Non-Equilibrium Continuum Physics **HW set #6**

TA: Avraham Moriel and Roy Wexler Submit to: groupbouchbinder@gmail.com

Due 26/07/2023 Submit as: NECP_HW6_⟨FirstName⟩_⟨LastName⟩.pdf

Plasticity - Solution

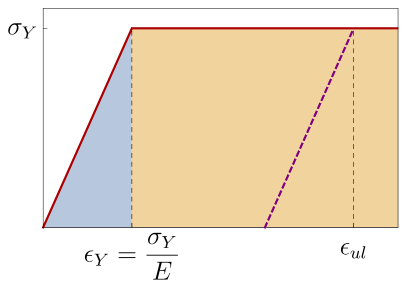
1. An incompressible elastic-perfect-plastic cylindrical rod, of Young's modulus E, yield stress $\sigma_Y \ll E$, length L and cross section A is compressed/pulled under uniaxial stress along its axis until its length is multiplied by a factor λ. How much work did the external loading preform? How much of it was dissipated? Work in the regime that $|\lambda - 1| \ll 1$, but plastic deformation does occur.

Solution

The work done by the loading is

$$\int F(x)dx = A\int \sigma_{zz}d(\epsilon_{zz}L) = AL\int \sigma_{zz}d\epsilon_{zz} \ .$$

This is simply the volume AL times the area under the stress-strain curve:



Stress-strain for elastic-perfect plastic material. The dashed purple line is an unloading curve.

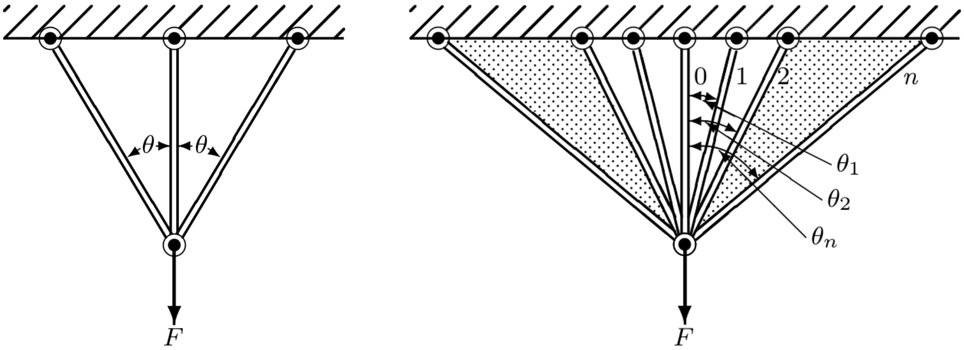
It has two contributions: the elastic part (blueish in the figure) equals $\frac{1}{2}\sigma_Y\epsilon_Y = \sigma_Y^2/2E$ and the plastic part (brownish in the figure) equals $(\epsilon - \epsilon_Y)\sigma_Y$. Of course, we need to use $\epsilon = \lambda - 1$. All the plastic part is dissipated, and all the elastic part is stored.

In the next question we will need to use the rest-length of the unloaded rod. Upon unloading, the response is elastic, and therefore the slope of the stress-strain curve is again E, as shown in the above figure (dashed purple line). The residual plastic strain upon unloading would therefore be $\epsilon = \epsilon_{ul} - \sigma_Y/E$ where ϵ_{ul} is the strain at which the unloading began. The new rest length will therefore be $L(1 + \epsilon_{ul} - \sigma_Y/E)$.

Note that we don't take into account the fact that the area A changes during deformation. This change will be first-order ($A \sim A(t = 0)(1 - 2\nu\epsilon_{zz})$ and since all the strains/stresses/energies/everything is already at least first order, this contribution is of a higher order and should be neglected. **This is generally true for all linear problems**, like we stressed many times in the course.

2. Consider the setting shown in Fig 1a: three elastic-perfect-plastic rods with cross sectional area A are connected with pins that can transfer only axial forces but no torques, and a vertical force F is exerted on them. The top pins are held at fixed positions to the ceiling (but not at a fixed

angle). All rods have Young's modulus E and yield stress $\sigma_Y \ll E$. When $F = 0$ the system is stress-free. Assume small deformations.



(a) Three rod setup

(b) Bonus question setup

Figure 1: n-rods setup.

