Tensor analysis - Solution

A general comment

The purpose of the HW exercises is to give you hands-on experience with the course materials. We try hard to ask questions that require a conceptual process of understanding, rather than technical computation. Whenever some complicated calculations are required, please remember that it is only in order to convey the mathematical structure of the physical problems that we tackle, a structure that might elude the "passive listener" in the classroom. Accordingly, in the answers you hand in we do not require detailed calculations, unless they are crucial for the understanding.

1 Isotropic tensors

We defined tensors as linear operators transforming n into m vectors. One can define a tensor as an object that under orthogonal (unitary) coordinate transformations (i.e. rotations) transforms as

$$A_{i_1 i_2 \ldots i_k} = Q_{i_1 j_1} Q_{i_2 j_2} \ldots Q_{i_k j_k} A_{j_1 j_2 \ldots j_k} \ , \tag{1}$$

where the $\boldsymbol{Q}$'s represent the orthogonal transformation from coordinates j to coordinates i. We will not bother with the distinction between covariant and contravariant degrees of freedom (though they are crucial in other fields of physics like general relativity).

A tensor is called *isotropic* if its coordinate representation is invariant under coordinate rotation. In this question, we will look at all the possible forms of isotropic tensors of low ranks in 3 dimensions.

(i) How do scalars change under rotations? Does a 0^{th} rank isotropic tensor, a.k.a a scalar, exist? If yes, give an example. If not, explain why.

Solution

A 0^{th} rank tensor, a.k.a a scalar, does not change under rotations, therefore all scalars are isotropic (surprise!).

(ii) A vector $\vec{v}$ is isotropic if for every rotation matrix R_{ij} we have $R_{ij}\, v_j = v_i$. Does a 1^{st} rank isotropic tensor, a.k.a an isotropic vector, exist? If yes, give an example. If not, explain why.

Solution

A vector $\vec{v}$ is isotropic if for every rotation matrix R_{ij} we have

$$R_{ij}\, v_j = v_i \ . \tag{2}$$

You can easily show that this condition is satisfied for arbitrary $\boldsymbol{R}$ only if $\vec{v} = 0$. So the zero vector is the only isotropic vector (surprise #2!!).

(iii) A matrix $\boldsymbol{A}$ is isotropic if for every rotation matrix $\boldsymbol{R}$ we have $A_{ij} = R_{ik}R_{jl}A_{kl}$, or in matrix notation:

$$\boldsymbol{R}\boldsymbol{A}\boldsymbol{R}^T = \boldsymbol{A} \ . \tag{3}$$

- Choose a specific rotation matrix, say a rotation of angle α around $\hat{z}$

$$\boldsymbol{R}^z(\alpha) \equiv \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{4}$$

Using this in the Eq. (3) will be complicated (you can go ahead and try). Instead — expand the matrix for small rotation angle α to linear order i.e. $\boldsymbol{R}^z(\alpha) \simeq \boldsymbol{M}_0 + \alpha\boldsymbol{M}_1$. What is the zeroth order matrix $\boldsymbol{M}_0$? What is the matrix $\boldsymbol{M}_1$? (hint: you may have encountered these objects before, e.g., in quantum mechanics courses).

Solution

We expand in orders of α to find:

$$\boldsymbol{R}^z(\alpha) \equiv \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \simeq \boldsymbol{I} + \alpha \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\equiv \boldsymbol{L}^z} + \mathcal{O}\left(\alpha^2\right) \ . \tag{5}$$

We see that $\boldsymbol{M}_0$ is the identity matrix, and $\boldsymbol{M}_1$ is the generator of rotations in the z direction, a.k.a. $\boldsymbol{L}^z$!

- Use the approximate matrix $\boldsymbol{M}_0 + \alpha\boldsymbol{M}_1$ in Eq. (3). Differentiate both sides of the equation with respect to α, and then substitute $\alpha = 0$. What conditions does the entries of the matrix $\boldsymbol{A}$ should satisfy?

