Non-Equilibrium Continuum Physics **HW set #5**

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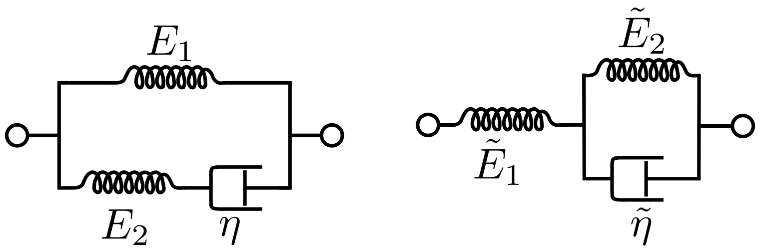
Due 12.07.23 Submit as: NECP_HW5_⟨FirstName⟩_⟨LastName⟩.pdf

Visco-Elasticity - Solution

1 Standard linear solid

Consider the one-dimensional Standard-Linear-Solid (SLS) model, described in the left panel of the Fig. below. In this exercise we will explore its visco-elastic properties in much the same way that we did in class for the Maxwell and Kelvin-Voigt models.



(a) The SLS model (b) The alternative model

Figure 1:

(i) Calculate $G^{\text{SLS}}(t)$ and $J^{\text{SLS}}(t)$.

Solution

We'll denote the stresses in the upper and lower branches of Fig. 1(a) by σ_1, σ_2. The upper branch is easy - we know that $\sigma_1(t) = E_1\varepsilon(t)$. In the lower branch, the spring and dashpot E_2, η will behave like a Maxwell material. Let's denote the strain of the spring by x and that of the dashpot by y. Then

$$\dot{x}(t) + \dot{y}(t) = \dot{\varepsilon}(t) \ , \tag{1}$$

$$\dot{x}(t) = \frac{\dot{\sigma}_2(t)}{E_2} \ , \qquad \dot{y} = \frac{\sigma_2}{\eta} \ , \tag{2}$$

$$\sigma = \sigma_1 + \sigma_2 \quad \Rightarrow \quad \sigma_2 = \sigma - E_1\varepsilon \ , \tag{3}$$

$$\sigma_2 = \eta\ \dot{y}(t) = \eta\left(\dot{\varepsilon}(t) - \dot{x}\right) = \eta\left(\dot{\varepsilon}(t) - \frac{\dot{\sigma}_2(t)}{E_2}\right) \ . \tag{4}$$

Plugging the expressions for $\dot{x}, \dot{y}, \sigma_2$ in (**??**) gives

$$\dot{\varepsilon} = \frac{\dot{\sigma}_2}{E_2} + \frac{\sigma_2}{\eta} = \frac{\dot{\sigma} - E_1\dot{\varepsilon}}{E_2} + \frac{\sigma - E_1\varepsilon}{\eta} \ . \tag{5}$$

This is more elegantly written as

$$\left(1 + \frac{E_1}{E_2}\right)\dot{\varepsilon} + \frac{E_1}{\eta}\varepsilon = \frac{1}{E_2}\dot{\sigma} + \frac{1}{\eta}\sigma \ . \tag{6}$$

This is the basic relation we work with. Suppose we impose a step strain $\varepsilon(t) = \varepsilon_0 H(t)$. For $t > 0$ we have $\dot{\varepsilon} = 0$, and therefore the relation reads $\tau_{\rm G}\, \dot{\sigma} + \sigma = E_1\varepsilon_0$, where $\tau_{\rm G} \equiv \frac{\eta}{E_2}$. This is easily solved, and the initial condition is $\sigma(0) = (E_2 + E_2)\varepsilon_0$, because at short time scales the dashpot is passive. The result is

$$G(t) = \frac{\sigma(t)}{\varepsilon_0} = E_1 + E_2\; e^{-t/\tau_{\rm G}}\ . \tag{7}$$

Also, if you don't like the way I "guessed" the initial conditions (although you should be able to guess so yourselves) you can instead solve the full equation, by setting $\varepsilon = \varepsilon_0 H(t)$, $\dot{\varepsilon} = \varepsilon_0\delta(t)$:

$$\tau_{\rm G}\, \dot{\sigma} + \sigma = E_1\varepsilon_0 H(t) + \eta\left(1 + \frac{E_1}{E_2}\right)\varepsilon_0\delta(t)\ , \tag{8}$$

with the initial condition $\sigma(t \le 0) = 0$.