(a) Denote the vertical displacement of the loading point by Δ. Calculate and plot $\Delta(F)$ (choose some values for the parameters you need). What is the maximal force F_E for which the response is elastic? What is the maximal force F_U that can be applied?

Solution

We begin by calculating the elastic solution. Let's denote the middle bar by 1 and the side bars by 2. We'll also denote the initial rest-lengths of the bars by L_1^0, L_2^0, and thus the force exerted by each bear is given by $|F_i| = EA\frac{L_i - L_i^0}{L_i^0}$. For the middle bar, this is easy:

$$F_1 = EA\frac{\Delta}{L_1^0} \ . \tag{1}$$

For the side bars, we need to use

$$L_2(\Delta) = \sqrt{(L_2^0 \cos\theta + \Delta)^2 + (L_2^0 \sin\theta)^2} = L_2^0 + \Delta\cos\theta + \mathcal{O}(\Delta^2) \tag{2}$$

$$F_2(\Delta) = EA\frac{\Delta}{L_2^0}\cos\theta = EA\frac{\Delta}{L_1^0}\cos^2\theta \tag{3}$$

Again, note that F is (obviously) linear in Δ, so for all calculations we don't need to take into account the change in θ, because this will give a contribution of order Δ^2. The total force is given by

$$\begin{aligned} F(\Delta) &= F_1 + 2F_2\cos\theta = EA\frac{\Delta}{L_1^0}\left(1 + 2\cos^3\theta\right) \\ \Delta(F) &= L_1^0 \frac{F}{EA\left(1 + 2\cos^3\theta\right)} \ . \end{aligned} \tag{4}$$

Avoiding direct reference to the rest-length, we can write (4) as

$$F_1 = F\frac{1}{1 + 2\cos^3\theta} \qquad\qquad F_2 = F\frac{\cos^2\theta}{1 + 2\cos^3\theta}$$

The stress ($\propto$ force) in the middle bar is larger, and therefore the system will yield when $F_1 \geq A\sigma_Y$. That is, the response will be elastic as long as

$$
\begin{aligned}
F &\leq F_E \equiv \sigma_Y A(1 + 2\cos^3\theta) \\
\Delta &\leq \Delta_E \equiv \frac{\sigma_Y}{E} L_1^0 = \epsilon_Y L_1^0 \ ,
\end{aligned}
$$

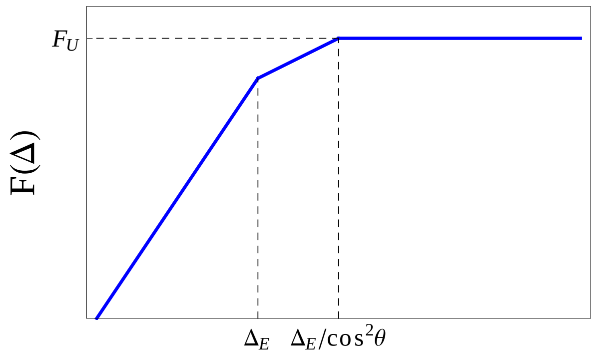
where we introduced the notation $\epsilon_Y \equiv \sigma_Y/E$. For $F > F_E$ the stress in the middle bar equals σ_Y and the force is thus $\sigma_Y A$. Static considerations still tell us that $F = F_1 + 2F_2\cos\theta$, which means

$$
F_2 = \frac{F - F_1}{2\cos\theta} = \frac{F - \sigma_Y A}{2\cos\theta} \ . \tag{5}
$$

Of course, this holds only as long as this value is lower than $\sigma_Y A$. Otherwise, the other beams yield too. This occurs when

$$
F = F_U \equiv \sigma_Y A(1 + 2\cos\theta) \ , \qquad \Delta = \Delta_U \equiv \frac{\Delta_E}{\cos^2\theta} \ . \tag{6}
$$

F_U is the ultimate possible force that can be exerted on the system.



When $\Delta_E \leq \Delta \leq \Delta_U$, the elastic deformation of the outer rods constrains the plastic deformation of the middle one. When $\Delta > \Delta_U$, the outer ones yield too, and the motion is unconstrained.

(b) Calculate the residual strains and stresses if the force is removed after the displacement was Δ.