Solution

Differentiating with respect to α and plugging $\alpha = 0$ gives

$$0 = \frac{\partial \boldsymbol{A}(0)}{\partial\alpha} = \left.\frac{\partial \boldsymbol{A}(\alpha)}{\partial\alpha}\right|_{\alpha=0} = \left.\frac{\partial \boldsymbol{R}^z(\alpha)}{\partial\alpha}\right|_{\alpha=0} \boldsymbol{A}\,\boldsymbol{R}^z(0) + \boldsymbol{R}^z(0)\,\boldsymbol{A}\,\left.\frac{\partial \boldsymbol{R}^z(\alpha)^T}{\partial\alpha}\right|_{\alpha=0} \ , \tag{6}$$

but since $\boldsymbol{R}^z(0)$ is the identity matrix, this reduces to the simple equation

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \boldsymbol{A} + \boldsymbol{A} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0 \ . \tag{7}$$

We see that equation $\boldsymbol{A}(0) = \boldsymbol{A}(\alpha)$ is equivalent to the much easier equation (notice the sign change)

$$\boldsymbol{A}(0) = \boldsymbol{A}(\alpha) \quad \Longleftrightarrow \quad [\boldsymbol{A}, \boldsymbol{L}^z] = 0 \ . \tag{8}$$

Explicitly calculating $[\boldsymbol{A}, \boldsymbol{L}^z]$ gives

$$[\boldsymbol{A}, \boldsymbol{L}^z] = \begin{pmatrix} -A_{12} - A_{21} & A_{11} - A_{22} & -A_{23} \\ A_{11} - A_{22} & A_{12} + A_{21} & A_{13} \\ -A_{32} & A_{31} & 0 \end{pmatrix} . \tag{9}$$

We see that commutation with $\boldsymbol{L}^z$ requires (a) $A_{13} = A_{31} = A_{23} = A_{32} = 0$ and (b) $A_{11} = A_{22}$.

- The choice of $\hat{z}$ was arbitrary. What conditions will you get if you were to repeat the above procedure for rotations around different axis?

Solution

If we repeat the above procedure for the other $\boldsymbol{L}$'s, the analog of (a) will be that all off-diagonal elements must vanish, and the analog of (b) will be that all diagonal elements must be equal.

- Does a 2nd rank isotropic tensor, a.k.a an isotropic matrix, exist? If yes, give an example. If not, explain why.

Solution

All of the above implies that a 2nd rank isotropic tensor has the form

$$A_{ij} \propto \delta_{ij} . \tag{10}$$

(iv) **Bonus I**: A 3rd rank tensor $\boldsymbol{A}$ is isotropic iff for every rotation matrix R_{ij} we have

$$R_{i\alpha}\, R_{j\beta}\, R_{k\gamma} A_{\alpha\beta\gamma} = A_{ijk} . \tag{11}$$

You can imagine the mess that comes out of this if you plug in a real rotation matrix with sines and cosines and whatnot, and then start using trig identities. Phew, no thanks!

- Instead, like before, choose $\boldsymbol{R} = \boldsymbol{R}^z(\alpha)$, differentiate, and set $\alpha = 0$. You should end up with

$$\begin{aligned} 0 = &\left(L^z_{i\alpha}\, \delta_{j\beta}\, \delta_{k\gamma} + \delta_{i\alpha}\, L^z_{j\beta}\, \delta_{k\gamma} + \delta_{i\alpha}\, \delta_{j\beta}\, L^z_{k\gamma}\right) A_{\alpha\beta\gamma} \\ = &\, L^z_{i\alpha}\, A_{\alpha jk} + L^z_{j\beta}\, A_{i\beta k} + L^z_{k\gamma}\, A_{ij\gamma} . \end{aligned} \tag{12}$$

Solution

Some algebra is required here, but the final form is given above.

