Non-Equilibrium Continuum Physics **HW set #2**

TA: Avraham Moriel and Roy Wexler Submit to:groupbouchbinder@gmail.com

Due 17.05.2023 Submit as: NECP_HW2_⟨FirstName⟩_⟨LastName⟩.pdf

Kinematics - Solution

1 Eulerian and Lagrangian frameworks

Consider the following 2D deformation:

$$x_1(t) = \cosh(t)X_1 + \sinh(t)X_2 \ , \qquad x_2(t) = \sinh(t)X_1 + \cosh(t)X_2 \ .$$

(i) Find the material velocity and the acceleration $\boldsymbol{V}, \boldsymbol{A}$ and express their spatial forms $\boldsymbol{v}, \boldsymbol{a}$. Remember to represent each field in the proper coordinates (i.e. $\boldsymbol{V}, \boldsymbol{A}$ in terms of $\boldsymbol{X}$ and $\boldsymbol{v}, \boldsymbol{a}$ in terms of $\boldsymbol{x}$). Plot schematically $\boldsymbol{V}$ and $\boldsymbol{v}$ at $t = -10, 0, 10$. Note how vastly different $\boldsymbol{V}$ and $\boldsymbol{v}$ are!

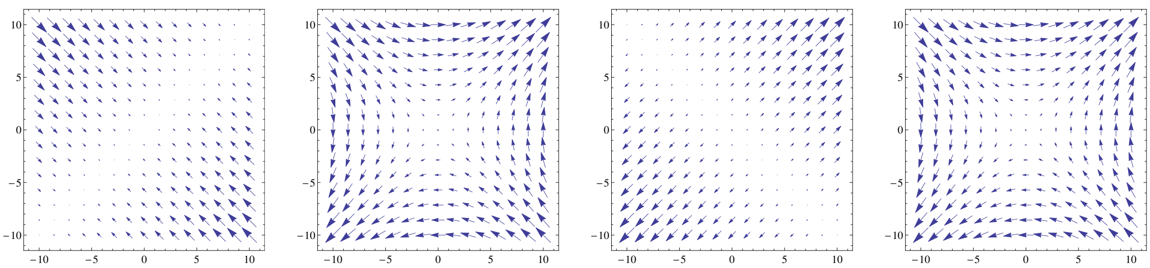
Solution

$$\boldsymbol{V} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} \sinh(t)X_1 + \cosh(t)X_2 \\ \cosh(t)X_1 + \sinh(t)X_2 \end{pmatrix} \ .$$

Note that this can be simply expressed as $\begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$, so we also found $\boldsymbol{v} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$.

Similarly,

$$\boldsymbol{A} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \begin{pmatrix} \cosh(t)X_1 + \sinh(t)X_2 \\ \sinh(t)X_1 + \cosh(t)X_2 \end{pmatrix} \ , \qquad \text{and } \boldsymbol{a} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$



$V(t=-10)$ $V(t=0)$ $V(t=10)$ v (for all t)

Note that $\boldsymbol{V}$ changes exponentially in time while $\boldsymbol{v}$ is constant (!!). This goes to show how different things may look like if they're presented as a function of $\boldsymbol{X}$ or $\boldsymbol{x}$.

(ii) The acceleration $\boldsymbol{a}$ can also be calculated as a material derivative of the velocity:

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} \ .$$

Calculate $\boldsymbol{a}$ using this expression, and show that the results coincide.

Solution

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} = \vec{0} + (v_1 \partial_{x_1} + v_2 \partial_{x_2})\, \boldsymbol{v} = (x_2 \partial_{x_1} + x_1 \partial_{x_2}) \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

(iii) Calculate $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$ and $J = \det \boldsymbol{F}$.

Solution

$$\boldsymbol{F} = \begin{pmatrix} \partial_{X_1} x_1 & \partial_{X_2} x_1 \\ \partial_{X_1} x_2 & \partial_{X_2} x_2 \end{pmatrix} = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} ,$$

and clearly $J = \det \boldsymbol{F} = 1$.

