Thermo-Elasticity

Note

This HW set, especially the last question, summarizes the first part of the course, and demands a working knowledge of linear elasticity. It also involves a small numeric calculation, allowing you to practice solving a physical problem with computational tools. We consider it a "semi-mid-term", and it will be given a larger weight in the final grade than the other sets. You are also given three weeks to complete it, so please start early and take it seriously.

1 Circular hole

Consider an infinite 2D material, from which a circular hole is taken out. The material is now heated by some amount. Will the hole shrink or expand?

2 Temperature and displacements

Consider a static infinite 3D material with a given arbitrary distribution of temperature $T(x,y,z)$, that decays at infinity: $T(\vec{r}) \to T_\infty$, as $|\vec{r}| \to \infty$. Before reading further it might by nice to try to estimate: if the temperature variation is localized, how does the displacement field decay at large $\boldsymbol{r}$? And the strain field?

Here's a nice way to gain intuition as to what temperature gradients do in thermo-elasticity: Prove that the displacement field is curl-free, i.e. is of the form $\vec{u} = \nabla\phi$, and that ϕ satisfies Poisson's equation $\nabla^2\phi = T$. Assuming you already have some intuition about electrostatics, this should help you gain intuition about thermo-elasticity.

Guidance: Begin with Navier-Lamé equation $(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = K\alpha\nabla T$. You can guess the correct form for u, and if it works then you're done because the solution is unique. Some vector-analysis identities might prove useful.

3 Thermally induced fracture

In 1993, Yuse & Sano published a remarkable paper regarding instabilities of thermally induced fracture (Yuse & Sano, Nature (362) 1993). They consider a strip of material which is pulled out of an oven **at a constant velocity** and cools down as it moves. The gradients of the temperature field induce fracture, as is seen in Figure 1.

To model the phenomenon, consider an infinite (in the x direction) 2D strip of width $2b$. The strip is subject to a y-independent temperature distribution $T(x)$, and is free of tractions at its boundaries $y = \pm b$. Fracture will be considered later in this course. For now, we'll limit ourselves to finding

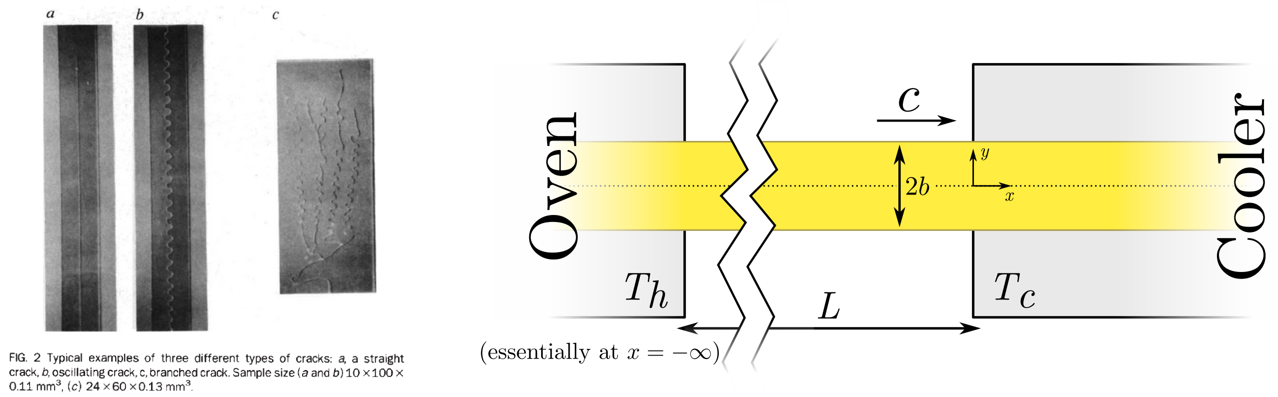

Figure 1: Left: Thermally induced fracture (from the paper). Right: the simplified model. Note the position of the origin of axes, and that the plot **is not to scale**: we assume $L \gg b$.

an expression for the stretching component σ_{yy} along the strip's symmetry axis $y = 0$. This is the driving force that induces fracture.

(a) We begin by finding the temperature distribution. Write the heat diffusion equation in both the material ($\boldsymbol{X}$) and laboratory ($\boldsymbol{x}$) coordinates, and solve it in the laboratory coordinates for our problem. Assume that the cooler and the oven are strong enough such that $T(x>0)=T_c$, and $T(x<-L)=T_h$. Assume also that L is much larger than any other length scale of the system. The heat diffusion constant D is of course given. Remember that you can always shift T by a global constant to get a simpler expression.

(b) Show that the equations of plane-stress combined with static thermo-elasticity are

$$\begin{aligned}
\varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu\sigma_{yy}\right] + \frac{1}{3}\alpha_T \Delta T, \\
\varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu\sigma_{xx}\right] + \frac{1}{3}\alpha_T \Delta T, \\
\varepsilon_{xy} &= \frac{1+\nu}{E}\sigma_{xy}
\end{aligned} \tag{1}$$

Guidance: Start with the known Hooke's law derived in class, $\boldsymbol{\sigma}(\boldsymbol{\varepsilon}, T) = \lambda \operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\,\boldsymbol{\varepsilon} - \alpha K T \boldsymbol{I}$, invert it to the compliance form $\boldsymbol{\varepsilon}(\boldsymbol{\sigma}, T)$, and use the relations between λ, μ, K to E, ν (which are summarized in a nice table in Wikipedia).

(c) Prove the compatibility relation

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2\frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y}\ , \tag{2}$$

and use it together with the definition of the Airy potential χ and Eqs. (1) to show that χ satisfies the equation

$$\nabla^2\nabla^2\chi = -\frac{1}{3}E\alpha_T\nabla^2 T \tag{3}$$

What is the symmetry of χ with respect to y? What are the boundary conditions that χ satisfies?

(d) Solve Eq. (3) by Fourier transforming it in the x direction and imposing the boundary conditions. Express $\sigma_{yy}(x, y = 0)$ in an integral form. You should obtain an expression of the form

$$\sigma_{yy}(x, y = 0) = \int_{-\infty}^{\infty} T(x')\Psi(x - x')dx' \equiv T * \Psi \ , \tag{4}$$

where $*$ denotes convolution, and $\Psi(x)$ is the convolution kernel, for which you should have a closed expression (as an integral of something).

(e) Calculate numerically $\sigma_{yy}(x, y = 0)$ for three cases: $b \ll D/c$ (very narrow strip), $b \approx D/c$ (intermediate) and $b \gg D/c$ (very wide strip). Is the scale of variation of σ determined by b or by D/c?

(f) BONUS: Try to solve (e) by guessing an ansatz of the form $\chi(x, y) = f(y)g(x)$ where g has the same x-dependence as $T(x)$. Plugging it into Eq. (3) should give you a differential equation on $f(y)$ which is solvable. The solution is drastically different from the one you obtained in (e), but is an exact solution of Eq. (3) in the region $x > 0$. How do you resolve this apparent contradiction?