- To see what kind of equation we got, let's choose $i = 1,\ j = 3,\ k = 3$. Since the only non-zero elements of $\boldsymbol{L}^z$ are L^z_{12} and L^z_{21}, we get

$$0 = L^z_{1\alpha}\, A_{\alpha 33} + L^z_{3\beta}\, A_{1\beta 3} + L^z_{3\gamma}\, A_{13\gamma} = A_{233} . \tag{13}$$

Similarly, by choosing different combinations of i, j, k and/or different $\boldsymbol{L}$'s, you get that $A_{ijk} = 0$ whenever i, j, k are not all different, that is, if (ijk) is not a permutation of (123).
Using this knowledge, we can choose now $i = 1,\ j = 1,\ k = 3$, and we get

$$A_{113} = 0 = L^z_{1\alpha} A_{\alpha 13} + L^z_{1\beta} A_{1\beta 3} + L^z_{3\gamma} A_{11\gamma} = A_{213} + A_{123} \ ,$$

or put differently, $A_{213} = -A_{123}$. Similarly, we can show that every time we flip two indices we get a minus sign. Can you guess what is this 3^{rd} rank isotropic tensor $\boldsymbol{A}$?

Solution

We guess that the only isotropic 3^{rd} rank tensor is equal, up to a multiplicative constant, to $\boldsymbol{\mathcal{E}}$,

$$\mathcal{E}_{ijk} = \begin{cases} 0 & (ijk) \text{ is not a permutation of } (123) \\ \text{sign of permutation} & \text{otherwise} \end{cases} \ . \tag{14}$$

As you probably know, $\boldsymbol{\mathcal{E}}$ is called the Levi-Civita completely anti-symmetric tensor[1].

(v) **Bonus II**: You have shown above (if done correctly) that in 3 dimensions a 2^{nd} rank isotropic tensor must be proportional to δ_{ij}, (in fact, this is true for all dimensions ≥ 3). However, in 2D this does not hold. Find the general form of an isotropic two-dimensional 2^{nd} rank tensor. What kind of symmetry do these tensors violate (those not proportional to the identity)?

Solution

In 2D, there's only one rotation, so A is isotropic iff

$$\left[\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right] = \begin{pmatrix} a_{12} + a_{21} & a_{22} - a_{11} \\ a_{22} - a_{11} & -a_{12} - a_{21} \end{pmatrix} = 0$$

That is, we A is isotropic iff $a_{12} = -a_{21}$ and $a_{11} = a_{22}$. Thus, the general form of an isotropic 2D tensor is $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ and is itself proportional to a rotation, because

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \frac{1}{\sqrt{a^2 + b^2}} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

for $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$. So it is not surprising that it commutes with other rotations.

Another way to look at the same thing: We can write the general isotropic tensor as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = a\,\boldsymbol{I} + b\,\boldsymbol{L}^z \ .$$

So we actually proved that there are exactly two independent isotropic tensors in 2D: $\boldsymbol{I}$ and $\boldsymbol{L}^z$. Both can be generalized to higher dimensions and ranks. In general, one can always construct an isotropic tensor with an even number of indices from copies of $\boldsymbol{I}$ (in our case, one copy), and this is not surprising. About $\boldsymbol{L}^z$, you may look at it as the *completely antisymmetric tensor of rank 2*, i.e. the 2-rank analogue of $\boldsymbol{\mathcal{E}}$. This is because L^z_{ij} is zero when $i = j$, 1 when (ij) is an even permutation of (12) and -1 when it's an odd permutation. In general, the completely antisymmetric tensor of rank k in k dimensions is isotropic. It's geometrical meaning is that

$$\mathcal{E}_{i_1,i_2,i_3,\ldots,i_k}\, v^1_{i_1}\, v^2_{i_2} \ldots v^k_{i_k} = \det \begin{pmatrix} - & v^1 & - \\ - & v^2 & - \\ & \vdots & \\ - & v^k & - \end{pmatrix} ,$$

and in our case

$$L^z_{ij} v_i u_j = v_1 u_2 - v_2 u_1 = \det \begin{pmatrix} v_1 & v_2 \\ u_1 & u_2 \end{pmatrix} .$$

Last comment: This is out of the scope of this course, but for those of you who are interested in this kinda stuff: The fact that there are non-trivial isotropic 2D matrices is closely related to the fact that SO_2 is a one-parameter Lie group, and hence abelian. SO_n for $n > 2$ is not Abelian.

Can you think of an example of an isotropic 2D tensor that is not diagonal, for a real physical system?