(iv) Calculate the Green-Lagrange strain tensor $\boldsymbol{E}$, and the Euler-Almansi strain tensor $\boldsymbol{e}$, and show that the results coincide.

Solution

We know that the Green-Lagrange strain tensor is expressed as $E = \frac{1}{2}\left(\boldsymbol{F}^T \boldsymbol{F} - \boldsymbol{I}\right)$. Taking $\boldsymbol{F}$ from the previous section, we obtain

$$\boldsymbol{E} = \begin{pmatrix} \sinh^2(t) & \sinh(t)\cosh(t) \\ \sinh(t)\cosh(t) & \sinh^2(t) \end{pmatrix} .$$

To obtain the Euler-Almansi strain tensor $\boldsymbol{e}$, we can either express the X's in terms of x's, giving $X_1 = \cosh(t) x_1 - \sinh(t) x_2$, and $X_2 = -\sinh(t) x_1 + \cosh(t) x_2$ — this is the inverse mapping $\boldsymbol{\phi}^{-1}$. Next, we define $\boldsymbol{f}$ to be the "equivalent" of $\boldsymbol{F}$ only in the Eulerian frame, as $\boldsymbol{f} \equiv \frac{\partial \boldsymbol{\phi}^{-1}}{\partial \boldsymbol{x}}$. It takes the form

$$\boldsymbol{f} = \begin{pmatrix} \cosh(t) & -\sinh(t) \\ -\sinh(t) & \cosh(t) \end{pmatrix} = \boldsymbol{F}^{-1} .$$

From here its rather easy to see that

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{f}^T \boldsymbol{f}\right) = \begin{pmatrix} -\sinh^2(t) & \sinh(t)\cosh(t) \\ \sinh(t)\cosh(t) & -\sinh^2(t) \end{pmatrix} = \frac{1}{2}\left(\boldsymbol{I} - \boldsymbol{F}^{-T} \boldsymbol{F}^{-1}\right) .$$

Note that $\boldsymbol{E}$ is given in the *Lagrangian* frame in terms of X's, while $\boldsymbol{e}$ is given in the *Eulerian* frame in terms of the x's (though the coordinates are absent for the motion given in this question).

To finally convince ourselves that we have the same "objects" here, lets look at the eigenvalues Λ of $\boldsymbol{E}$ and $\boldsymbol{e}$. For $\boldsymbol{E}$, we have $\Lambda_{\boldsymbol{E}} = \frac{1}{2}\left(e^{\pm 2t} - 1\right)$, while for $\boldsymbol{e}$ we have $\Lambda_{\boldsymbol{e}} = \frac{1}{2}\left(1 - e^{\pm 2t}\right)$. We expect $\Lambda_{\boldsymbol{E}} = \frac{1}{2}\left(\lambda^2 - 1\right)$ (Eq.(5) from TA 1), so that $\lambda = e^{\pm t}$. Then for $\Lambda_{\boldsymbol{e}}$ we should have $\Lambda_{\boldsymbol{e}} = \frac{1}{2}\left(1 - \lambda^{-2}\right)$ (Eq.(6) from TA 1) which is exactly the relation we have here.

2 Apparent contradictions

Solve these apparent contradictions:

(i) One may claim that $\nabla_{\boldsymbol{x}}\boldsymbol{v} \equiv 0$ because

$$\nabla_{\boldsymbol{x}}\boldsymbol{v} = \nabla_{\boldsymbol{x}}\frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial}{\partial x_j}\frac{\partial x_i}{\partial t} = \frac{\partial}{\partial t}\frac{\partial x_i}{\partial x_j} = \frac{\partial \delta_{ij}}{\partial t} = 0 \ ,$$

is this true (hint: no)? What is wrong with this reasoning?

