \nu\sigma_{yy}}{E_s} \approx \frac{\sigma_{xx}}{E_h} + \frac{\sigma_{yy}}{E_s} \ . \tag{45}$$

Similarly, plugging (43)-(44) into (33),

$$\begin{aligned}\varepsilon_{yy} &\approx \phi\frac{\sigma_{yy}-\nu\sigma_{xx}^{(h)}}{E_h}+(1-\phi)\frac{\sigma_{yy}-\nu\sigma_{xx}^{(s)}}{E_s} \\ &\approx \frac{\sigma_{yy}-(\sigma_{xx}+\sigma_{yy})}{E_h}+\frac{\sigma_{yy}-\left(\frac{E_s}{E_h}\sigma_{xx}+\sigma_{yy}\right)}{E_s}\approx\frac{\sigma_{yy}}{E_s}+\frac{\sigma_{xx}}{E_s}\ .\end{aligned} \tag{46}$$

In addition, Eq. (40) shows that $\mu_{\text{eff}}\sim E_s$. We conclude that the energy functional takes the approximate form

$$u=\frac{1}{2}\left(\sigma_{xx}\varepsilon_{xx}+\sigma_{yy}\varepsilon_{yy}+2\sigma_{xy}\varepsilon_{xy}\right)\approx\frac{\sigma_{xx}^2}{E_h}+\frac{\sigma_{yy}^2}{E_s}+\frac{\sigma_{xx}\sigma_{yy}}{E_s}+\frac{\sigma_{xy}^2}{E_s}\ , \tag{47}$$

thus all the responses except the one in the xx direction are dominated by the soft material.