Non-Equilibrium Continuum Physics **HW set #1**

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Tensor analysis

A general comment

The purpose of the HW exercises is to give you hands-on experience with the course materials. We try hard to ask questions that require a conceptual process of understanding, rather than technical computation. Whenever some complicated calculations are required, please remember that it is only in order to convey the mathematical structure of the physical problems that we tackle, a structure that might elude the "passive listener" in the classroom. Accordingly, in the answers you hand in we do not require detailed calculations, unless they are crucial for the understanding.

1 Isotropic tensors

We defined tensors as linear operators transforming n into m vectors. One can define a tensor as an object that under orthogonal (unitary) coordinate transformations (i.e. rotations) transforms as

$$A_{i_1 i_2 \ldots i_k} = Q_{i_1 j_1} Q_{i_2 j_2} \ldots Q_{i_k j_k} A_{j_1 j_2 \ldots j_k} \ , \tag{1}$$

where the $\boldsymbol{Q}$'s represent the orthogonal transformation from coordinates j to coordinates i. We will not bother with the distinction between covariant and contravariant degrees of freedom (though they are crucial in other fields of physics like general relativity).

A tensor is called *isotropic* if its coordinate representation is invariant under coordinate rotation. In this question, we will look at all the possible forms of isotropic tensors of low ranks in 3 dimensions.

(i) How do scalars change under rotations? Does a 0^{th} rank isotropic tensor, a.k.a a scalar, exist? If yes, give an example. If not, explain why.

(ii) A vector $\vec{v}$ is isotropic if for every rotation matrix R_{ij} we have $R_{ij}\, v_j = v_i$. Does a 1^{st} rank isotropic tensor, a.k.a an isotropic vector, exist? If yes, give an example. If not, explain why.

(iii) A matrix $\boldsymbol{A}$ is isotropic if for every rotation matrix $\boldsymbol{R}$ we have $A_{ij} = R_{ik} R_{jl} A_{kl}$, or in matrix notation:

$$\boldsymbol{R}\boldsymbol{A}\boldsymbol{R}^T = \boldsymbol{A} \ . \tag{2}$$

- Choose a specific rotation matrix, say a rotation of angle α around $\hat{z}$

$$\boldsymbol{R}^z(\alpha) \equiv \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{3}$$

  Using this in the Eq. (2) will be complicated (you can go ahead and try). Instead — expand the matrix for small rotation angle α to linear order i.e. $\boldsymbol{R}^z(\alpha) \simeq \boldsymbol{M}_0 + \alpha \boldsymbol{M}_1$. What is the zeroth order matrix $\boldsymbol{M}_0$? What is the matrix $\boldsymbol{M}_1$? (hint: you may have encountered these objects before, e.g., in quantum mechanics courses).

- Use the approximate matrix $\boldsymbol{M}_0+\alpha\boldsymbol{M}_1$ in Eq. (2). Differentiate both sides of the equation with respect to α, and then substitute $\alpha=0$. What conditions does the entries of the matrix $\boldsymbol{A}$ should satisfy?
- The choice of $\hat{z}$ was arbitrary. What conditions will you get if you were to repeat the above procedure for rotations around different axis?
- Does a 2^{nd} rank isotropic tensor, a.k.a an isotropic matrix, exist? If yes, give an example. If not, explain why.

(iv) **Bonus I**: A 3^{rd} rank tensor $\boldsymbol{A}$ is isotropic iff for every rotation matrix R_{ij} we have

$$R_{i\alpha}\,R_{j\beta}\,R_{k\gamma}A_{\alpha\beta\gamma}=A_{ijk}\ . \tag{4}$$

You can imagine the mess that comes out of this if you plug in a real rotation matrix with sines and cosines and whatnot, and then start using trig identities. Phew, no thanks!

- Instead, like before, choose $\boldsymbol{R}=\boldsymbol{R}^z(\alpha)$, differentiate, and set $\alpha=0$. You should end up with

$$\begin{aligned}0&=\Big(L^z_{i\alpha}\,\delta_{j\beta}\,\delta_{k\gamma}+\delta_{i\alpha}\,L^z_{j\beta}\,\delta_{k\gamma}+\delta_{i\alpha}\,\delta_{j\beta}\,L^z_{k\gamma}\Big)A_{\alpha\beta\gamma}\\&=L^z_{i\alpha}\,A_{\alpha jk}+L^z_{j\beta}\,A_{i\beta k}+L^z_{k\gamma}\,A_{ij\gamma}\ .\end{aligned} \tag{5}$$

- To see what kind of equation we got, let's choose $i=1,\ j=3,\ k=3$. Since the only non-zero elements of $\boldsymbol{L}^z$ are L^z_{12} and L^z_{21}, we get

$$0=L^z_{1\alpha}\,A_{\alpha 33}+L^z_{3\beta}\,A_{1\beta 3}+L^z_{3\gamma}\,A_{13\gamma}=A_{233}\ . \tag{6}$$

  Similarly, by choosing different combinations of i,j,k and/or different $\boldsymbol{L}$'s, you get that $A_{ijk}=0$ whenever i,j,k are not all different, that is, if (ijk) is not a permutation of (123).
  Using this knowledge, we can choose now $i=1,\ j=1,\ k=3$, and we get

$$A_{113}=0=L^z_{1\alpha}\,A_{\alpha 13}+L^z_{1\beta}\,A_{1\beta 3}+L^z_{3\gamma}\,A_{11\gamma}=A_{213}+A_{123}\ ,$$

  or put differently, $A_{213}=-A_{123}$. Similarly, we can show that every time we flip two indices we get a minus sign. Can you guess what is this 3^{rd} rank isotropic tensor $\boldsymbol{A}$?