Solution

Imagine that after we unload the system, we disconnect the rods. What would be their new rest-lengths? If $\Delta < \Delta_E$, clearly there are no residual stresses/strains. If $\Delta_E \leq \Delta \leq \Delta_U$ then the outer beams responded elastically, and therefore their rest-lengths did not change. Using the answer to Q1, the new rest-length of the middle beam, which we denote by $\tilde{L}_1^0$, is

$$
\tilde{L}_1^0 = L_1^0(1 + \epsilon_{ul} - \epsilon_Y) = L_1^0\left(1 + \frac{\Delta}{L_1^0} - \epsilon_Y\right) = L_1^0 + \Delta - \Delta_E \tag{7}
$$

Let's denote the residual displacement of the loading point by δ. We assume that during the unloading everything is elastic and does not re-enter the plastic regime (we will check this assumption a posteriori). We need to find the value δ such that the system will be in mechanical equilibrium. The length of the middle rod in equilibrium is $\delta + L_1^0$ and thus the force it exerts is

$$F_1 = EA\frac{L_1^0 + \delta - \tilde{L}_1^0}{L_1^0} = EA\frac{\delta - (\Delta - \Delta_E)}{L_1^0} \ .$$

The outer rods are simply elastic, so it follows Eq. (3):

$$F_2 = EA\frac{\delta}{L_1^0}\cos^2\theta \ .$$

Note that we expect F_1 to be negative (compression) and F_2 to be positive (extension). That is, we expect to find $0 \le \delta \le (\Delta - \Delta_E)$. We seek a static solution, i.e. $F_1 + 2F_2\cos\theta = 0$, which is solved for δ:

$$\delta = \frac{\Delta - \Delta_E}{1 + 2\cos^3\theta} \ . \tag{8}$$

We see that our expectations were fulfilled. From this the stresses and strains are easily calculated:

$$\begin{aligned} F_1 &= \frac{EA}{L_1^0}\left(\frac{\Delta - \Delta_E}{1 + 2\cos^3\theta} - (\Delta - \Delta_E)\right) = -\frac{EA}{L_1^0}(\Delta - \Delta_E)\frac{2\cos^3\theta}{1 + 2\cos^3\theta} \ , \\ F_2 &= \frac{EA}{L_1^0}(\Delta - \Delta_E)\frac{\cos^2\theta}{1 + 2\cos^3\theta} \ . \end{aligned}$$

Is this solution elastic? For the stress in rod 1 to be plastic we need to have$|F_1| > \sigma_Y A$. Plugging that into the expression for F_1 and solving for Δ gives

$$\Delta > \Delta_E\frac{1 + 4\cos^3\theta}{2\cos^3\theta} \tag{9}$$

However, simple algebra shows that it is impossible to satisfy this condition while respecting the assumption $\Delta < \Delta_U$ (cf. Eq. (6)). That is, this solution is always elastic as far as F_1 is concerned. Similar analysis shows that F_2 cannot be plastic neither.

Now consider the case that we stretched the material to the ultimate force, i.e. $\Delta \ge \Delta_U$. In this case all rods have changed their rest-lengths. The middle rod's rest-length is still given by (7). From Eq. (2) we understand that the strain of the outer rods is $\epsilon = \frac{\Delta}{L_2^0}\cos\theta$ and therefore their rest-length upon unloading will be

$$\tilde{L}_2^0 = L_2^0\left(1 + \frac{\Delta}{L_2^0}\cos\theta - \epsilon_Y\right) \ . \tag{10}$$

The forces in the rods after unloading are thus given by

$$\begin{aligned} F_1 &= EA\frac{\delta - (\Delta - \Delta_E)}{L_1^0} \ , \\ F_2 &= EA\frac{L_2 - \tilde{L}_2^0}{L_2^0} = EA\frac{L_2^0 + \delta\cos\theta - L_2^0\left(1 + \frac{\Delta}{L_2^0}\cos\theta - \epsilon_Y\right)}{L_2^0} = EA\frac{\cos^2\theta(\delta - \Delta) + \Delta_E}{L_1^0} \ . \end{aligned}$$

As before, solving $F_1 + 2F_2 \cos\theta = 0$ for δ we obtain

$$\delta = \Delta - \Delta_E \frac{1 + 2\cos\theta}{1 + 2\cos^3\theta} \ . \tag{11}$$

Plugging this back in to calculate the forces, we get

$$F_1 = -\sigma_Y A \frac{2\sin^2(\theta)\cos(\theta)}{2\cos^3(\theta) + 1} \qquad F_2 = \sigma_Y A \frac{2\sin^2(\theta)}{2\cos^3(\theta) + 1} \tag{12}$$

Simple algebra again shows that $|F_1/A|$ and $|F_2/A|$ are smaller than σ_Y so the assumption that everything was elastic is OK. Note that the residual stresses are *independent of* Δ but the residual strains are not. Does this surprise you?