On the other hand, if we impose a step stress $\sigma(t) = \sigma_0 H(t)$, then Eq. (**??**) reduces to (for $t > 0$)

$$\tau_{\rm J}\dot{\varepsilon} + \varepsilon = \frac{\sigma_0}{E_1}\ , \tag{9}$$

with $\tau_{\rm J} = \eta\left(\frac{1}{E_1} + \frac{1}{E_2}\right) = \tau_{\rm G} + \frac{\eta}{E_1}$. As before, the initial condition is $\varepsilon(0) = \frac{\sigma_0}{E_1+E_2}$ and is obtained by neglecting the dashpot's compliance. The solution is

$$J(t) = \frac{\varepsilon(t)}{\sigma_0} = \frac{1}{E_1} - \left(\frac{1}{E_1} - \frac{1}{E_1 + E_2}\right) e^{-t/\tau_{\rm J}}\ . \tag{10}$$

Note that both G and J should be multiplied by a step function, as they both obviously vanish for negative arguments. I didn't write the step function in the above but you should know that it is always there.

There is also a different, more general, way to calculate $G(t)$, somewhat analogous to Kirchoff's law for electrical circuits. The total stress in the material is the sum of the stresses in both branches, and the strain is the same for both branches. Therefore, $G(t)$ is given by the sum of the G's for each branch: for top branch it's purely elastic (i.e. $G = E_1$) and for the bottom branch it's a Maxwell material (i.e. $G = E_2 e^{-t/t_{\rm G}}$). Having obtained $G(t)$, $J(t)$ can be calculated from G by using the Laplace-transform identity of Q2.

(ii) Calculate $G^*(\omega)$. Find the effective Young's modulus for very short and very long time scales. Denote these quantities by E_0 and E_∞, respectively. Check that your finding agrees with the proper limits of the results of Q1(i). (writing J and G in terms of E_0 and E_∞ might prove to be more elegant than with E_1 and E_2).

Solution

We denote $\tilde{\omega} \equiv \omega\tau_{\mathrm{G}}$. From the definition of $G^*(\omega)$:

$$\begin{aligned} G^*(\omega) &= i\omega \int_0^\infty G(t) e^{-i\omega t} dt = i\omega \int_0^\infty \left[E_1\, e^{-i\omega t} + E_2\, e^{-(i\omega + 1/\tau_{\mathrm{G}})t}\right] dt \\ &= i\omega \left(\frac{E_1}{i\omega} + \frac{E_2}{i\omega + 1/\tau_{\mathrm{G}}}\right) = E_1 + E_2 \frac{i\tilde{\omega}}{1+i\tilde{\omega}} \\ &= \underbrace{E_1 + E_2 \frac{\tilde{\omega}^2}{1+\tilde{\omega}^2}}_{G'} + i\, \underbrace{E_2 \frac{\tilde{\omega}}{1+\tilde{\omega}^2}}_{G''} \ . \end{aligned} \tag{11}$$

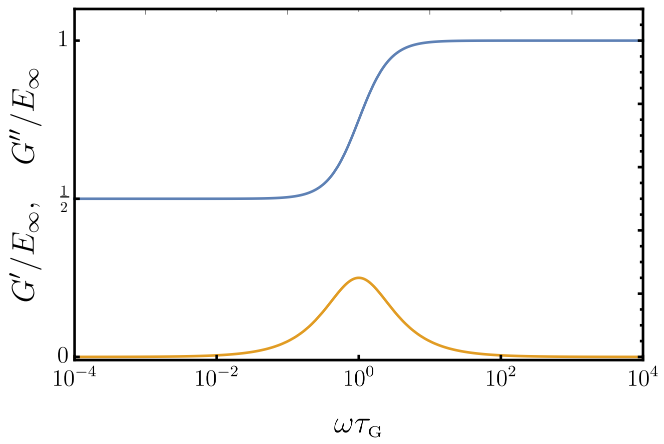
For $\tilde{\omega} \to 0$ we have $G^* = E_\infty \equiv E_1$ and for $\tilde{\omega} \to \infty$ we have $G^* = E_0 \equiv E_1 + E_2$. Note that with this notation we can rewrite J and G as

