Non-Equilibrium Continuum Physics **HW set #3**

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Due 31.05.23 Submit as: NECP_HW3_⟨FirstName⟩_⟨LastName⟩.pdf

Linear elasticity - Solution

1 Uniaxially-stressed box

Consider a 3D rectangular box, subject to uniaxial stress σ_0 in the z direction, as shown in Fig. 1. The faces in the x, y directions are traction-free. The rest-lengths of the boxes sides are a, b. Calculate the slope of the dashed line as a function of σ_0, E and ν. When $\sigma_0 \geq E$ something weird happens. Is linear elasticity wrong?

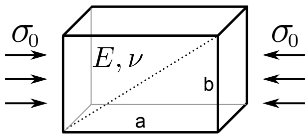


Figure 1: Box under uniaxial compression.

Solution

This is simple uniaxial compression. We solved that in class (Eq. (5.14) in the lecture notes). We have

$$\epsilon_{zz} = \frac{\Delta a}{a} = -\frac{\sigma_0}{E}, \qquad \epsilon_{yy} = \frac{\Delta b}{b} = \nu \epsilon_{zz} = \nu \frac{\sigma_0}{E}$$

The slope is thus

$$\frac{b + \Delta b}{a + \Delta a} = \frac{b}{a} \frac{1 + \nu \frac{\sigma_0}{E}}{1 - \frac{\sigma_0}{E}} .$$

When $\sigma_0 = E$ the box shrinks to be 2 dimensional. This is clearly unphysical - one has to remember that σ_0/E is a small parameter, and this was the assumption in the derivation of linear elasticity.

2 Hertzian contact problem

As a reminder, the Hertz Problem concerns two elastic bodies in contact. This problem was initially posed by Hertz when he considered Newton diffraction rings on the contact of two lenses. He solved the problem on his Xmas vacation in 1880, when he was 23 years old[1], and his treatment became canonical. In class, Eran "solved" this problem scaling-wise, and here you'll work out the solution a bit more carefully.

The generic case is that of two paraboloids, and is justified by the assumption that near the contact point the surface of both bodies can be expanded to second order. The two bodies may have different radii of curvature and also different elastic properties. This case is completely tractable, and

[1] What did you do by the time you were 23? I definitely did nothing as impressive.

the procedure goes through reducing the problem to the case of contact between a rigid half-plane and an elastic sphere with some effective radius and elastic properties. We will not go through this reduction, and assume at the outset that this is the case - a sphere of radius R and elastic moduli E and ν is pressed against a rigid half plane with force F.

The force induces a global displacement of δ. That is, the displacement of distant points in the body is $\boldsymbol{u}\approx\delta\hat{z}$. Cylindrical symmetry tells us that the area of contact will be a circle of radius a. The problem is to find the relationship between a, F, and δ, and also to find the pressure distribution on the contact. We will assume that $a \ll R$ (otherwise the strain is not small).

(i) Assuming we have pressed the sphere down by an amount δ, creating an area of contact in a circle of radius a. What are the boundary conditions on the sphere now? Notice that we have a mixed BC - that is one BC in part of the system, and another BC in the other.

Solution

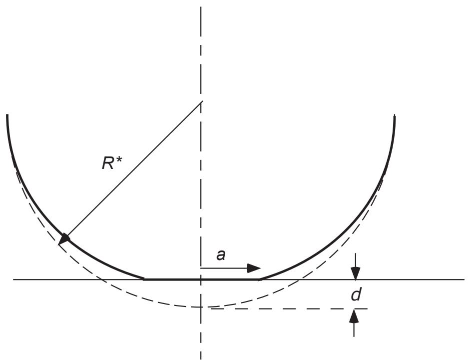
First let us consider the BC in the area which is in contact. By drawing the system, it is easily seen that

$$u_z(r, z=0) = \delta - \frac{r^2}{2R} \qquad , \qquad r < a \; , \tag{1}$$

where we defined $r = \sqrt{x^2+y^2}$. Outside of the contact area, we cannot know a-priori the displacement, only bound it

$$u_z(r^2 + (z-\delta)^2 = R^2) < \delta - \frac{r^2}{2R} \qquad , \qquad r > a \; , \tag{2}$$

which of course cannot be a BC. But not all is lost! As we know that there is no contact, for r¿a we just assume a free interface, namely $\boldsymbol{\sigma}\cdot\hat{\boldsymbol{n}} = 0$. In spherical coordinates $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\rho}}$ this is can be written explicitly as $\sigma_{\rho\rho}\left(\rho = R, \theta > \sin^{-1}\left(\frac{R}{a}\right)\right) = \sigma_{\rho\theta}\left(\rho = R, \theta > \sin^{-1}\left(\frac{R}{a}\right)\right) = \sigma_{\rho\phi}\left(\rho = R, \theta > \sin^{-1}\left(\frac{R}{a}\right)\right) = 0$. In the next part of the question we will assume that $a \ll R$, thus in the vicinity of the contact area this can be approximated to leading order in $\frac{a}{R}$ as $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$.