Solution

Three generic (and closely related) examples would be:

(i) The conductivity tensor in a 2D plate when a perpendicular magnetic field is present,

(ii) The matrix that relates the velocity of a charged particle in 2D with the Lorentz force,

(iii) The tensor that relates the velocity to the Coriolis force in a rotating disc.

These tensors violate reflection symmetry in the 3^{rd} dimension, which is also related to time-reversal symmetry.

2 Tensor integration — Archimedes law

Fluids exert forces on bodies that are submerged in them. At each point on the body's surface, denote the local normal by $\hat{\boldsymbol{n}}$. The force per unit area exerted by the fluid is given by $f_i = \sigma_{ij} n_j$, where the index j is summed over, and the index i is not. $\boldsymbol{\sigma}$ is called the *stress tensor* of the fluid, and we'll deal with it extensively in the course. The component σ_{ij} denote the force in the i direction applied to areal element whose normal is in the j direction. Consider a stationary (hydro-static), isotropic fluid that occupies the bottom half-space $z < 0$. The fluid is subjected to a constant gravitational field $-g\hat{\boldsymbol{z}}$. At $z=0$, we have $\sigma_{ij}=0$; that is, the surface of the fluid is stress-free (we neglect air pressure).

(i) The off-diagonal elements of σ_{ij} are called *shear stresses*. Almost by definition, in a stationary fluid the shear stresses must vanish. Therefore, for $i \neq j$ we must have $\sigma_{ij}=0$ *for every choice of coordinate system*. Prove that this implies $\sigma_{ij}=-p(\boldsymbol{r})\,\delta_{ij}$, where $p(\boldsymbol{r})$ is a scalar field (hint: think about isotropic tensors). Note: $p=-\frac{1}{3}\operatorname{tr}(\boldsymbol{\sigma})$ is called *pressure*.

Solution

Choose any coordinate system. Since the shear stresses vanish, we can write

$$\boldsymbol{\sigma}=\begin{pmatrix}\sigma_{11} & 0 & 0\\ 0 & \sigma_{22} & 0\\ 0 & 0 & \sigma_{33}\end{pmatrix}$$

Let's rotate around the z axis, by:

$$\boldsymbol{R}=\begin{pmatrix}\cos\alpha & -\sin\alpha & 0\\ \sin\alpha & \cos\alpha & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$\tilde{\boldsymbol{\sigma}}=\boldsymbol{R}^T\boldsymbol{\sigma}\boldsymbol{R}=\begin{pmatrix}\sigma_{11}\cos\alpha^2+\sigma_{22}\sin\alpha^2 & (\sigma_{22}-\sigma_{11})\cos\alpha\sin\alpha & 0\\ (\sigma_{11}-\sigma_{22})\cos\alpha\sin\alpha & \sigma_{22}\cos\alpha^2+\sigma_{11}\sin\alpha^2 & 0\\ 0 & 0 & \sigma_{33}\end{pmatrix}.$$

Since the off-diagonal terms must vanish in this basis too, we have $\sigma_{11}=\sigma_{22}$. The same works for σ_{33}, and we see that $\boldsymbol{\sigma}$ is proportional to δ_{ij}.

A different way, in the spirit of the previous question: differentiate $\boldsymbol{\sigma}(\alpha)$ with respect to α and plug in $\alpha=0$:

$$\partial_\alpha\boldsymbol{\sigma}|_{\alpha=0}=\boldsymbol{L}^z\boldsymbol{\sigma}-\boldsymbol{\sigma}\boldsymbol{L}^z=[\boldsymbol{\sigma},\boldsymbol{L}^z]=\begin{pmatrix}0 & \sigma_{yy}-\sigma_{xx} & 0\\ -\sigma_{yy}+\sigma_{xx} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

And since $\boldsymbol{\sigma}$ is always be diagonal, its derivative cannot have off-diagonal elements, so we must have $\sigma_{yy}=\sigma_{xx}$.

(ii) By considering the force balance on a small cube of fluid and the translational symmetries of the system (in the $x-y$ plane), show that the stress field satisfies the equation

$$\partial_z\sigma_{zz}(x,y,z)=-\rho g$$

where ρ is the fluid's density. Together with the results of (i), conclude that the stress tensor is given by $\sigma_{ij}=-\rho g z\delta_{ij}$ (note that you satisfy both the equation and the boundary conditions).