$$G(t) = E_\infty - (E_\infty - E_0) e^{-t/\tau_{\mathrm{G}}} \ , \qquad J(t) = E_\infty^{-1} - \left(E_\infty^{-1} - E_0^{-1}\right) e^{-t/\tau_{\mathrm{J}}} \ . \tag{12}$$

(iii) `QUALITATIVE QUESTION:` Plot the loss and storage moduli $G'(\omega)$ and $G''(\omega)$ on a logarithmic ω scale for the case $E_1 = E_2$. If you were to transfer waves through an SLS material, which frequencies would be transmitted and which attenuated?

Solution

As I try to stress time and again in this course, it is always a good habit to work in dimensionless units, and especially to plot things in dimensionless units. What is the natural time unit of ω that we need to use? There's a single time scale in $G(t)$ (or $G^*(\omega)$) which is τ_{G}. So we'll plot everything in as a function of the dimensionless quantity $\omega\tau_{\mathrm{G}}$. For non-dimensionalising G you can choose E_1, E_2, E_0 or E_∞, all of which make sense. Here I chose E_∞. Also, in this case it makes much more sense to plot the frequency in a logarithmic axis.



G' is plotted in blue and G'' in yellow. It is seen that the loss modulus is non negligible only for $\omega\tau_{\mathrm{G}} \approx 1$, so these are the frequencies that will be attenuated, and other frequencies, either much faster or much slower, will be transmitted.

(iv) Calculate $\varepsilon(t)$ when the stress increases from 0 to σ_0 over a time scale T. To make things concrete, take the stress to be

$$\sigma(t) = \begin{cases} 0 & t < 0 \\ \sigma_0 \left(1 - e^{-t/T}\right) & t > 0 \end{cases} .$$

For simplicity, also set $E_1 = E_2$. Plot $\varepsilon(t)$ for the cases that T is (i) much larger than, (ii) much smaller than, and (iii) roughly equal to the relevant internal time-scale of the system (for the two extreme cases you should know the answer without any further algebra!).

Solution

As always, one should start by non-dimensionalising the equations, and finding the important dimensionless quantities. Since here we have a step stress it seems reasonable that τ_J will be the relevant time scale. Defining $\tilde{t} = t/\tau_J$, $\alpha = \tau_J/T$, $\tilde{E}_i = E_i/E_\infty$ and $\tilde{\varepsilon} = \frac{E_\infty}{\sigma_0}\varepsilon$ we have

$$\begin{aligned} \varepsilon(t) &= \int_{-\infty}^{t} J(t-t')\dot{\sigma}(t')dt' \\ &= \int_0^t \left[\frac{1}{E_\infty} - \left(\frac{1}{E_\infty} - \frac{1}{E_0}\right) e^{-(t-t')/\tau_J}\right] \frac{\sigma_0}{T} e^{-t'/T} dt' , \end{aligned} \tag{13}$$

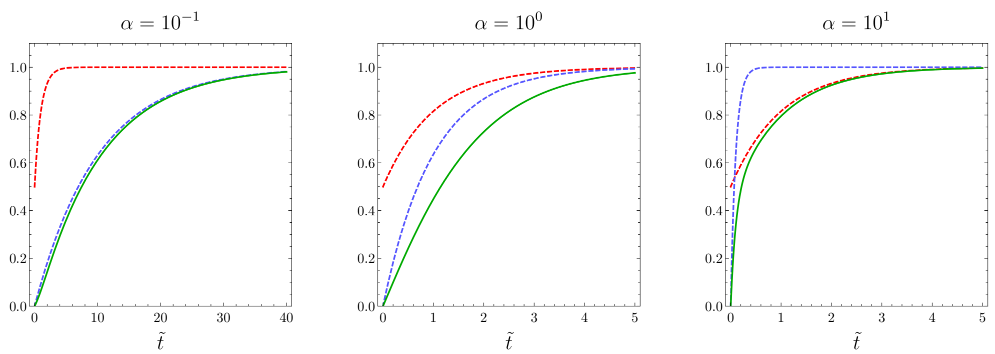
$$\tilde{\varepsilon}(\tilde{t}) = \alpha \int_0^{\tilde{t}} \left[1 - \left(1 - \frac{1}{\tilde{E}_0}\right) e^{-(\tilde{t}-\tilde{t}')}\right] e^{-\alpha\tilde{t}'} d\tilde{t}' . \tag{14}$$

