Finite elasticity and incompressibility

In this TA we'll be looking at non-linear elasticity, a.k.a finite elasticity. We'll also discuss how we handle incompressible systems.

1 2D plane-stress

As a first example, we'll consider a 2D plane-stress problem of an incompressible neo-Hookean material. The Neo-Hookean energy functional is (Eq. (7.7) in Eran's notes)

$$u(\boldsymbol{F}_{3D}) = \frac{\mu}{2}\left[\mathrm{tr}(\boldsymbol{F}_{3D}^T\boldsymbol{F}_{3D}) - 3\right] \ , \tag{1}$$

together with the requirement that $J_{3D} \equiv \det \boldsymbol{F}_{3D} = 1$. We want to consider now the case where the stresses are only in-plane. If we consider now λ_i, the principle values of $\boldsymbol{F}_{3D}$, we may write

$$u = \frac{\mu}{2}\left[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right] \underbrace{=}_{\lambda_1\lambda_2\lambda_3=1} \frac{\mu}{2}\left[\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} - 3\right] \ . \tag{2}$$

If we now consider the two-dimensional deformation gradient tensor $\boldsymbol{F}_{2D}$, we find that the Neo-Hookean energy functional for plane-stress is

$$u(\boldsymbol{F}_{2D}) = \frac{\mu}{2}\left[\mathrm{tr}(\boldsymbol{F}_{2D}^T\boldsymbol{F}_{2D}) + (\det \boldsymbol{F}_{2D})^{-2} - 3\right] \ . \tag{3}$$

Note that $\det \boldsymbol{F}_{2D}$ appears in the elastic energy functional due to the incompressibility condition. For the rest of the section I'll drop the subscript and just write $\boldsymbol{F}$, as we will be dealing with 2D problems, but keep in mind that it is $\boldsymbol{F}_{2D}$.

1.1 The first Piola-Kirchhoff stress tensor

To calculate $\boldsymbol{P} \equiv \frac{\partial u}{\partial \boldsymbol{F}}$ note that

$$\frac{\partial\, \mathrm{tr}(\boldsymbol{F}^T\boldsymbol{F})}{\partial \boldsymbol{F}} = 2\boldsymbol{F}, \quad \frac{\partial(\det \boldsymbol{F})^{-2}}{\partial \boldsymbol{F}} = -2(\det \boldsymbol{F})^{-3}\left(\mathrm{tr}\, \boldsymbol{F}\ \boldsymbol{I} - \boldsymbol{F}^T\right) \ , \tag{4}$$

where the latter can be easily obtained using the identity (valid in 2D only)

$$\det \boldsymbol{F} = \frac{1}{2}\left[(\mathrm{tr}\, \boldsymbol{F})^2 - \mathrm{tr}\, \boldsymbol{F}^2\right] \ . \tag{5}$$

Therefore, we have

$$\boldsymbol{P} = \frac{\partial u}{\partial \boldsymbol{F}} = \mu\left[\boldsymbol{F} - (\det \boldsymbol{F})^{-3}\left(\mathrm{tr}\, \boldsymbol{F}\ \boldsymbol{I} - \boldsymbol{F}^T\right)\right] \ , \tag{6}$$

$$\boldsymbol{P} = \mu\left[\begin{pmatrix} \partial_X\varphi_x & \partial_Y\varphi_x \\ \partial_X\varphi_y & \partial_Y\varphi_y \end{pmatrix} - J^{-3}\begin{pmatrix} \partial_Y\varphi_y & -\partial_X\varphi_y \\ -\partial_Y\varphi_x & \partial_X\varphi_x \end{pmatrix}\right] \ , \tag{7}$$

where $J \equiv \det \boldsymbol{F} = \partial_X\varphi_x(X,Y)\partial_Y\varphi_y(X,Y) - \partial_Y\varphi_x(X,Y)\partial_X\varphi_y(X,Y)$.