(c) Suppose no force is applied, but the temperature is increased (or decreased) by ΔT. Calculate the minimal temperature difference ΔT_E that causes plastic deformation (assume α_T, σ_Y, E are T-independent).

Solution

Following the same philosophy, the rest-lengths of the rods are now

$$\tilde{L}_1^0 = L_1^0\left(1 + \frac{\alpha_T}{3}\Delta T\right) \qquad \tilde{L}_2^0 = L_2^0\left(1 + \frac{\alpha_T}{3}\Delta T\right) = \frac{\tilde{L}_1^0}{\cos\theta} \tag{13}$$

If the displacement of the bottom point is δ, then the forces are

$$\begin{aligned} F_1 &= EA\frac{(L_1^0 + \delta) - L_1^0\left(1 + \frac{\alpha_T}{3}\Delta T\right)}{L_1^0} = EA\frac{\delta - L_1^0\frac{\alpha_T \Delta T}{3}}{L_1^0} \\ F_2 &= EA\frac{(L_2^0 + \delta\cos\theta) - L_2^0\left(1 + \frac{\alpha_T}{3}\Delta T\right)}{L_2^0} = EA\frac{\delta\cos\theta - L_2^0\frac{\alpha_T}{3}\Delta T}{L_2^0} \\ &= EA\frac{\delta\cos^2\theta - L_1^0\frac{\alpha_T}{3}\Delta T}{L_1^0} \ . \end{aligned} \tag{14}$$

Again, we solve for equilibrium $F_1 + 2F_2\cos\theta = 0$ to get

$$\delta = L_1^0\frac{\alpha_T \Delta T}{3}\left(\frac{1 + 2\cos\theta}{1 + 2\cos^3\theta}\right) \tag{15}$$

Plugging this solution back in the expressions for the forces, we get that

$$F_1 = -2\cos(\theta)\, F_2 \qquad F_2 = -\frac{\alpha_T \Delta T}{3} EA \frac{\sin^2(\theta)}{2\cos^3(\theta) + 1} \tag{16}$$

One sees that we need to divide to two cases. If $\theta < 60°$ then $|F_1| > |F_2|$ so the middle rod will yield first. Solving the above equation with $F_1 = A\sigma_Y$ for ΔT, we obtains

$$\Delta T = \frac{3}{\alpha_T}\frac{\sigma_Y}{E}\frac{1 + 2\cos^3(\theta)}{2\sin^2\theta\cos\theta} \tag{17}$$

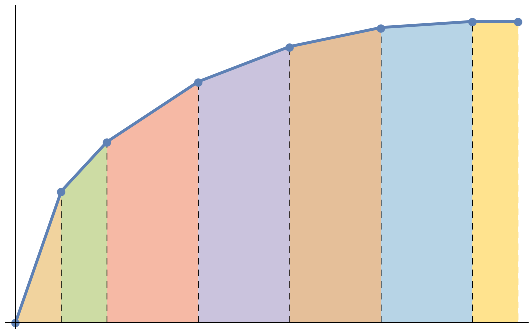
Note that this diverges when $\theta \to 0$, do you understand why?

If $\theta > 60°$ then you need to solve $F_2 = \sigma_Y A$. From (16) it's clear that you get the same ΔT as before, but multiplied by $2\cos\theta$.

(d) Bonus: repeat (a) for the case where there are 5 bars, or better yet, $2n+1$. The setup is shown in Figure 1b. Assume the system is symmetric with respect to horizontal reflection.

Solution

See *Plasticity Theory*, Jacob Lubliner, 1990 section 4.1.4 (pg. 185). The solution is not as detailed as the one I gave above, but it suffices for you to complete the the details. The bottom line is that you get a piecewise-linear stress-strain curve such that first the middle rod yields, then the closest-to-the-middle, then the second-closest-to-the-middle and so on. Between two successive yield events the function is linear. An example is plotted here:



3. In class, we've found the elasto-plastic solution for a spherical shell. We now look at some interesting aspects of the results.