I hope you see how simpler everything is when non-dimensional units are used. The integral is trivial, and the result is

$$\tilde{\varepsilon}(\tilde{t}) = 1 + \frac{\alpha(\tilde{E}_0 - 1)}{(1-\alpha)\tilde{E}_0} e^{-\tilde{t}} + \frac{\alpha - \tilde{E}_0}{(1-\alpha)\tilde{E}_0} e^{-\alpha\tilde{t}} . \tag{15}$$

Sanity check: for all α, in the limit $\tilde{t} \to \infty$, we have $\tilde{\varepsilon} \to 1$, or in other words $\varepsilon \to \sigma_0/E_\infty$, which is exactly as we (should) expect. In the opposite limit $\tilde{t} \to 0$ we have $\varepsilon \to 0$, which is also what we expect.

Now for numerical results: In our case we have $E_1 = E_2$ and thus $\tilde{E}_0 = 2$. For $\alpha \ll 1$ (very slow loading) we expect the dynamics to be quasi-static. That is, the strain should be given by $\varepsilon(t) = \sigma(t)/E_\infty$. For $\alpha \gg 1$ (very fast loading) we effectively have $\sigma = \sigma_0 H(t)$ and we therefore expect to have $\varepsilon = \sigma_0 J(\tilde{t})$. Both these predictions are numerically verified:



$\tilde{\varepsilon}(\tilde{t})$ is plotted in solid green, $J(\tilde{t})/E_\infty$ in red and $\sigma(\tilde{t})/\sigma_0$ in blue. Note that $J(\tilde{t})$ does not depend on α but $\sigma(\tilde{t})$ does.

(v) Consider an alternative model, defined in Fig. 1(b). Show that it is exactly equivalent to the SLS model, but with renormalized visco-elastic constants. That is, show that one can choose $\tilde{E}_1, \tilde{E}_2, \tilde{\eta}$ so that this model will have the same rheological properties as the SLS model with parameters E_1, E_2, η.

Solution

Following the same procedure as before, we'll denote the strain of the spring E_1 by ε_1, the strain of the Kelvin-Voigt module by ε_2, and the stresses in the spring and dashpot of the KV module by x, y respectively. We have

$$\varepsilon_1 = \frac{\sigma}{E_1} \ , \tag{16}$$

$$x = E_2\varepsilon_2, \qquad y = \eta\dot{\varepsilon}_2 \ , \tag{17}$$

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \quad \Rightarrow \quad \varepsilon_2 = \varepsilon - \frac{\sigma}{E_1} \ , \tag{18}$$

$$\sigma = x + y = E_2\varepsilon_2 + \eta\dot{\varepsilon}_2 = E_2\left(\varepsilon - \frac{\sigma}{E_1}\right) + \eta\left(\dot{\varepsilon} - \frac{\dot{\sigma}}{E_1}\right) \ . \tag{19}$$

Reorganizing the equation gives

$$\dot{\varepsilon} + \frac{E_2}{\eta}\varepsilon = \frac{1}{E_1}\dot{\sigma} + \frac{\left(1 + \frac{E_2}{E_1}\right)}{\eta}\sigma \ . \tag{20}$$

analogously to the Kirchoff law for the original SLS, here we have components attached in a series and therefore, their J's should be additive. One is a spring, for which $J = E_1 H(t)$, and the other is a KV material for which $J = E_2^{-1}\left(1 - e^{-E_2 t/\eta}\right)$. We therefore have

$$J = E_1^{-1} + E_2^{-1}\left(1 - e^{-E_2 t/\eta}\right) \ .$$

The substitution $E_1 \to E_1 + E_2$, $E_2 \to \frac{E_1}{E_2}(E_1 + E_2)$ and $\eta \to \eta\left(1 + \frac{E_1}{E_2}\right)^2$ makes this equation coincide with Eq. (??). Since each of J or G carries all the information, it is sufficient to show that the J coincide.