1.2 Linearized energy functional

Before going fully non-linear, let's examine the linearized version our equations to see if we get something that we recognize. Assume for simplicity that the axes are chosen in parallel to the the principal stretches, i.e.

$$\boldsymbol{F} = \begin{pmatrix} 1+\varepsilon_x & 0 \\ 0 & 1+\varepsilon_y \end{pmatrix} \ .$$

The energy density is then

$$u = \frac{\mu}{2}\left[(1+\varepsilon_x)^2 + (1+\varepsilon_y)^2 + \frac{1}{(1+\varepsilon_x+\varepsilon_y+\varepsilon_x\varepsilon_y)^2} - 3\right] \ . \tag{8}$$

Expanding to second order in the ε_i (we need second order because we develop the energy, which has quadratic terms in the stretch) we have

$$\begin{aligned} &(1+\varepsilon_x)^2 = 1 + 2\varepsilon_x + \varepsilon_x^2 \qquad , \qquad (1+\varepsilon_y)^2 = 1 + 2\varepsilon_y + \varepsilon_y^2 \ , \\ &\frac{1}{(1+\varepsilon_x+\varepsilon_y+\varepsilon_x\varepsilon_y)^2} = 1 - 2(\varepsilon_x+\varepsilon_y) + 3(\varepsilon_x^2+\varepsilon_y^2) + 4\varepsilon_x\varepsilon_y + \mathcal{O}(\varepsilon^3) \ , \end{aligned} \tag{9}$$

all in all we get

$$u = \mu\left(\varepsilon_x^2 + \varepsilon_y^2 + (\varepsilon_x+\varepsilon_y)^2\right) = \mu \operatorname{tr}\boldsymbol{\varepsilon}^2 + \mu \operatorname{tr}^2\boldsymbol{\varepsilon} \ . \tag{10}$$

So we see that the linear form of the energy functional is the familiar and expected form $u = \frac{1}{2}(2\tilde{\mu}\operatorname{tr}\boldsymbol{\varepsilon}^2 + \tilde{\lambda}\operatorname{tr}(\boldsymbol{\varepsilon})^2)$. This also means that our material has $\tilde{\lambda} = 2\tilde{\mu}$ which implies

$$\tilde{\nu} = \frac{\tilde{\lambda}}{2(\tilde{\lambda}+\tilde{\mu})} = \frac{1}{3} \ . \tag{11}$$

This value of ν should come as a surprise because we started with an incompressible material, so we should expect to have $\nu = \frac{1}{2}$. What went wrong? Keep in mind that the energy functional (3) is the result of the reduction of a set of 3D equations to 2D. We have done this in detail in the linear case, and we all remember well that the elastic constants are not the same as the 3D ones, but renormalized ones (see Eq. (5.62) in the lecture notes, or Sec. 3 in the TA session #5). The relation between the renormalized elastic constants to the real ones is

$$\tilde{\mu} = \mu, \qquad \tilde{\lambda} = \frac{2\nu\mu}{1-\nu} \ , \tag{12}$$

rearranging the latter, we get

$$\nu = \frac{\tilde{\lambda}}{\tilde{\lambda}+2\tilde{\mu}} \ . \tag{13}$$

Plugging in our result $\tilde{\lambda} = 2\tilde{\mu}$ gives

$$\nu = \frac{2\tilde{\mu}}{2\tilde{\mu}+2\tilde{\mu}} = \frac{1}{2} \ . \tag{14}$$

What a relief. The real Poisson ratio is 1/2, which means that the material is indeed incompressible. The fact that the "apparent" 2D Poisson's ration is different that 1/2 means that in-plane compressibility is allowed. This is because the material expands in the third direction, which is unaccounted for in the 2D description.

1.3 The equations of motion in our system

We remind ourselves that in the material coordinates the equations of motion read (see TA session #4).

$$\rho_0 \dot{\boldsymbol{V}} = \nabla_{\boldsymbol{X}} \cdot \boldsymbol{P} \ , \tag{15}$$

plugging in our expression for $\boldsymbol{P}$, Eq. (7), we get

$$\begin{aligned} \frac{\rho_0}{\mu}\ddot{\varphi}_x &= \nabla^2 \varphi_x - \frac{\partial \varphi_y}{\partial Y}\frac{\partial J^{-3}}{\partial X} + \frac{\partial \varphi_y}{\partial X}\frac{\partial J^{-3}}{\partial Y} \ , \\ \frac{\rho_0}{\mu}\ddot{\varphi}_y &= \nabla^2 \varphi_y - \frac{\partial \varphi_x}{\partial X}\frac{\partial J^{-3}}{\partial Y} + \frac{\partial \varphi_x}{\partial Y}\frac{\partial J^{-3}}{\partial X} \ . \end{aligned} \tag{16}$$