(ii) OK, but what next? Hertz pulled a dirty trick here. Since $a \ll R$, he assumed that the sphere can be treated as a half plane for the purposes of calculating the elastic responses. That is, he invoked the Green's function for a point load on a half surface (Eq. (5.26) in Eran's lecture notes). Following his path, use the Green's function formulation to write an integral equation for the normal stress field p_z, using the results of the previous question. Write down the results in polar coordinates, including the boundaries of integration.

Solution

Eq. (5.26) in Eran's lectures reads

$$u_z(x, y, z{=}0) = \frac{1-\nu^2}{\pi\, E} \int \frac{p_z(x', y', z' = 0)dx'dy'}{\sqrt{(x-x')^2 + (y-y')^2}} \ . \tag{3}$$

To get to a simpler (polar) representation, we can choose without loss of generality (since the system is axially symmetric) $\{x, y\} = \{r, 0\}$. We are hence left with

$$\delta - \frac{r^2}{2R} = \frac{1-\nu^2}{\pi\, E} \int_0^{2\pi} \int_0^a \frac{p_z(\rho)\rho d\rho d\phi}{\sqrt{\rho^2 + r^2 - 2r\rho\cos(\phi)}} \ , \tag{4}$$

applicable of course only for $r < a$.

(iii) The form of p_z that solves the equation is

$$p_z(r) = p_0\sqrt{1 - \frac{r^2}{a^2}} \qquad , \qquad r < a \ , \tag{5}$$

the proof of which is very cumbersome, so I won't give here. Of course, for $r > a$ $p_z = 0$. If any of you are interested, it can be found in K. L. Johnson's Contact mechanics, which is a great book with many solutions to all kind of contact problems. Setting Eq. (5) in Eq. (5.26) in Eran's lecture notes leads to

$$u_z(r) = \frac{\pi\,(1-\nu^2)\,p_0}{4aE}\left(2a^2 - r^2\right) \qquad , \qquad r < a \ . \tag{6}$$

Now finish it off and write down a, δ, and p_0 in terms of the material parameters, the loading force F, and the radius of the sphere R. Verify that it indeed agrees with Eran's scaling analysis.

Solution

Since Eq. (6) should agree with Eq. (1), we conclude

$$\delta = \frac{\pi\,(1-\nu^2)\,p_0 a}{2E} \ , \tag{7}$$

$$\frac{\pi\,(1-\nu^2)\,p_0}{4aE} r^2 = \frac{r^2}{2R} \ . \tag{8}$$

The force can be calculated by integrating p_z

$$F = \int_{\rho<a} p_z(\rho) d^2\rho = \frac{2}{3} p_0 \pi a^2 \ . \tag{9}$$

Combining all of the above, we get

$$a = \left(\frac{3\left(1-\nu^2\right)FR}{4E}\right)^{1/3} , \tag{10}$$

$$\delta = \left(\frac{9\left(1-\nu^2\right)^2 F^2}{16RE^2}\right)^{1/3} , \tag{11}$$

$$p_0 = \left(\frac{6FE^2}{\pi^3\left(1-\nu^2\right)^2 R^2}\right)^{1/3} , \tag{12}$$

in accordance with Eran's scaling analysis.

(iv) Small bonus - consider and tell us about your greatest achievement by the time you were 23. How does this compare with finding the solution to the elastic contact problem?

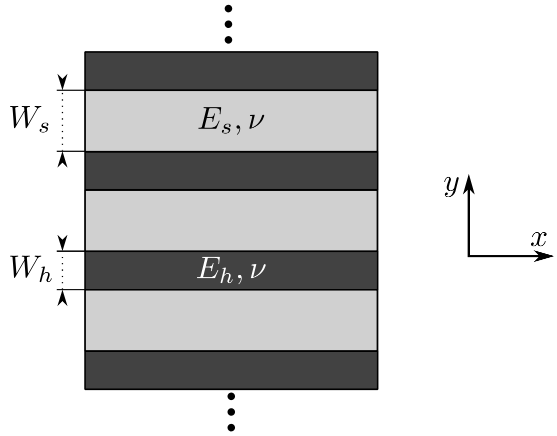
(v) Large bonus - Try to prove that the stress distribution given in Eq. (5) actually leads to the displacement given in Eq. (6). Keep in mind that it's rather long and complicated, so do it only if you really like this kind of stuff.

