Non-Equilibrium Continuum Physics **HW set #2**

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Kinematics

1 Eulerian and Lagrangian frameworks

Consider the following 2D deformation:

$$x_1(t) = \cosh(t)X_1 + \sinh(t)X_2 \ , \qquad x_2(t) = \sinh(t)X_1 + \cosh(t)X_2 \ .$$

(i) Find the material velocity and the acceleration $\boldsymbol{V}, \boldsymbol{A}$ and express their spatial forms $\boldsymbol{v}, \boldsymbol{a}$. Remember to represent each field in the proper coordinates (i.e. $\boldsymbol{V}, \boldsymbol{A}$ in terms of $\boldsymbol{X}$ and $\boldsymbol{v}, \boldsymbol{a}$ in terms of $\boldsymbol{x}$). Plot schematically $\boldsymbol{V}$ and $\boldsymbol{v}$ at $t = -10, 0, 10$. Note how vastly different $\boldsymbol{V}$ and $\boldsymbol{v}$ are!

(ii) The acceleration $\boldsymbol{a}$ can also be calculated as a material derivative of the velocity:

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} \ .$$

Calculate $\boldsymbol{a}$ using this expression, and show that the results coincide.

(iii) Calculate $\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}$ and $J = \det \boldsymbol{F}$.

(iv) Calculate the Green-Lagrange strain tensor $\boldsymbol{E}$, and the Euler-Almansi strain tensor $\boldsymbol{e}$, and show that the results coincide.

2 Apparent contradictions

Solve these apparent contradictions:

(i) One may claim that $\nabla_{\boldsymbol{x}} \boldsymbol{v} \equiv 0$ because

$$\nabla_{\boldsymbol{x}} \boldsymbol{v} = \nabla_{\boldsymbol{x}} \frac{\partial \boldsymbol{x}}{\partial t} = \frac{\partial}{\partial x_j} \frac{\partial x_i}{\partial t} = \frac{\partial}{\partial t} \frac{\partial x_i}{\partial x_j} = \frac{\partial \delta_{ij}}{\partial t} = 0 \ ,$$

is this true (hint: no)? What is wrong with this reasoning?

(ii) Throughout the course, we use the fact that $D_t \boldsymbol{x} = \boldsymbol{v}$. One may claim that there's a factor of 2 missing, since

$$D_t \boldsymbol{x} \equiv \partial_t \boldsymbol{x} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{x} = \boldsymbol{v} + \boldsymbol{v} \boldsymbol{I} = 2\boldsymbol{v} \ .$$

Is this true (hint: no)? What is wrong with this reasoning?

3 Invertibility of the deformation gradient

We use quite freely in class $\boldsymbol{F}^{-1}$ and $\boldsymbol{F}^{-T}$ and so on. What is the physical meaning of the assumption that $\boldsymbol{F}$ is always an invertible matrix?

4 Spherical cavity

Consider a material that fills the whole space, except for a spherical cavity of initial radius Q, centered at the origin. At time $t = 0$ an explosive is detonated in the cavity and its radius varies as some specified function $q(t)$, resulting in a sphero-symmetric motion. That is, the motion is given by

$$\begin{aligned} &\boldsymbol{x}(t) = \frac{r(t)}{R}\boldsymbol{X} = \frac{f(R,t)}{R}\boldsymbol{X} \ , \\ &r(t) = f(R,t) = |\boldsymbol{x}(R,t)| \ , \\ &R(\boldsymbol{X}) = |\boldsymbol{X}| \ , \\ &f(R=Q,t) = q(t) \ . \end{aligned}$$

(i) Show that the deformation gradient is given by

$$\boldsymbol{F} = \nabla_{\boldsymbol{X}}\boldsymbol{x} = \frac{\partial f}{\partial R}\hat{\boldsymbol{r}}\otimes\hat{\boldsymbol{r}} + \frac{f}{R}(\hat{\boldsymbol{\phi}}\otimes\hat{\boldsymbol{\phi}} + \hat{\boldsymbol{\theta}}\otimes\hat{\boldsymbol{\theta}}) \ , \tag{1}$$

where $\hat{\boldsymbol{r}} = R^{-1}\boldsymbol{X} = r^{-1}\boldsymbol{x}$, and $\hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}$ are the spherical unit vectors.
Hints:

- For a spherically symmetric function $g(r)$, $\nabla_{\boldsymbol{X}} g = \frac{\partial g}{\partial R}\hat{\boldsymbol{r}}$.
- $\boldsymbol{I} = \sum_i \boldsymbol{e}_i \otimes \boldsymbol{e}_i$ for any set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ of orthonormal vectors.

(ii) If the motion is isochoric (volume-preserving), show that

$$f(R,t) = \sqrt[3]{R^3 + q(t)^3 - Q^3} \ .$$

You can show that either by using Eq.(1) to calculate the volume change, or by direct computation without going knowing the explicit form of $\boldsymbol{F}$ (doing both is better!).

(iii) Calculate $\boldsymbol{v}$, expressed in terms of q and $\partial_t q(t)$.

5 Acceleration, stress and force fields

Solve these two *unrelated* questions:

(i) Consider the following velocity field $\boldsymbol{v}$ in the Eulerian description:

$$\boldsymbol{v} = Ce^{-at}\left(x^3 + xy^2, -x^2y - y^3, 0\right)^T \ , \tag{2}$$

where C and a are constants. Find the acceleration $\boldsymbol{a}$ at point $(1,1,0)$ at time $t{=}0$

(ii) If the stress field is given by the matrix:

$$\boldsymbol{\sigma} = C\begin{pmatrix} x^2y & (a^2-y^2)\,x & 0 \\ (a^2-y^2)\,x & \frac{1}{3}\left(y^2 - 3a^2y\right) & 0 \\ 0 & 0 & 2az^2 \end{pmatrix} \ , \tag{3}$$

find the body force field necessary for the stress field to be in equilibrium.