1.4 Small-on-Large waves

Consider then a homogeneously deformed body with principal stretches λ_x and λ_y. On this stretched state we superimpose a small displacement $\boldsymbol{\Delta}(\boldsymbol{X}, t)$. The deformation $\boldsymbol{\varphi}(\boldsymbol{X}, t)$ is thus

$$\begin{aligned} \varphi_X(\boldsymbol{X}, t) &= \lambda_x X + \Delta_X(\boldsymbol{X}, t) \ , \\ \varphi_Y(\boldsymbol{X}, t) &= \lambda_y Y + \Delta_Y(\boldsymbol{X}, t) \ . \end{aligned} \tag{17}$$

Note that the homogeneous solution $\boldsymbol{\Delta} = 0$ satisfies the equations of motion. The motion gradient reads

$$\boldsymbol{F} = \begin{pmatrix} \lambda_x + \partial_X \Delta_X & \partial_Y \Delta_X \\ \partial_X \Delta_Y & \lambda_y + \partial_Y \Delta_Y \end{pmatrix} \ . \tag{18}$$

We want to look at small perturbations on the stretched state, that is, we want to expand the equations of motion to first order in $\boldsymbol{\Delta}(\boldsymbol{X}, t)$. First, we calculate

$$\det \boldsymbol{F} \simeq (\lambda_x + \partial_X \Delta_X)(\lambda_y + \partial_Y \Delta_Y) \approx \lambda_x \lambda_y \left(1 + \frac{\partial_X \Delta_X}{\lambda_x} + \frac{\partial_Y \Delta_Y}{\lambda_y}\right) + \mathcal{O}\left(\boldsymbol{\Delta}^2\right) \tag{19}$$

$$(\det \boldsymbol{F})^{-3} \simeq \frac{1}{\lambda_x^3 \lambda_y^3}\left(1 - 3\frac{\partial_X \Delta_X}{\lambda_x} - 3\frac{\partial_Y \Delta_Y}{\lambda_y}\right) + \mathcal{O}\left(\boldsymbol{\Delta}^2\right) \ . \tag{20}$$

The equations of motions are thus, to linear order,

$$\begin{aligned} \nabla^2 \Delta_X + 3\frac{\partial_{XX}\Delta_X}{\lambda_x^4 \lambda_y^2} + 3\frac{\partial_{XY}\Delta_Y}{\lambda_x^3 \lambda_y^3} &= \frac{\rho}{\mu}\ddot{\Delta}_X = c_s^{-2}\ddot{\Delta}_X \ , \\ \nabla^2 \Delta_Y + 3\frac{\partial_{YY}\Delta_Y}{\lambda_x^2 \lambda_y^4} + 3\frac{\partial_{XY}\Delta_X}{\lambda_x^3 \lambda_y^3} &= \frac{\rho}{\mu}\ddot{\Delta}_Y = c_s^{-2}\ddot{\Delta}_Y \ , \end{aligned} \tag{21}$$

where $c_s \equiv \sqrt{\frac{\mu}{\rho}}$. Assume then a solution in the form of plane waves

$$\begin{aligned} \Delta_X(\boldsymbol{X}, t) &= a_X e^{iK(\boldsymbol{N}\cdot\boldsymbol{X} - ct)} \ , \\ \Delta_Y(\boldsymbol{X}, t) &= a_Y e^{iK(\boldsymbol{N}\cdot\boldsymbol{X} - ct)} \ , \end{aligned} \tag{22}$$

where $\boldsymbol{N} = (\cos\theta, \sin\theta)$ is the direction of propagation in the undeformed coordinates, K is the wavenumber in the undeformed coordinates, and c is the (yet unknown) speed. What kind of waves are there in the system? what is (are) the wavespeed(s)?