Solution

As this is long and complicated, with a few transformations and changed of variables, I'm not going to repeat it here, you can check it out in Contact mechanics, Sections 3.4 and 4.2.

3 Simple composite material (an example from past exam)

Composite materials are materials that have a microscopic structure. They are abundant in nature, and examples include bone, wood, dentin (the material your teeth are made of), and many more (graphic examples will be shown in class). In the last few decades there are also many man-made composite materials. The vast advancement in composite material technology is one of the most influential revolutions in modern technology, and it allows manufacturing materials that have desirable characteristics – light-weight, high strength, shape memory, etc. – which are orders of magnitude better than homogeneous materials. This is a fascinating topic which is the subject of huge and very active ongoing research. In this short exercise we'll examine some simple outcomes of a simple model of a simple composite material. This model was given as a question in the final exam of the 2012 course.



Consider a material that is composed of layers of two linear-elastic isotropic materials, one is hard and the other soft. The hard material has a Young's modulus E_h and the soft material has a Young's modulus of E_s. For simplicity, we'll assume that both materials have the same Poisson's ratio ν. The width of each layer is denoted by W_i, and the layers are glued perfectly to each other. We assume the material is infinite in all directions, has a periodic structure in the y direction and is translationally invariant in the x direction. We also assume the z direction is very thin. We define the volume ratio of the hard material by

$$\phi = \frac{W_h}{W_h + W_s} \ . \tag{13}$$

Our goal is to calculate the coarse-grained Hooke's law of the composite system, that is, it's (linear) response to loading on length-scales much larger than the scale of the microscopic structure (in our case - W_s and W_h).

(i) Consider the overall geometry of the described material. Should one use the plane-stress or plane-strain conditions? Why?

Solution

As the z direction is very thin, we shall be using plane-stress conditions in the z direction.

(ii) Consider a small segment of the material over which the stress and strain field are constant *in each material*. We denote these fields by $\boldsymbol{\sigma}^{(i)}$, $\boldsymbol{\varepsilon}^{(i)}$ where $i = h, s$. Explicitly write down the relation between the stress field $\boldsymbol{\sigma}^{(i)}$ and strain field $\boldsymbol{\varepsilon}^{(i)}$ for each material separately (i.e. write $\boldsymbol{\sigma}^{(i)}\left(\boldsymbol{\varepsilon}^{(i)}\right)$ using the components of the tensors). Make sure you use either the plane-stress / plane-strain conditions (according to your answer above). You can write the relations only once, and use the index $\bullet^{(i)}$ for the type of material.

Solution

For the plane-stress conditions, each of the materials obeys the following relations

$$\begin{pmatrix} {\varepsilon_{xx}}^{(h,s)} \\ {\varepsilon_{yy}}^{(h,s)} \\ {\varepsilon_{xy}}^{(h,s)} \end{pmatrix} = \frac{1}{E^{(h,s)}} \begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{pmatrix} \begin{pmatrix} {\sigma_{xx}}^{(h,s)} \\ {\sigma_{yy}}^{(h,s)} \\ {\sigma_{xy}}^{(h,s)} \end{pmatrix} , \tag{14}$$

$$\begin{pmatrix} {\sigma_{xx}}^{(h,s)} \\ {\sigma_{yy}}^{(h,s)} \\ {\sigma_{xy}}^{(h,s)} \end{pmatrix} = \frac{E^{(h,s)}}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \begin{pmatrix} {\varepsilon_{xx}}^{(h,s)} \\ {\varepsilon_{yy}}^{(h,s)} \\ {\varepsilon_{xy}}^{(h,s)} \end{pmatrix} . \tag{15}$$

(iii) Now consider the boundary between the two materials, say at $y=0$. What can you say about the displacement fields $\boldsymbol{u}^{(s)}$ and $\boldsymbol{u}^{(h)}$ across the interface? Are they continuous / discontinuous? (Recall the interface is *glued*.) What can you then say about $\partial_x \boldsymbol{u}$?

Solution

Clearly, the displacement field $\boldsymbol{u}$ must be continuous across the interface, because we assumed perfect bonding between the materials, which means no relative displacement. Let's say that some of the interfaces lies on $y = 0$. As said before, $\boldsymbol{u}$ is continuous across the interface and therefore

$$u_i^{(s)}(x, y = 0) = u_i^{(h)}(x, y = 0) , \tag{16}$$

for all x. Specifically, this means that $\partial_x u_i$ is continuous across the interface. From this we conclude immediately that $\varepsilon_{xx}^{(s)} = \varepsilon_{xx}^{(h)}$.