Solution

$\partial_t(\cdot)$ is defined to be $\left.\frac{\partial(\cdot)}{\partial t}\right|_{\boldsymbol{X}}$. Thus, ∂_t and ∂_x do not commute , but ∂_t and ∂_X do. To see this more explicitly, note that the expression $\nabla_{\boldsymbol{x}}\boldsymbol{v}$ is actually shorthand for

$$\nabla_{\boldsymbol{x}}\boldsymbol{v}(\boldsymbol{x},t) = \nabla_{\boldsymbol{x}}\partial_t\boldsymbol{\varphi}(\boldsymbol{X}(\boldsymbol{x},t),t)$$

so you see that $\boldsymbol{x}$ is also time dependent.

(ii) Throughout the course, we use the fact that $D_t\boldsymbol{x} = \boldsymbol{v}$. One may claim that there's a factor of 2 missing, since

$$D_t\boldsymbol{x} \equiv \partial_t\boldsymbol{x} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{x} = \boldsymbol{v} + \boldsymbol{v}\boldsymbol{I} = 2\boldsymbol{v} \ .$$

Is this true (hint: no)? What is wrong with this reasoning?

Solution

Remind yourselves the derivation of the equation for the material derivative, Eqs. (3.7-8) in Eran's notes:

$$\begin{aligned}\frac{Df(\boldsymbol{x},t)}{Dt} &= \left(\frac{\partial f(\boldsymbol{\varphi}(\boldsymbol{X},t),t)}{\partial t}\right)_{\boldsymbol{X}=\boldsymbol{\varphi}^{-1}(\boldsymbol{x},t)} \\ &= \left(\frac{\partial f(\boldsymbol{x},t)}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial f(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\right)_t \left(\frac{\partial \boldsymbol{\varphi}(\boldsymbol{X},t)}{\partial t}\right)_{\boldsymbol{X}=\boldsymbol{\varphi}^{-1}(\boldsymbol{x},t)} \ . \end{aligned} \tag{1}$$

That is, in the above we should interpret $\partial_t\boldsymbol{x}$ as the time derivative of $\boldsymbol{x}$ when $\boldsymbol{x}$ is kept constant. In other words, it is strictly zero.

3 Invertibility of the deformation gradient

We use quite freely in class $\boldsymbol{F}^{-1}$ and $\boldsymbol{F}^{-T}$ and so on. What is the physical meaning of the assumption that $\boldsymbol{F}$ is always an invertible matrix?

Solution

$\det \boldsymbol{F}$ is the ratio of an infinitesimal volume element in the material coordinates to its volume in the deformed configuration. If $\boldsymbol{F}$ is non invertible, i.e. $\det \boldsymbol{F} = 0$, then the motion takes an infinitesimal volume and "squishes" it to a plane (or a line, or a point). That is, if $\boldsymbol{F}$ is non-invertible the motion maps a triad of basis vectors in the material coordinates $\{\boldsymbol{X}_1, \boldsymbol{X}_2, \boldsymbol{X}_3\}$ to a linearly dependent set $\{\boldsymbol{F}\boldsymbol{X}_1, \boldsymbol{F}\boldsymbol{X}_2, \boldsymbol{F}\boldsymbol{X}_3\}$ and the images of the basis vectors are co-planar and do not span a volume. Since we do not allow such a situation (what would you do with mass conservation then?), we assume that $\boldsymbol{F}$ is invertible.

Note that demanding that $\boldsymbol{F}$ is invertible is a stronger assumption than assuming that $\boldsymbol{\varphi}$ is invertible. Consider the motion

$$x_1 = X_1^3 \ , \qquad x_2 = X_2 \ , \qquad x_3 = X_3 \ .$$

This is clearly an invertible motion but $\det \boldsymbol{F}$ vanishes at $\boldsymbol{X} = 0$.