Solution

Due to symmetry, $\partial_x \sigma_{ij} = \partial_y \sigma_{ij} = 0$. The total force on a small cube of fluid must vanish, because the situation is static. The gravitational force is $\rho g \cdot dx \cdot dy \cdot dz$. The vertical force exerted on this cube from the lower side is $\sigma_{zz}(x, y, z)dx \cdot dy$, and the force exerted by the upper side is $-\sigma_{zz}(x, y, z + dz)$. Therefore, force balance tells us that

$$[-\sigma_{zz}(x, y, z + dz) + \sigma_{zz}(x, y, z)]\, dx \cdot dy = \rho g\, dx \cdot dy \cdot dz$$

$$\frac{\sigma_{zz}(x, y, z + dz) - \sigma_{zz}(x, y, z)}{dz} = -\rho g$$

Taking the limit $dz \to 0$, one gets the differential equation $\partial_z \sigma_{zz} = -\rho g$, the solution of which is clearly $\sigma_{zz} = -\rho g z$. Since we know already that $\boldsymbol{\sigma} \propto \delta_{ij}$ we immediately have $\sigma_{ij} = -\rho g z \delta_{ij}$.

(iii) Consider an imaginary surface within the fluid, of arbitrary shape and volume V. Calculate the magnitude and direction of the **total** force exerted by the surrounding fluid on the enclosed fluid by integrating $\sigma_{ij} n_j$ over the imaginary surface (hint: recall Gauss' tensorial theorem). This force is called the *Buoyancy force*.

Solution

Denote the space occupied by the enclosed fluid by V and its surface by ∂V. The total force is

$$\vec{F} = -\int_{\partial V} \boldsymbol{\sigma} \hat{\boldsymbol{n}} dS \ .$$

Note the minus sign, because $\boldsymbol{\sigma}\hat{\boldsymbol{n}}$ (for outwards-pointing normal) is the force exerted by the internal fluid on the outer one, and we want the force exerted by the fluid on the enclosed fluid. By Gauss' theorem this is equal to

$$\vec{F} = -\int_V \mathrm{div}\, \boldsymbol{\sigma} dV = \int_V \rho g \vec{e}_z dV = \rho g \vec{e}_z \int_V dV = \rho g V \vec{e}_z$$

where $\vec{e}_z$ is the unit vector in the z direction.

(iv) Take the same shape and volume of (iii) and replace it with a solid body of arbitrary mass density ρ_s, and hold it in its place (within the surrounding fluid). What forces are needed to keep this body within the fluid? (hint: the situation is *static*). What would happen if you let the solid body go?

Solution

The force exerted on the solid must exactly balance its weight. The gravitational force due to the solid is

$$\vec{F}_s = -mg\vec{e}_z = -\rho_s g V \vec{e}_z \ .$$

The fluid gives $\vec{F}_f = \rho_f g V \vec{e}_z$, such that overall the net force is

$$F = (\rho_f - \rho_s)\, gV\vec{e}_z \ .$$

In order to have a static situation, one would have to apply an external force that is exactly opposite of this net force, to produce force balance. That is, the external force needed to create a static scenario is $F_{ext} = -\,(\rho_f - \rho_s)\, gV\vec{e}_z$. If one then releases the solid (i.e. taking away the external force), the solid will start sinking if $\rho_f < \rho_s$, while if $\rho_f > \rho_s$ the fluid will push the body up.

(v) **Bonus**: Demonstrate this effect using your favorite solids and fluids. Stand up and shout out loud "Eurika!!" (*Note: only filmed evidence will be considered for bonus purposes*).

Solution

There are many possible solutions to this question. One of them is shown here: https://www.youtube.com/watch?v=JUxJBgJ5FJ4. It's not the best one, it's simply the only one that was uploaded to YouTube.

3 Invariants

A scalar function of a tensor $f(\boldsymbol{A}) = f(A_{ij})$ or of a vector $g(\vec{v}) = g(v_i)$ is called invariant if its value is independent of the choice of basis. That is, if it has a proper geometric meaning which is independent of the particular basis that one happens to choose. Later in this course, we will be interested in scalar invariants of tensors. For example, the elastic energy is a scalar invariant of the strain tensor.