(vi) `QUALITATIVE QUESTION:` The name "SLS" model suggests that it describes a solid. What is the basic property of solids that the SLS model possess but that Maxwell model doesn't? And what is the problem with Kelvin-Voigt model?

Solution

The Maxwell model cannot statically hold stresses. That is, it has no elastic response at long time scales. The Kelvin-Voigt model does not feature an elastic response at short time scales. The SLS model features an elastic response at both limits, and shows dissipation in the intermediate scales.

2 Creep compliance and stress relaxation moduli

Prove the general identity

$$\int_0^t G(t-t')J(t')dt' = t \ , \tag{21}$$

and verify it explicitly for the KV and M models. Hint: Laplace transform.

Solution

For a function $f(t)$, we'll denote $\hat{f}(s) = \mathcal{L}\{f\} = \int_0^\infty f(t)e^{-st}dt$. We'll define

$$\chi(t) \equiv \int_0^t G(t-t')J(t')dt' = \int_{-\infty}^\infty G(t-t')J(t')dt' \ .$$

One can integrate for all t because both $J(t)$ and $G(t)$ vanish for negative t. Then from the properties of the Laplace transform, we know that $\hat{\chi}(s) = \hat{G}(s)\hat{J}(s)$. But recall that $G(t)$ is *defined* by the equation

$$\sigma(t) = \int_{-\infty}^t G(t-t')\dot{\varepsilon}(t')dt' \ . \tag{22}$$

If we choose $\varepsilon(t\leq 0) = \sigma(t \leq 0) = 0$, this definition is exactly the truncated convolution that one needs for the Laplace transform. So we have $\hat{\sigma} = \hat{G}\cdot\mathcal{L}\{\dot{\varepsilon}(t)\} = \hat{G}(s)\cdot s\hat{\varepsilon}(s)$. Similarly, $\hat{\varepsilon} = \hat{J}\cdot s\hat{\sigma}$. Isolating $\hat{J}$ and $\hat{G}$ and plugging in the equation for χ, we get

$$\hat{\chi}(s) = \hat{G}(s)\hat{J}(s) = \left(\frac{\hat{\sigma}}{s\hat{\varepsilon}}\right)\left(\frac{\hat{\varepsilon}}{s\hat{\sigma}}\right) = \frac{1}{s^2} \ . \tag{23}$$

Note that $s^{-2} = \mathcal{L}\{t\}$, and thus the identity is proven.

3 Stored elastic energy

Define W_{sto} as the elastic energy stored in a quarter of an oscillatory cycle. Recall the definition of the phase δ (Eqs. (9.39)-(9.40) in Eran's lecture notes) and show that

$$\frac{W_{dis}}{W_{sto}} \sim \tan\delta \ . \tag{24}$$

Solution

The *total* power, P^{tot}, invested by the loading is calculated from

$$\dot{\varepsilon}(t) = \mathcal{R}\left\{i\omega\varepsilon_0 e^{i\omega t}\right\} = -\omega\varepsilon_0 \sin\omega t \ , \tag{25}$$

$$\sigma(t) = \mathcal{R}\left\{|G^*|e^{i\delta}\varepsilon_0 e^{i\omega t}\right\} = |G^*|\varepsilon_0 \cos(\omega t + \delta) \ , \tag{26}$$

$$P^{tot} = \sigma : \dot{\varepsilon} = -\varepsilon_0^2 \omega |G^*| \sin(\omega t)\cos(\omega t + \delta) \ , \tag{27}$$