Plugging in the ansatz (22) into the equations of motion (21) we get

$$\begin{aligned} a_X + 3\frac{\cos^2\theta}{\lambda_x^4\lambda_y^2}a_X + 3\frac{\sin\theta\cos\theta}{\lambda_x^3\lambda_y^3}a_Y - \frac{c^2}{c_s^2}a_X = 0 \ , \\ a_Y + 3\frac{\sin^2\theta}{\lambda_x^2\lambda_y^4}a_Y + 3\frac{\sin\theta\cos\theta}{\lambda_x^3\lambda_y^3}a_X - \frac{c^2}{c_s^2}a_Y = 0 \ , \end{aligned} \tag{23}$$

which is more concisely written as

$$\underbrace{\begin{pmatrix} 1 + \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2} - \frac{c^2}{c_s^2} & \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} \\ \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} & 1 + \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} - \frac{c^2}{c_s^2} \end{pmatrix}}_{\equiv \boldsymbol{M}} \begin{pmatrix} a_X \\ a_Y \end{pmatrix} = 0 \ . \tag{24}$$

Similarly to what we've done with Rayleigh waves, solutions are obtained when the determinant vanishes. This condition reads

$$\det \boldsymbol{M} = \left(1 - \frac{c^2}{c_s^2}\right)\left(1 + \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} + \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2} - \frac{c^2}{c_s^2}\right) = 0 \ , \tag{25}$$

so you immediately see that there are two families of solutions,

$$c = \pm c_s \ , \qquad \text{and} \qquad c = \pm c_s\sqrt{1 + \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} + \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2}} \ . \tag{26}$$

The first family are shear-like waves and their velocity is independent on direction. In order to see that they are shear waves, note that the amplitudes a_X, a_Y can be obtained, up to a multiplicative factor, by the kernel of the matrix $\boldsymbol{M}(c = c_s)$, which is

$$(a_X, a_Y) \in \ker \begin{pmatrix} \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2} & \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} \\ \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} & \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} \end{pmatrix} \propto (\lambda_x \sin\theta, -\lambda_y\cos\theta) \ . \tag{27}$$

These waves are "almost transverse" because $(a_X, a_Y)\cdot\boldsymbol{N} \propto (\lambda_x - \lambda_y)\sin(2\theta)$. Therefore, they are purely transverse for $\theta = 0, \frac{\pi}{2}$ (i.e. in the X or Y directions) or when $\lambda_x = \lambda_y$. Note that this also means that the shape of the waves will depend on the direction of propagation.

The other family of solutions has a direction-dependent velocity, which is an interesting situation which is not uncommon of anisotropic systems. Following the same logic as above, the amplitudes of these waves is, up to a multiplicative factor

$$(a_X, a_Y) \propto (\lambda_y\cos\theta, \lambda_x\sin\theta) \ , \tag{28}$$

such that $(a_X, a_Y)\times\boldsymbol{N} \propto (\lambda_x - \lambda_y)\sin(2\theta)$ and again these waves are purely longitudinal for waves propagating in the X or Y direction, or for $\lambda_x = \lambda_y$.

1.5 Example

Consider a uniaxial pre-stress (applied λ_y), for which we have (due to incompressibility)

$$\lambda_x = \lambda_y^{-1/2} \ . \tag{29}$$

With this setup, the longitudinal wavespeed will be

$$c = \pm c_s \sqrt{1 + 3\left(1 + \left(\frac{1}{\lambda_y^3} - 1\right)\sin^2\theta\right)} \ . \tag{30}$$

This function is plotted in Fig. 1.

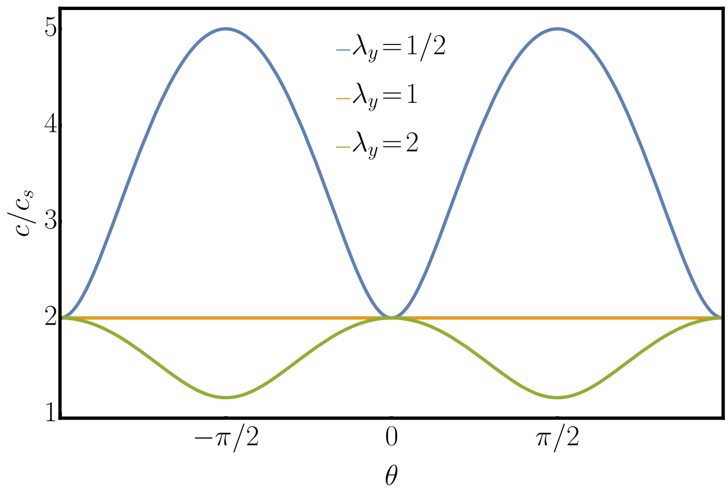

Figure 1: The longitudinal wavespeed (in units of c_s) as a function of the propagation direction θ for three values of λ_y.