(i) Show that the trace is the only linear invariant scalar of a 2^{nd} rank tensor $\boldsymbol{A}$. That is, show that if $f(\boldsymbol{A})$ is an invariant function that is linear in $\boldsymbol{A}$'s entries, it can be written as $f(\boldsymbol{A}) = \lambda \operatorname{tr} \boldsymbol{A}$ for some constant λ. Assume the dimension is ≥ 3.

Solution

Let $f(A_{ij})$ be a scalar invariant that is linear in the entries of $\boldsymbol{A}$. We know f is linear means that we can write it as a linear combination of the entries of $\boldsymbol{A}$, i.e.

$$f(A_{ij}) = C_{ij}A_{ij} = \operatorname{tr}\left(\boldsymbol{C}\boldsymbol{A}^T\right) = \boldsymbol{C} : \boldsymbol{A}$$

where C_{ij} is some matrix. The invariance of f means that $f(A)$ should be unchanged when applying a rotation $\boldsymbol{Q}$:

$$\operatorname{tr}\left(\boldsymbol{C}\tilde{\boldsymbol{A}}^T\right) = \operatorname{tr}\left(\boldsymbol{C}\boldsymbol{Q}^T\boldsymbol{A}^T\boldsymbol{Q}\right) = \operatorname{tr}\left(\boldsymbol{Q}\boldsymbol{C}\boldsymbol{Q}^T\boldsymbol{A}^T\right)$$

This should be true for all $\boldsymbol{A}$, which means $\boldsymbol{QCQ}^T = \boldsymbol{C}$. Or in other words, this means that $\boldsymbol{C}$ rotates like a proper tensor, and is isotropic. As we've seen, the only 2^{nd} rank isotropic tensor is δ_{ij}, up to a multiplicative constant. Therefore

$$f(\boldsymbol{A}) \propto \delta_{ij} A_{ij} = A_{ii} = \mathrm{tr}(\boldsymbol{A})$$

As you know by now, this only works for $d \geq 3$ unless we demand that f should also be reflection invariant. Otherwise, $\boldsymbol{L}^z : \boldsymbol{A}$ is also invariant.

(ii) Show that the only quadratic invariants of a 2^{nd} rank tensor $\boldsymbol{A}$ are $\mathrm{tr}\,(\boldsymbol{A}^2)$, $(\mathrm{tr}\,\boldsymbol{A})^2$, and $\mathrm{tr}\left(\boldsymbol{A}\boldsymbol{A}^T\right)$. That is, show that if $f(\boldsymbol{A})$ is invariant and quadratic in $\boldsymbol{A}$'s entries, it can be written as $f(\boldsymbol{A}) = \lambda_1 \,\mathrm{tr}\,(\boldsymbol{A}^2) + \lambda_2 \,(\mathrm{tr}\,\boldsymbol{A})^2 + \lambda_3 \,\mathrm{tr}\left(\boldsymbol{A}\boldsymbol{A}^T\right)$ (hint: think about the isotropic tensors of question 1).

Solution

Exactly like before, if f is a quadratic function in the entries of $\boldsymbol{A}$, it must be of the form

$$f(\boldsymbol{A}) = C_{ijkl} A_{ij} A_{kl}$$

for some tensor $\boldsymbol{C}$, which must be isotropic. There are exactly 3 options for such a tensor, as we saw in class:

$$A_{ij} A_{kl} \delta_{ij}\, \delta_{kl} = A_{ii} A_{kk} = (\mathrm{tr}\,\boldsymbol{A})^2$$
$$A_{ij} A_{kl} \delta_{il}\, \delta_{jk} = A_{ij} A_{ji} = \mathrm{tr}\left(\boldsymbol{A}^2\right)$$
$$A_{ij} A_{kl} \delta_{ik}\, \delta_{kj} = A_{ij} A_{ij} = \mathrm{tr}\left(\boldsymbol{A}\boldsymbol{A}^T\right)$$

All quadratic invariant functions are linear combination of these.