The energy invested after at time t is thus

$$\begin{aligned} E &= \int_0^t P^{tot}(t)dt = \int_0^t -\varepsilon_0^2\omega|G^*|\sin(\omega t)\cos(\omega t+\delta)dt \\ &= -\frac{\varepsilon_0^2\omega|G^*|}{2}\left[\omega t\sin(\delta) + \sin(\omega t)\sin(\omega t+\delta)\right] \ . \end{aligned} \tag{28}$$

The first term is a linear function of t, which we already identified in class as the dissipative part. The second term is therefore the stored part. For a quarter cycle, $\omega t = \pi/2$, we the ratio between them is

$$\frac{\frac{\pi}{2}\sin\delta}{\sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}+\delta\right)} = \frac{\pi}{2}\frac{\sin\delta}{\cos\delta} = \frac{\pi}{2}\tan\delta \ . \tag{29}$$

4 Distribution of relaxation times

For some systems which exhibit a wide distribution of relaxation times the stress relaxation modulus can be written as a weighted sum of exponential decays with different decay rates:

$$G(t) = \int_\tau f(\tau)e^{-\frac{t}{\tau}}d\tau \ , \tag{30}$$

where $f(\tau)$ is the relaxation times distribution function. If the relaxation times depend on some energy barrier Δ, this can be written as

$$G(t) = G_0\int_\Delta P(\Delta)e^{-\frac{t}{\tau(\Delta)}}d\Delta \ , \tag{31}$$

where P is the energy barrier distribution function. When the transitions are thermally–activated and the energy barriers are much larger than the thermal energy scale ($\Delta \gg k_BT$) the rate of escape times varies exponentially with the barrier height:

$$\tau(\Delta) \simeq \tau_0 e^{\frac{\Delta}{k_BT}} \ . \tag{32}$$

This is a fundamental result, derived by Arrhenius (1889, for chemical reactions) Eyring (1935) and Kramers (1940, for Brownian motion under an external potential) and the factor $e^{\frac{\Delta}{k_BT}}$ is usually termed "Arrhenius factor" or "rate factor".

(i) Assume that Δ is uniformly distributed between Δ_{min} and Δ_{max}, and calculate $G(t)$. Express your answer using rate variable $\nu(\Delta) \equiv 1/\tau(\Delta)$. You might find that the exponential integral function, $E_1(x) \equiv \int_1^\infty \frac{e^{-xy}}{y}dy$, is useful.

Solution

Defining $\nu_0 = 1/\tau_0$ we have

$$\nu(\Delta) = \frac{1}{\tau(\Delta)} = \nu_0 e^{-\frac{\Delta}{kT}} \ , \tag{33}$$

$$d\nu = -\frac{\nu(\Delta)}{kT} d\Delta \ . \tag{34}$$

Also, we know that $P(\Delta) = \frac{1}{\Delta_{max} - \Delta_{min}}$ for $\Delta_{min} \leq \Delta \leq \Delta_{max}$ and it vanishes elsewhere. Therefore,

$$G(t) = \frac{G_0}{\Delta_{max} - \Delta_{min}} \int_{\Delta_{min}}^{\Delta_{max}} e^{-\nu(\Delta)t} d\Delta \tag{35}$$

$$= -\tilde{G}_0 \int_{\nu_{min}}^{\nu_{max}} \frac{e^{-\nu t}}{\nu} d\nu = -\tilde{G}_0 \left(\int_{\nu_{min}}^{\infty} - \int_{\nu_{max}}^{\infty} \right) \frac{e^{-\nu t}}{\nu} d\nu \tag{36}$$

$$= \tilde{G}_0 \left[E_1(\nu_{max} t) - E_1(\nu_{min} t) \right] \ , \tag{37}$$

where we defined $\tilde{G}_0 \equiv G_0 \frac{kT}{\Delta_{max} - \Delta_{min}}$.

(ii) Choose $\nu_{min} \equiv \nu(\Delta_{min})$ and $\nu_{max} \equiv \nu(\Delta_{max})$ to be well separated and plot $G(t)$. What is special about this result? How does it differ from standard relaxation?