(a) Examine numerically Eq. (11.38) from the lecture notes. For the case that $b = 10a$, plot c as a function of p. Can you analytically explain what happens when $p \to p_U$? (hint: yes you can).

Solution

The equation is

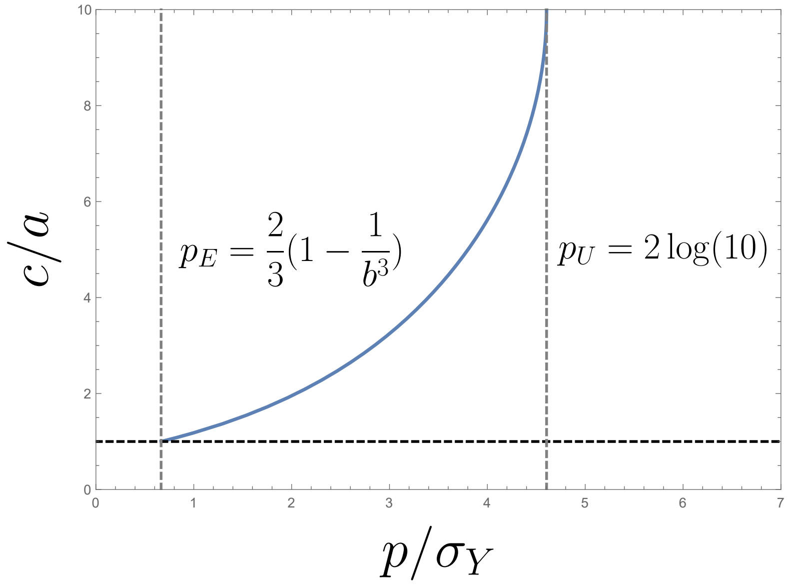
$$p = \frac{2\sigma_y}{3}\left[1 - \frac{c^3}{b^3} + 3\log\left(\frac{c}{a}\right)\right] .$$

As always, we should non-dimensionalize the equation. Measuring stresses in terms of σ_Y

and lengths in terms of a, the equation becomes

$$p = \frac{2}{3}\left[1 - \frac{c^3}{b^3} + 3\log(c)\right] \ , \tag{18}$$

where all quantities should have tildes. In these units, $b = 10a$ actually means $b = 10$. Inverting this relation numerically gives the following dependence:



The slope of the curve is

$$\frac{\partial c}{\partial p} = \left(\frac{\partial p}{\partial c}\right)^{-1} = \frac{1}{2c^2}\left(\frac{1}{c^3} - \frac{1}{b^3}\right)^{-1} \tag{19}$$

As $p \to p_U$, we have $c \to b$ so the term in parentheses vanishes and the slope diverges (but the curve reaches the finite value b/a). This happens when $\frac{p}{\sigma_Y} = 2\log\left(\frac{b}{a}\right)$.

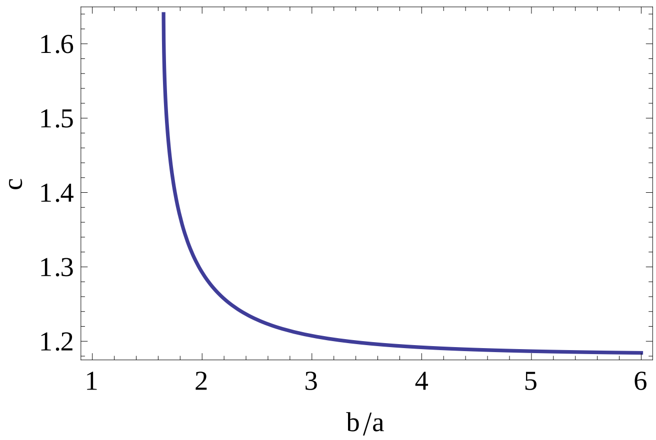
(b) For the case that $p = \sigma_Y$, plot c/a as a function of b/a. What is the asymptotic value of c when $b/a \to \infty$?