A side note for the rigorous-mathematics-oriented students: We just saw that the fact that $\boldsymbol{\varphi}$ is invertible does not imply that $\boldsymbol{F}$ is invertible. However, the other direction kind of works: the inverse-function theorem says that if $\det \boldsymbol{F} \neq 0$ then $\boldsymbol{\varphi}$ is *locally* invertible (i.e. that if $\det \boldsymbol{F} \neq 0$ at a point then there's a small environment around this point where $\boldsymbol{\varphi}$ is invertible).

4 Spherical cavity

Consider a material that fills the whole space, except for a spherical cavity of initial radius Q, centered at the origin. At time $t = 0$ an explosive is detonated in the cavity and its radius varies as some specified function $q(t)$, resulting in a sphero-symmetric motion. That is, the motion is given by

$$\begin{aligned} &\boldsymbol{x}(t) = \frac{r(t)}{R}\boldsymbol{X} = \frac{f(R,t)}{R}\boldsymbol{X} \ , \\ &r(t) = f(R,t) = |\boldsymbol{x}(R,t)| \ , \\ &R(\boldsymbol{X}) = |\boldsymbol{X}| \ , \\ &f(R = Q, t) = q(t) \ . \end{aligned}$$

(i) Show that the deformation gradient is given by

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{x} = \frac{\partial f}{\partial R}\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + \frac{f}{R}(\hat{\boldsymbol{\phi}} \otimes \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}}) \ , \qquad (2)$$

where $\hat{\boldsymbol{r}} = R^{-1}\boldsymbol{X} = r^{-1}\boldsymbol{x}$, and $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ are the spherical unit vectors.
Hints:

- For a spherically symmetric function $g(r)$, $\nabla_{\boldsymbol{X}} g = \frac{\partial g}{\partial R}\hat{\boldsymbol{r}}$.
- $\boldsymbol{I} = \sum_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i$ for any set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of orthonormal vectors.

Solution

Direct calculation gives simply

$$
\begin{aligned}
\boldsymbol{F} = \nabla_{\boldsymbol{X}} \boldsymbol{x} = \nabla_{\boldsymbol{X}} \frac{f(\boldsymbol{X}, t)}{R} \boldsymbol{X} &= \frac{\boldsymbol{X}}{R} \nabla_{\boldsymbol{X}} f + f \boldsymbol{X} \nabla_{\boldsymbol{X}} \left(\frac{1}{R}\right) + \frac{f}{R} \nabla_{\boldsymbol{X}} \boldsymbol{X} \\
&= \frac{\boldsymbol{X}}{R} \otimes \partial_R f \hat{\boldsymbol{r}} + f \boldsymbol{X} \otimes \left(-\frac{\hat{\boldsymbol{r}}}{R^2}\right) + \frac{f}{R} \boldsymbol{I} \\
&= \partial_R f \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + f \hat{\boldsymbol{r}} \otimes \left(-\frac{\hat{\boldsymbol{r}}}{R}\right) + \frac{f}{R} \left(\hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + \hat{\boldsymbol{\phi}} \otimes \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}}\right) \\
&= \nabla_{\boldsymbol{X}} \boldsymbol{x} = \frac{\partial f}{\partial R} \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} + \frac{f}{R} (\hat{\boldsymbol{\phi}} \otimes \hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}})
\end{aligned}
$$

Where the 3rd line is obtained by the definition $\hat{\boldsymbol{r}} \equiv \boldsymbol{X}/R$.

(ii) If the motion is isochoric (volume-preserving), show that

$$f(R, t) = \sqrt[3]{R^3 + q(t)^3 - Q^3} \ .$$

You can show that either by using Eq.(2) to calculate the volume change, or by direct computation without going knowing the explicit form of $\boldsymbol{F}$ (doing both is better!).