(v) **Bonus II**: You have shown above (if done correctly) that in 3 dimensions a 2^{nd} rank isotropic tensor must be proportional to δ_{ij}, (in fact, this is true for all dimensions ≥ 3). However, in 2D this does not hold. Find the general form of an isotropic two-dimensional 2^{nd} rank tensor. What kind of symmetry do these tensors violate (those not proportional to the identity)?

Can you think of an example of an isotropic 2D tensor that is not diagonal, for a real physical system?

2 Tensor integration — Archimedes law

Fluids exert forces on bodies that are submerged in them. At each point on the body's surface, denote the local normal by $\hat{\boldsymbol{n}}$. The force per unit area exerted by the fluid is given by $f_i=\sigma_{ij}n_j$, where the index j is summed over, and the index i is not. $\boldsymbol{\sigma}$ is called the *stress tensor* of the fluid, and we'll deal with it extensively in the course. The component σ_{ij} denote the force in the i direction applied to areal element whose normal is in the j direction. Consider a stationary (hydro-static), isotropic fluid that occupies the bottom half-space $z < 0$. The fluid is subjected to a constant gravitational field $-g\hat{\boldsymbol{z}}$. At $z = 0$, we have $\sigma_{ij} = 0$; that is, the surface of the fluid is stress-free (we neglect air pressure).

(i) The off-diagonal elements of σ_{ij} are called *shear stresses*. Almost by definition, in a stationary fluid the shear stresses must vanish. Therefore, for $i \neq j$ we must have $\sigma_{ij} = 0$ *for every choice of coordinate system*. Prove that this implies $\sigma_{ij} = -p(\boldsymbol{r})\,\delta_{ij}$, where $p(\boldsymbol{r})$ is a scalar field (hint: think about isotropic tensors). Note: $p = -\frac{1}{3}\,\mathrm{tr}(\boldsymbol{\sigma})$ is called *pressure*.

(ii) By considering the force balance on a small cube of fluid and the translational symmetries of the system (in the $x - y$ plane), show that the stress field satisfies the equation

$$\partial_z \sigma_{zz}(x, y, z) = -\rho g$$

where ρ is the fluid's density. Together with the results of (i), conclude that the stress tensor is given by $\sigma_{ij} = -\rho g z \delta_{ij}$ (note that you satisfy both the equation and the boundary conditions).

(iii) Consider an imaginary surface within the fluid, of arbitrary shape and volume V. Calculate the magnitude and direction of the **total** force exerted by the surrounding fluid on the enclosed fluid by integrating $\sigma_{ij} n_j$ over the imaginary surface (hint: recall Gauss' tensorial theorem). This force is called the *Buoyancy force*.

(iv) Take the same shape and volume of (iii) and replace it with a solid body of arbitrary mass density ρ_s, and hold it in its place (within the surrounding fluid). What forces are needed to keep this body within the fluid? (hint: the situation is *static*). What would happen if you let the solid body go?

(v) **Bonus**: Demonstrate this effect using your favorite solids and fluids. Stand up and shout out loud "Eurika!!" (*Note: only filmed evidence will be considered for bonus purposes*).

3 Invariants

A scalar function of a tensor $f(\boldsymbol{A}) = f(A_{ij})$ or of a vector $g(\vec{v}) = g(v_i)$ is called invariant if its value is independent of the choice of basis. That is, if it has a proper geometric meaning which is independent of the particular basis that one happens to choose. Later in this course, we will be interested in scalar invariants of tensors. For example, the elastic energy is a scalar invariant of the strain tensor.

(i) Show that the trace is the only linear invariant scalar of a 2^{nd} rank tensor $\boldsymbol{A}$. That is, show that if $f(\boldsymbol{A})$ is an invariant function that is linear in $\boldsymbol{A}$'s entries, it can be written as $f(\boldsymbol{A}) = \lambda\,\mathrm{tr}\,\boldsymbol{A}$ for some constant λ. Assume the dimension is ≥ 3.

(ii) Show that the only quadratic invariants of a 2^{nd} rank tensor $\boldsymbol{A}$ are $\mathrm{tr}\,(\boldsymbol{A}^2)$, $(\mathrm{tr}\,\boldsymbol{A})^2$, and $\mathrm{tr}\left(\boldsymbol{A}\boldsymbol{A}^T\right)$. That is, show that if $f(\boldsymbol{A})$ is invariant and quadratic in $\boldsymbol{A}$'s entries, it can be written as $f(\boldsymbol{A}) = \lambda_1\,\mathrm{tr}\,(\boldsymbol{A}^2) + \lambda_2\,(\mathrm{tr}\,\boldsymbol{A})^2 + \lambda_3\,\mathrm{tr}\left(\boldsymbol{A}\boldsymbol{A}^T\right)$ (hint: think about the isotropic tensors of question 1).