Solution

The dimensionless pressure is 1, so our equation takes the form

$$1 = \frac{2}{3}\left[1 - \frac{c^3}{b^3} + 3\log(c)\right] \ , \tag{20}$$

and the solution is shown here:



When $b/a = \tilde{b} \to \infty$ Eq. (20) reduces to

$$1 = \frac{2}{3}(1 + 3\log(c)) \ , \tag{21}$$

which is solved by $c = e^{1/6} \approx 1.18$.

(c) Find the displacement field $u_r(r)$ (from symmetry, $\vec{u}$ is a function of r only and other components vanish). Is the stress/strain/displacement field continuous/differentiable across the elasto-plastic boundary?

Guidance: In the elastic region, there's a particularly simple relation between u_r and some of the strain components. In the plastic region, the volumetric part of the deformation is still elastic - we still have $\operatorname{tr}\boldsymbol{\sigma} = K \operatorname{tr}\boldsymbol{\epsilon}$, where K is the bulk modulus.

Solution

In the elastic domain we have $\epsilon_{\theta\theta} = \epsilon_{\phi\phi} = u_r/r$ (that's a general kinematic formula for radial motion). Since $\epsilon_{\theta\theta} = E^{-1}[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{\phi\phi})]$, we obtain

$$u_r = \frac{r}{E}\left((1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right) \ . \tag{22}$$

Plugging in Eqs. (11.28)-(11.29) we get

$$u_r = \frac{r}{E}\,\frac{p_c}{b^3/c^3 - 1}\left(1 - 2\nu + (1+\nu)\frac{b^3}{2r^3}\right) \tag{23}$$

In the plastic regime, the volumetric response is elastic, that is $\operatorname{tr}\boldsymbol{\sigma} = 3K \operatorname{tr}\boldsymbol{\epsilon}$, with $K = \frac{E}{3(1-2\nu)}$:

$$\operatorname{tr}\boldsymbol{\epsilon} = \epsilon_{\theta\theta} + \epsilon_{\phi\phi} + \epsilon_{rr} = \frac{\partial u_r}{\partial r} + 2\frac{u}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) \tag{24}$$

$$\operatorname{tr}\boldsymbol{\sigma} = \sigma_{rr} + 2\sigma_{\theta\theta} = (3\sigma_{rr} + 2\sigma_Y) \tag{25}$$

Where we used the fact that in the plastic zone we have $\sigma_{\theta\theta} = \sigma_{rr} + \sigma_Y$ (Eq. (11.37)). Plugging in the expression for σ_{rr} (Eq. (11.36)) we arrive at

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = \frac{2(1-2\nu)\sigma_Y}{E}\left[\frac{c^3}{b^3} - 3\log\left(\frac{c}{r}\right)\right] \tag{26}$$

Which is solved by

$$u_r = \frac{A}{r^2} + \frac{2(1-2\nu)\sigma_Y r}{E}\left[\frac{1}{3}\left(\frac{c^3}{b^3}-1\right) - \log\left(\frac{c}{r}\right)\right] \tag{27}$$

The integration constant A is determined form continuity at $r = c$. The stress field is continuous across the boundary. This is because σ_{rr} must be continuous for static equilibrium to exist, and the other components of the stress depend continuously on σ_r (remember that $\sigma_{\theta\theta} = \sigma_{rr} + \sigma_Y$). The strain is not continuous, and neither the stress nor the strain are differentiable.

4. Continuing our TA session, consider an elastic-perfect-plastic 2D annulus with internal and external radii a, b, subject to internal pressure p and zero outer pressure, under *plane-stress* conditions. Use the Tresca yield criterion, and preform the analysis that was done in class:

   (a) Find the stress field $\sigma_{ij}(r)$, the minimal internal pressure that induces plastic flow (p_E), the ultimate pressure for which the entire annulus is plastic p_U, and give an equation that determines the radius of the elasto-plastic boundary c. Try and solve this in a different method than the one showed in the TA session.

Solution

The purpose of this exercise was that you redo the algebra in a slightly different setting. The calculations are practically the same, so I will only give hints here. The full thing is derived in Lubliner's book (section 4.3.5).