(iv) Next, consider force balance on a small volume element of length L and infinitesimal height, which is half in the soft region and half in the hard region. What is the total force in the y direction on this small volume? Given that the situation is *static*, what can you say about σ_{iy} (continuous/discontinuous)?

Solution

Think of a small volume element of length L and infinitesimal height, which is half in the soft region and half in the hard region. The vertical forces applied to it (per unit thickness in the z direction) sum up to

$$L \left(\sigma_{yy}^{(h)} - \sigma_{yy}^{(s)}\right) \hat{y} + L \left(\sigma_{xy}^{(h)} - \sigma_{xy}^{(s)}\right) \hat{x} , \tag{17}$$

and since the situation is static, we conclude that σ_{iy} is continuous across the interface.

(v) Now lets clean things and ensure things are correct up to this point — explain why the displacement fields u_x, u_y, the stresses σ_{xy}, σ_{yy} and the strain ε_{xx} are continuous across the boundaries.

Solution

The explanations were provided above. To summarize:

- $\boldsymbol{u}$, σ_{yy}, σ_{xy}, ε_{xx} are continuous across the interface.
- σ_{xx}, ε_{yy}, ε_{xy} experience a jump across the interface. The jump can be calculated from Hooke's law.

(vi) We now want to describe the large-scale fields $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in terms of the small-scale structure of the constituent materials $\boldsymbol{\sigma}^{(i)}$ and $\boldsymbol{\varepsilon}^{(i)}$. Consider the macroscopic strain ε_{xx} on the material. As ε_{xx} is continuous across the boundaries, we have $\varepsilon_{xx} = \varepsilon_{xx}^{(s)} = \varepsilon_{xx}^{(h)}$. Use your answer about the relation of $\varepsilon_{xx}^{(s)}$ and $\varepsilon_{xx}^{(h)}$, and express this relation using $\boldsymbol{\sigma}^{(s)}$ and $\boldsymbol{\sigma}^{(h)}$.

Solution

The macroscopic strain in the x direction is $\varepsilon_{xx} = \varepsilon_{xx}^{(h)} = \varepsilon_{xx}^{(s)}$, which translates through the microscopic Hooke's law to

$$\varepsilon_{xx} = \varepsilon_{xx}^{(h)} = \varepsilon_{xx}^{(s)} = \frac{\sigma_{xx}^{(h)} - \nu\sigma_{yy}^{(h)}}{E_h} = \frac{\sigma_{xx}^{(s)} - \nu\sigma_{yy}^{(s)}}{E_s} \ . \tag{18}$$

(vii) Now consider the macroscopic stress σ_{xx}. This stress is equal to the total force exerted by all the different layers, i.e. $\sigma_{xx} = \phi\sigma_{xx}^{(h)} + (1-\phi)\sigma_{xx}^{(s)}$. This item and the previous one were example cases of a continuous quantity and of a discontinuous one. Apply the same principles to the quantities ε_{yy} and σ_{yy} (this time, express these relations in terms of $\boldsymbol{\varepsilon}^{(i)}$'s). Note that in the y directions things are exactly the other way around (stresses are continuous — equal —, and the strains are weighted sums).

Solution

In the y direction things are exactly the other way around:

$$\sigma_{yy} = \sigma_{yy}^{(h)} = \sigma_{yy}^{(s)} = \frac{E^{(s)}}{1-\nu^2}\left(\varepsilon_{yy}^{(s)} + \nu\varepsilon_{xx}^{(s)}\right) = \frac{E^{(h)}}{1-\nu^2}\left(\varepsilon_{yy}^{(h)} + \nu\varepsilon_{xx}^{(h)}\right) \ , \tag{19}$$

$$\varepsilon_{yy} = \phi\ \varepsilon_{yy}^{(h)} + (1-\phi)\varepsilon_{yy}^{(s)} \ . \tag{20}$$

(viii) Combining the equations for σ_{xx}, σ_{yy}, ε_{xx} and ε_{yy} obtained in the last two items. Eliminate the 4 variables $\sigma_{xx}^{(h,s)}$ and $\varepsilon_{yy}^{(h,s)}$ from these equations, and put the remaining relations in the form

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \end{pmatrix} \ . \tag{21}$$