Note that if you'd go to the lab and measure the wave speeds, you'll find different results, because these wave speeds are given in the *material coordinates*, and not in the deformed (lab) coordinates. Also, the absence of anisotropy in the shear wave-speed is a special case specific to this constitutive law and not a general feature of finite elasticity.

2 The Rivlin instability

This example was taken from R.S. Rivlin, Large elastic deformation of isotropic materials. II. Some uniqueness theorems for pure homogeneous deformation. Philos. Trans. Roy. Soc. Lond. A 240, 491-508, 1948. Consider a cube of an incompressible material subject to a hydrostatic stress. An obvious state of equilibrium is that the cube remains undeformed. Can the cube deform into a rectangular block?

In this case, we need to return to the 3D formulation, as we have stresses in the z direction. While in the 2D plane stress case the incompressibility was automatically satisfied, here the issue is more delicate. Using Eq. (1) directly is problematic, as the motion in each direction isn't independent. One way to handle such situations is to use a Lagrange multiplier, so that

$$u(\boldsymbol{F}) = \frac{\mu}{2}\left[\mathrm{tr}(\boldsymbol{F}^T\boldsymbol{F}) - 3\right] - \alpha\,(J-1) \ , \tag{31}$$

here we use $\boldsymbol{F}$ to denote the 3D tensor. From this energy density we get three equations, which together with $J=1$ give us four equations for four fields: three components of motion and α. From this density we find

$$\boldsymbol{P} \equiv \frac{\partial u}{\partial \boldsymbol{F}} = \mu \boldsymbol{F} - \alpha J \boldsymbol{F}^{-T} \ , \tag{32}$$

where we used the identify you proved in your second homework $\frac{\partial \det \boldsymbol{A}}{\partial \boldsymbol{A}} = \det(\boldsymbol{A})\boldsymbol{A}^{-T}$.

Next, consider a cube with sides of length L. If we assume that the deformation is homogeneous and without any shear components, the motion is

$$\varphi_x = \lambda_x X \quad , \quad \varphi_y = \lambda_y Y \quad , \quad \varphi_z = \lambda_z Z \ , \tag{33}$$

leading to

$$\boldsymbol{F} = \begin{pmatrix} \lambda_x & 0 & 0 \\ 0 & \lambda_y & 0 \\ 0 & 0 & \lambda_z \end{pmatrix} \ , \ \boldsymbol{P} = \begin{pmatrix} \mu\lambda_x - \alpha\lambda_y\lambda_z & 0 & 0 \\ 0 & \mu\lambda_y - \alpha\lambda_x\lambda_z & 0 \\ 0 & 0 & \mu\lambda_z - \alpha\lambda_x\lambda_y \end{pmatrix} \ . \tag{34}$$

Considering a hydrostatic pressure $\boldsymbol{\sigma} = -p\boldsymbol{I}$, and using $\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$ (Eq. 3.38 in Eran's notes), we find

$$\boldsymbol{P} = -pJ\boldsymbol{F}^{-T} = -p \begin{pmatrix} \lambda_y\lambda_z & 0 & 0 \\ 0 & \lambda_x\lambda_z & 0 \\ 0 & 0 & \lambda_x\lambda_y \end{pmatrix} \ , \tag{35}$$

so we have indeed four equations

$$\begin{aligned} (p-\alpha)\lambda_y\lambda_z + \mu\lambda_x = 0 \quad &, \quad (p-\alpha)\lambda_x\lambda_z + \mu\lambda_y = 0, \\ (p-\alpha)\lambda_x\lambda_y + \mu\lambda_z = 0 \quad &, \quad \lambda_x\lambda_y\lambda_z = 1 \ , \end{aligned} \tag{36}$$

which have only one unique solution (recalling that all the λ's have to be positive)

$$\lambda_x = \lambda_y = \lambda_z = 1 \ , \tag{37}$$

As you'd expect from an incompressible material in a setting which can only compress it. Notice that this solution does not depend on the sign of p, so in general a hydrostatic *tension* would also lead to the same solution.