Elastic solution

The elastic solution is obtained in the following way. In 2D the force balance equation (11.14) takes the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 \ . \tag{28}$$

As in 3D, we use Hooke's law, combined with the compatibility equation, to obtain the equivalent of Eq. (11.18):

$$\frac{\partial}{\partial r}(\sigma_r + \sigma_\theta) = 0 \tag{29}$$

This is solved under the proper boundary conditions to yield

$$\sigma_r = -\frac{p}{b^2/a^2 - 1}\left[\frac{b^2}{r^2} - 1\right] \tag{30}$$

$$\sigma_\theta = \frac{p}{b^2/a^2 - 1}\left[\frac{b^2}{r^2} + 1\right] \tag{31}$$

The maximal value of $\sigma_\theta - \sigma_r$ is obtained at $r = a$ where it equals $\frac{2p}{1-a^2/b^2}$ and therefore the system will begin to yield when

$$p = p_E \equiv \sigma_Y\left(1 - \frac{a^2}{b^2}\right) \ . \tag{32}$$

Elasto-Plastic solution

The elastic part of the elasto-plastic solution is obtained by substituting b with c and p with p_c in the above equations. The plastic part is obtained by assuming that

$$\sigma_r < \sigma_z = 0 < \sigma_\theta \tag{33}$$

(this will be checked later for consistency) and therefore the Tresca criterion reads

$$|\sigma_\theta - \sigma_r| = \frac{2p}{1 - c^2/b^2} = 2\sigma_Y \quad \Rightarrow \quad p_c = \sigma_Y \left(1 - \frac{c^2}{b^2}\right) \tag{34}$$

p_E is obtained by plugging $c \to a$ in the above. Eq. (28) can then be integrated to give

$$\sigma_r = -p + \sigma_Y \log \frac{r^2}{a^2} \ . \tag{35}$$

Continuity of stresses then yields the transcendental equation for c:

$$p = \sigma_Y \left(1 - \frac{c^2}{b^2} + \log \frac{c^2}{a^2}\right) \tag{36}$$

$p_U = 2\sigma_Y \log \frac{b}{a}$ is the solution of this equation for $c = b$. Plugging (36) into (35), and using $\sigma_\theta = \sigma_r + 2\sigma_Y$ in the plastic zone, we get the stress field:

$$\sigma_r = \sigma_Y \left(\frac{c^2}{b^2} - \log \frac{c^2}{r^2} - 1\right) \tag{37}$$

$$\sigma_\theta = \sigma_Y \left(\frac{c^2}{b^2} - \log \frac{c^2}{r^2} + 1\right) \tag{38}$$

(b) Show that your solution is valid only if

$$1 + \frac{c^2}{b^2} - \log \frac{c^2}{a^2} \geq 0 \ . \tag{39}$$

What happens if this criterion is not satisfied? Why is this problem not present in plane strain conditions?

Solution

Take a look at Eq. (38) and remind yourself that we assumed $\sigma_\theta > 0$. If this is not the case, then $\sigma_z = 0 > \sigma_\theta$ and then the form of the Tresca criterion changes and everything we did is invalid. The smallest value of σ_θ occurs on $r = a$ so in order for our solution to be valid we need to demand $\sigma_\theta(r = a) > 0$, and this is exactly the condition (39).

In plane-strain conditions, we have $\sigma_z = \nu(\sigma_r + \sigma_\theta)$. At $r = c$ we have

$$\begin{aligned} \sigma_r &= \frac{p_c}{(b/c)^2 - 1}\left(-\frac{b^2}{c^2} + 1\right) \\ \sigma_\theta &= \frac{p_c}{(b/c)^2 - 1}\left(\frac{b^2}{c^2} + 1\right) \\ \sigma_z &= 2\nu\frac{p_c}{(b/c)^2 - 1} \end{aligned} \tag{40}$$

and since $1-\frac{b^2}{c^2} < 2\nu < 1+\frac{b^2}{c^2}$ our assumption is always valid (remember that $0 < \nu < \frac{1}{2}$).

(c) Considering this, what is the condition on a/b that ensures that p_U exists? Give an equation that describes, for a given value of a/b, the maximal possible value of c/a. What is this value when $b/a \to \infty$?

Solution

p_U describes the situation that the entire disk can become plastic, that is, $c{=}b$. plugging that in the condition, we get $1 - \log(b/a) \geq 0$, or more nicely $b/a \leq e$. For larger values of b/a our solution breaks down before the entire disk have flowed.

The maximal possible value of c is obtained by turning the condition (39) into an equality. In the limit $b \gg a$ (a hole in an infinite plane), this turns to be $1 - 2\log(c/a)$, and the limiting value is therefore $c = a\sqrt{e}$.