Solution

Let's count to see that we already have all the ingredients. We use the fact that $\varepsilon_{xx} = \varepsilon_{xx}^{(h,s)}$ and $\sigma_{yy} = \sigma_{yy}^{(h,s)}$ to summarize our equations as

$$\sigma_{xx} = \phi\,\sigma_{xx}^{(h)} + (1-\phi)\sigma_{xx}^{(s)} \;, \tag{22}$$

$$\sigma_{yy} = \frac{E^{(s)}}{1-\nu^2}\left(\varepsilon_{yy}^{(s)} + \nu\varepsilon_{xx}\right) = \frac{E^{(h)}}{1-\nu^2}\left(\varepsilon_{yy}^{(h)} + \nu\varepsilon_{xx}\right) \;, \tag{23}$$

$$\varepsilon_{yy} = \phi\;\varepsilon_{yy}^{(h)} + (1-\phi)\varepsilon_{yy}^{(s)} \;, \tag{24}$$

$$\varepsilon_{xx} = \frac{\sigma_{xx}^{(h)} - \nu\sigma_{yy}}{E_h} = \frac{\sigma_{xx}^{(s)} - \nu\sigma_{yy}}{E_s} \;. \tag{25}$$

These are linear 6 equations, and we want to eliminate 4 variables: $\sigma_{xx}^{(h,s)}$ and $\varepsilon_{yy}^{(h,s)}$. Thus, we expect to finish with two linear equations, which we will be able to put in the desired form

$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\end{pmatrix} = \begin{pmatrix}\cdot & \cdot\\ \cdot & \cdot\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\end{pmatrix} \;. \tag{26}$$

The algebra is not very interesting, but the main point is that the obtained relations can be put in this standard linear form.

(ix) Now lets check the results — consider the following three cases: (a) $E_h = E_s$, (b) $\phi = 1$ and (c) $\phi = 0$. Do you get plausible results?

Solution

All three cases yield a homogeneous material. Case (a) yields a material with $E = E_h = E_s$, case (b) yields a hard material, and case (c) reproduces a soft material.

(x) Lets consider the case of $E_h \gg E_s$. Express $\sigma_{xx}^{(s)}$ and $\sigma_{xx}^{(h)}$ and expand these to first order in the small ratio E_s/E_h. Plug these results into your expression for ε_{xx} and ε_{yy}. If we impose σ_{xx}, which material (hard/soft) governs the response of ε_{xx} and ε_{yy}? What happens if we impose σ_{yy}?

Solution

We are only interested in approximate trends, so we'll assume ϕ is not to close to 0 or to 1, and omit all pre-factors of the form $\phi, \phi/(1-\phi)$ and so on in our final estimation.

We rewrite the relations of the stresses above as

$$\sigma_{xx}^{(h)} = \frac{\sigma_{xx} - (1-\phi)\sigma_{xx}^{(s)}}{\phi} \;, \tag{27}$$

$$\sigma_{xx}^{(s)} = \frac{\sigma_{xx} + \phi\,\nu\left(\frac{E_h}{E_s} - 1\right)\sigma_{yy}}{\phi\,\frac{E_h}{E_s} + (1-\phi)} \;. \tag{28}$$

Expanding these to order in E_s/E_h, we get

$$\sigma_{xx}^{(s)} \approx \nu\sigma_{yy} + \frac{E_s}{E_h}\frac{1}{\phi}\left(\sigma_{xx} - \nu\sigma_{yy}\right) \ , \tag{29}$$

$$\sigma_{xx}^{(h)} \approx -\frac{1-\phi}{\phi}\left(\sigma_{xx} + \nu\sigma_{yy}\right) - \frac{1-\phi}{\phi^2}\frac{E_s}{E_h}\left(\sigma_{xx} - \nu\sigma_{yy}\right) \ . \tag{30}$$

Plugging this into Hooke's law we get

$$\varepsilon_{xx} \approx \frac{1}{\phi}\frac{\sigma_{xx} - \nu\sigma_{yy}}{E_h} \sim \frac{\sigma_{xx} - \nu\sigma_{yy}}{E_h} \ . \tag{31}$$

Similarly, plugging (29)-(30) into the expressions for the strains $\boldsymbol{\varepsilon}$ we get

$$\begin{aligned} \varepsilon_{yy} &\approx \phi\frac{\sigma_{yy} - \nu\sigma_{xx}^{(h)}}{E_h} + (1-\phi)\frac{\sigma_{yy} - \nu\sigma_{xx}^{(s)}}{E_s} \\ &\approx \frac{\sigma_{yy}}{E_s}(1-\phi)\left(1-\nu^2\right) + \frac{1}{E_h}\left[\sigma_{yy}\left(\phi^2 + \nu^2 - \phi^2\nu^2\right) - \nu\sigma_{xx}\right] \\ &\sim \frac{\sigma_{yy}}{E_s} - \frac{\nu\sigma_{xx}}{E_h} \ . \end{aligned} \tag{32}$$