What will happen if instead of a hydrostatic pressure we instead apply a triaxial force f? By which we mean that we apply a force f on each side. In the linear elastic case there is no difference, as the stress is $\boldsymbol{\sigma} = fL^{-2}\boldsymbol{I}$. What happens in the non-linear case? Here, we need to account for the change in shape, so

$$\boldsymbol{\sigma} = \frac{f}{L^2} \begin{pmatrix} \frac{1}{\lambda_y\lambda_z} & 0 & 0 \\ 0 & \frac{1}{\lambda_x\lambda_z} & 0 \\ 0 & 0 & \frac{1}{\lambda_x\lambda_y} \end{pmatrix} \ , \tag{38}$$

and

$$\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \lambda_x\lambda_y\lambda_z \frac{f}{L^2} \begin{pmatrix} \frac{1}{\lambda_y\lambda_z} & 0 & 0 \\ 0 & \frac{1}{\lambda_x\lambda_z} & 0 \\ 0 & 0 & \frac{1}{\lambda_x\lambda_y} \end{pmatrix} \begin{pmatrix} \frac{1}{\lambda_x} & 0 & 0 \\ 0 & \frac{1}{\lambda_y} & 0 \\ 0 & 0 & \frac{1}{\lambda_z} \end{pmatrix} = \frac{f}{L^2}\boldsymbol{I} \ . \tag{39}$$

equating Eq. (34) and (39), we find a different set of equations

$$\mu\lambda_x - \alpha\lambda_y\lambda_z = \mu\lambda_y - \alpha\lambda_x\lambda_z = \mu\lambda_z - \alpha\lambda_x\lambda_y = \frac{f}{L^2}, \quad \lambda_x\lambda_y\lambda_z = 1 \ . \tag{40}$$

These equations have one trivial solution $\lambda_x = \lambda_y = \lambda_z = 1$ and $\alpha = \mu - \frac{f}{L^2}$. Are there any more? Actually yes. To make things simples, let's consider case where the block is in a state of equal-biaxial stretch, i.e. $\lambda_x = \lambda_y \equiv \lambda$. This of course entails $\lambda_z = \lambda^{-2}$. setting this in to Eq. (40) we are left with two equations for α and λ, and we find

$$\frac{f}{\mu L^2} = \frac{1}{\lambda^2} + \lambda \ , \tag{41}$$

together with $\alpha = -\mu/\lambda$. This equation can be solved in closed form for λ in general, since it is a third order polynomial, but the result isn't very interesting. To understand the physics of this results look at Fig 2. The first thing we notice is that we have a

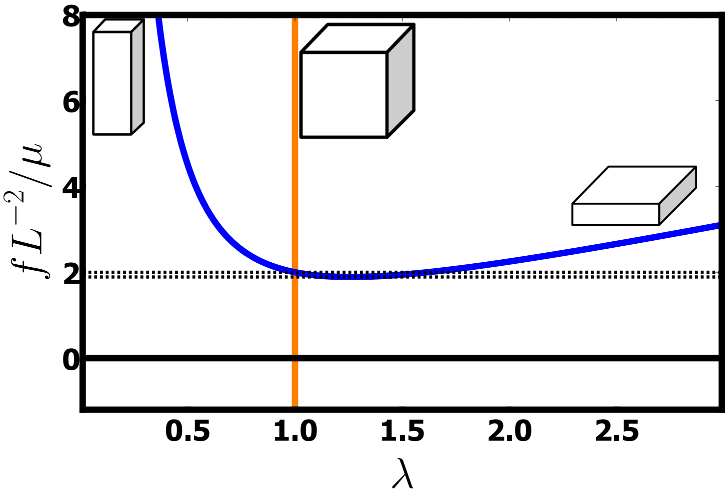

Figure 2: Solution in $\frac{f}{\mu L^2}$, Λ space. The blue line depicts Eq. (41) and the orange line is the no-deformation solution. The dashed lines depict the two bifurcations.

minimum at $\lambda = 2^{1/3}$, so for $f < \frac{3\mu L^2}{2^{2/3}}$ the only solution is $\lambda = 1$, i.e. no deformation. When increasing f past $\frac{3\mu L^2}{2^{2/3}}$ we have a bifurcation, a qualitative change in the solutions, as two new solutions are born. As we pass through $2\mu L^2$, we have another bifurcation, as two solutions coalesce and then split again. The actual mode that will be selected will depend upon the original inhomogeneity of the box/loading.