Solution

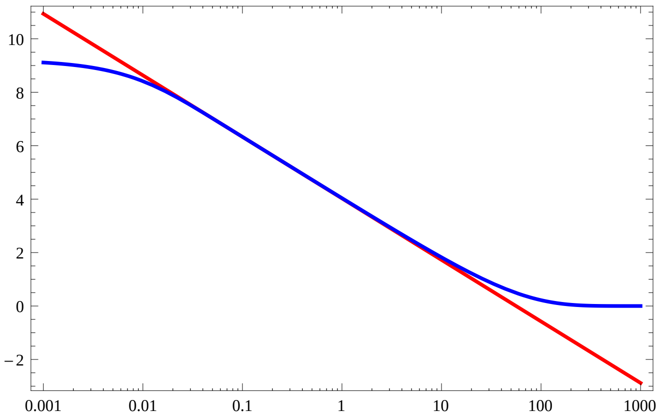


Figure S1: $G(t)/\tilde{G}_0$ (in blue) and $-\gamma - \log(\nu_{max} t)$ (in red). I chose $\nu_{min} = 10^2$ and $\nu_{max} = 10^{-2}$. Time axis is logarithmic.

This is fundamentally different from (and much slower than) usual relaxation because the relaxation is *logarithmic in time.*

(iii) Obtain an analytic expression (you are allowed to be wrong by an additive constant) for $G(t)$

for an intermediate asymptotic regime, $\nu_{min}^{-1} \ll t \ll \nu_{max}^{-1}$. This is a simple mathematical model for slow/glassy relaxation emerging from a broad distribution of relaxation times (activation barriers in this case).

For the interested, such a model has been proposed already by Primak, Phys. Rev. **100**, 1677 (1955), and later on by Kimmel and Uhlmann, J. Appl. Phys. **40**, 4254 (1969). This model has been revisited by Ariel Amir, Yuval Oreg and Joe Imry from the institute. See for example Amir, Oreg and Imry, PNAS **109**, 1850–1855 (2012).

Solution

Since obviously $\lim_{x\to\infty} E_1(x) = 0$, and since $\nu_{min}t \gg 1$, we can neglect the term $E_1(\nu_{min}t)$. The Taylor series of $E_1(x)$ is given by $E_1(x) = -\gamma - \log(x) + O(x)$, where γ is the Euler-Mascheroni constant, and we therefore have

$$\frac{G(t)}{\tilde{G}_0} \approx -\gamma - \log\left(\nu_{max}t\right) \ . \tag{38}$$

Comment: If you don't know the Taylor expansion of $E_1(x)$, you can do it yourself. Defining $u = xy$, we have $\frac{du}{u} = \frac{dy}{y}$ and

$$\begin{aligned} E_1(x) &= \int_1^\infty e^{-xy}\frac{dy}{y} = \int_x^\infty e^{-u}\frac{du}{u} = \int_1^\infty \frac{e^{-u}}{u}du + \int_x^1 \frac{e^{-u}}{u}du = \\ & E_1(1) + \int_x^1 \frac{e^{-u}}{u}du = E_1(1) + \int_x^1 \frac{1}{u}du - \sum_{n=0}^\infty \frac{(-1)^n}{(n+1)!}\int_x^1 u^n du \\ &= E_1(1) - \log(x) + \sum_{n=1}^\infty \frac{(-1)^n\left(1-x^n\right)}{nn!} \\ &= E_1(1) + \sum_{n=1}^\infty \frac{(-1)^n}{n\ n!} - \log(x) - \sum_{n=1}^\infty \frac{(-1)^n x^n}{n\ n!} \ . \end{aligned} \tag{39}$$

So we see that indeed we have the $-\log(x)$ term, and then an infinite polynomial which we may neglect at small x. We are left with showing that

$$E_1(1) + \sum_{n=1}^\infty \frac{(-1)^n}{n\ n!} = -\gamma \ , \tag{40}$$

which is true according to Mathematica, and you may can try to achieve from any of the identities regarding this constant. There are all kind of identities regarding the connection between the E_1 and Γ functions of all kinds of sorts, knock yourself out. You can find more details in chapter 13.6 in Arfken's "Mathematical methods for physicists"