Solution

If the motion is volume-preserving, then

$$\det \boldsymbol{F} = \left(\frac{\partial f}{\partial R}\right) \left(\frac{f}{R}\right)^2 = 1$$

which can be written as a differential equation for f:

$$f^2 df = R^2 dR \quad \Rightarrow \quad f(R)^3 = R^3 + C$$

where C is an integration constant. Since $f(R = Q) = q$, we can get the value of C:

$$f(Q)^3 = Q^3 + C = q^3 \quad \Rightarrow \quad C = q^3 - Q^3$$

and we conclude that

$$f(R) = \left(R^3 + q^3 - Q^3\right)^{1/3} \ .$$

The other way of doing this is as follows. Before the expansion, the volume inside a sphere of radius $R > Q$ was

$$\frac{4\pi}{3} \left(R^3 - Q^3\right) \ .$$

At time t, the volume is

$$\frac{4\pi}{3} \left(f(R, t)^3 - f(Q, t)^3\right) = \frac{4\pi}{3} \left(f(R, t)^3 - q^3\right)$$

Equating the two, we have

$$f^3 = R^3 + q^3 - Q^3$$

as needed.

(iii) Calculate $\boldsymbol{v}$, expressed in terms of q and $\partial_t q(t)$.

Solution

Since $\boldsymbol{x} = \frac{f(R,t)}{R}\boldsymbol{X}$, we have $\partial_t x = \partial_t f \frac{\boldsymbol{X}}{R} = \partial_t f \hat{\boldsymbol{r}}$. From our formula for f we have

$$\partial_t f = \frac{1}{3}\left(R^3 + q^3 - Q^3\right)^{-2/3}\left(3q^2\right)\partial_t q = f^{-2} q^2 \partial_t q$$

Substituting, we get

$$\boldsymbol{V}(\boldsymbol{X}, t) = \left(\frac{q}{f(|\boldsymbol{X}|, t)}\right)^2 \partial_t q \hat{\boldsymbol{r}}$$

Switching to the spatial coordinates, we simply use $|\boldsymbol{x}| = f$ to get

$$\boldsymbol{v}(\boldsymbol{x}, t) = \left(\frac{q}{|\boldsymbol{x}|}\right)^2 \partial_t q \hat{\boldsymbol{r}}$$

5 Acceleration, stress and force fields

Solve these two *unrelated* questions:

(i) Consider the following velocity field $\boldsymbol{v}$ in the Eulerian description:

$$\boldsymbol{v} = Ce^{-at}\left(x^3 + xy^2, -x^2y - y^3, 0\right)^T \ , \tag{3}$$

where C and a are constants. Find the acceleration $\boldsymbol{a}$ at point $(1, 1, 0)$ at time $t = 0$

Solution

As mentioned in the first question, we can calculate the accelaration from $\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v}$. Doing this, we get

$$\boldsymbol{a}(\boldsymbol{x}, t) = Ce^{-2at}\begin{pmatrix} -x\left(x^2 + y^2\right)\left[ae^{at} + C\left(y^2 - 3x^2\right)\right] \\ y\left(x^2 + y^2\right)\left[ae^{at} - C\left(x^2 - 3y^2\right)\right] \\ 0 \end{pmatrix} .$$

Evaluating this at $(1, 1, 0)$ at time 0 we get

$$\boldsymbol{a}(x = 1, y = 1, z = 0, t = 0) = 2C\begin{pmatrix} 2C - a \\ 2C + a \\ 0 \end{pmatrix} .$$

(ii) If the stress field is given by the matrix:

$$\boldsymbol{\sigma} = C\begin{pmatrix} x^2y & \left(a^2 - y^2\right)x & 0 \\ \left(a^2 - y^2\right)x & \frac{1}{3}\left(y^2 - 3a^2y\right) & 0 \\ 0 & 0 & 2az^2 \end{pmatrix} , \tag{4}$$

find the body force field necessary for the stress field to be in equilibrium.

Solution

To satisfy the static momentum balance equation we demand $\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = 0$, that is $\boldsymbol{b} = -\nabla \cdot \boldsymbol{\sigma}$. We obtain

$$\boldsymbol{b} = \begin{pmatrix} 0 \\ -\frac{y}{3}\left(2 - 3y\right) \\ -4az \end{pmatrix} \ .$$