There are also other solutions, where $\lambda_x \neq \lambda_y$, but the solutions are messy. The threshold $\frac{3\mu L^2}{2^{2/3}}$ still exists, as there are no solution below it other that the no deformation one. This prediction, however, has not been examined experimentally. Unfortunately, no one has actually managed to come us with a way to apply such a triaxial tensile force without creating lateral forces at the interface. But you guys are all young and creative, maybe one of you guys will solve this 70 years old mystery!

3 Divergence in spherical coordinates

In many problems, the geometry of the problem clearly suggests that spherical coordinates should be used. So let's take the opportunity to discuss how to derive the equation of motion $\nabla \cdot \boldsymbol{\sigma} = \rho \ddot{\boldsymbol{u}}$ in curvilinear coordinates. What do we mean when we write a tensor $\boldsymbol{A}$ in Cartesian coordinates as

$$\boldsymbol{A} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix} ? \tag{42}$$

This is a shorthand notation for $\boldsymbol{A} = \sum_{ij} A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ with $i, j \in \{x, y, z\}$ and $\boldsymbol{e}_i$ is the unit vector in the i direction. Writing the tensor in, say, spherical coordinates, means to write it in terms of the unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi$ which are *space-dependent*. Since spherical coordinates are orthonormal, we know that *locally* the transformation from Cartesian to spherical coordinates is given by a rotation,

$$[\boldsymbol{A}]_{r,\phi,\theta} = \boldsymbol{R}(\phi, \theta)^T [\boldsymbol{A}]_{x,y,z} \boldsymbol{R}(\phi, \theta) \ , \tag{43}$$

but you have to remember that the rotation matrix $\boldsymbol{R}$ is different in different points in space.

When you calculate derivative of the tensor in curvilinear coordinates you need to keep track of the fact that not only the components of the tensor change in space, but also the unit vectors themselves change. This amounts to differentiating Eq. (43) and remembering to differentiate both copies of $\boldsymbol{R}(\phi, \theta)$, because ϕ and θ are space-dependent. Doing this properly is a long and technical calculation which we will not do here, but you should be able in principle to do it, and you should definitely understand it's algebraic structure. The bottom line is that the divergence of a tensor in spherical coordinates is

$$\begin{aligned} \nabla \cdot \boldsymbol{A} = & \\ & \left[\frac{\partial A_{rr}}{\partial r} + 2\frac{A_{rr}}{r} + \frac{1}{r}\frac{\partial A_{\theta r}}{\partial \theta} + \frac{\cot\theta}{r} A_{\theta r} + \frac{1}{r\sin\theta}\frac{\partial A_{\phi r}}{\partial \phi} - \frac{1}{r}(A_{\theta\theta} + A_{\phi\phi}) \right] \boldsymbol{e}_r \\ & + \left[\frac{\partial A_{r\theta}}{\partial r} + 2\frac{A_{r\theta}}{r} + \frac{1}{r}\frac{\partial A_{\theta\theta}}{\partial \theta} + \frac{\cot\theta}{r} A_{\theta\theta} + \frac{1}{r\sin\theta}\frac{\partial A_{\phi\theta}}{\partial \phi} + \frac{A_{\theta r}}{r} - \frac{\cot\theta}{r} A_{\phi\phi} \right] \boldsymbol{e}_\theta \\ & + \left[\frac{\partial A_{r\phi}}{\partial r} + 2\frac{A_{r\phi}}{r} + \frac{\sin\theta}{r}\frac{\partial A_{\theta\phi}}{\partial \theta} + \frac{\cos\theta}{r} A_{\theta\phi} + \frac{1}{r\sin\theta}\frac{\partial A_{\phi\phi}}{\partial \phi} + \frac{1}{r}(A_{\phi r} + A_{\phi\theta}) \right] \boldsymbol{e}_\phi \ . \end{aligned} \tag{44}$$

You can find similar expressions for other differential operators (Laplacian, gradient, material derivative, etc.) for both spherical and cylindrical coordinates on this page in wikipedia.