Plasticity

1 Unloading, residual stresses, shakedown (auto-frettage)

We consider a cylindrical shell under internal pressure, similar to the spherical problem fully solved by Eran. It's important to note that here we define $p \equiv -\sigma_{rr}(r=a)$, which is exactly the pressure in the usual stress, but is obviously related to it, and is the main physical player in the driving. The equations of plane strain ($\epsilon_{zz} = 0$) are

$$\epsilon_{rr} = \frac{1+\nu}{E}\left((1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}\right) \ , \tag{1}$$

$$\epsilon_{\theta\theta} = \frac{1+\nu}{E}\left((1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right) \ . \tag{2}$$

Plugging these into the compatibility equation, which in polar coordinates reads $\epsilon_{rr} = \frac{d}{dr}(r\epsilon_{\theta\theta})$, gives

$$\frac{d}{dr}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right] = \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \ . \tag{3}$$

Together with the force balance $\partial_r\sigma_{rr} + (\sigma_{rr} - \sigma_{\theta\theta})/r = 0$, we get

$$\frac{d}{dr}(\sigma_{rr} + \sigma_{\theta\theta}) = 0 \ . \tag{4}$$

1.1 Recap - elastic solution

The elastic solution may be obtained either by using the Airy stress fuction χ, or by substitution and symmetry considerations. Let us quickly recap how this is done — using χ we have $\sigma_{rr} = \frac{\partial_r\chi}{r} + \frac{\partial_{\theta\theta}\chi}{r^2}$, $\sigma_{r\theta} = -\partial_r\left(\frac{\partial_\theta\chi}{r}\right)$, and $\sigma_{\theta\theta} = \partial_{rr}\chi$. Due to the azimutal symmetry, we are looking for θ-independent solution. Substituting we have

$$\frac{\partial_r\chi}{r} + \partial_{rr}\chi = C_1 \ . \tag{5}$$

Solving the homogeneous equation we have $\chi = C_2\log(r) + C_3$. Together with the private solution, we have $\chi = C_2\log(r) + C_3 + \frac{C_1}{4}r^2$.

To find the various constants, we need to use the boundary conditions. Take $r=a$, where we know $\sigma_{rr}(r=a) = \frac{C_1}{2} + \frac{C_2}{a^2} = -p$, which is one relation. Using the outside, traction-free, boundary we have that $\sigma_{rr}(r=b) = \frac{C_1}{2} + \frac{C_2}{b^2} = 0$. Solving for the coefficients from these two equations, we have $C_1 = -\frac{2a^2p}{a^2-b^2}$ and $C_2 = \frac{a^2b^2p}{a^2-b^2}$, and altogether

$$\sigma_{rr} = p\cdot\frac{a^2}{a^2-b^2}\left(\frac{b^2}{r^2} - 1\right) \ , \ \sigma_{\theta\theta} = -p\cdot\frac{a^2}{a^2-b^2}\left(\frac{b^2}{r^2} + 1\right) \ . \tag{6}$$

1.2 Plasticity

We now introduce the Tresca Criterion for plasticity, i.e. that $\frac{1}{2}\max\left(|\sigma_i - \sigma_j|_{i\neq j}\right) = \sigma_y$. As the loading happens from the *inside* we know the elastic region will be in the outer part, up to some internal radius c. In this elastic region $c < r < b$ we have

$$\sigma_{rr} = \sigma_y \left(\frac{c^2}{b^2} - \frac{c^2}{r^2}\right) \ , \qquad \sigma_{\theta\theta} = \sigma_y \left(\frac{c^2}{b^2} + \frac{c^2}{r^2}\right) \ , \tag{7}$$

and in the plastic region $a < r < c$ it is

$$\sigma_{rr} = \sigma_y \left[\frac{c^2}{b^2} - \log\left(\frac{c^2}{r^2}\right) - 1\right] \ , \qquad \sigma_{\theta\theta} = \sigma_y \left[\frac{c^2}{b^2} - \log\left(\frac{c^2}{r^2}\right) + 1\right] \ , \tag{8}$$

where the logarithmic contribution arises from the different constitutive law in the plastic zone, and the quasi-static momentum balance $\partial_r\sigma_{rr} + (\sigma_{rr} - \sigma_{\theta\theta})/r = 0$. Since we are in plane-strain conditions, we the zz component of the stress is given by $\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$.

We now evaluate p_E — the pressure above which the solution is no longer elastic (i.e. the onset of plasticity at $r\!=\!a$) —, p_U — the pressure for which the whole shell becomes plastic —, and p_c, as

$$p_c = -\sigma_{rr}(r = c) = \sigma_y \left(1 - \frac{c^2}{b^2}\right) \ , \quad p_E = \sigma_y \left(1 - \frac{a^2}{b^2}\right) \ , \quad p_U = \sigma_y \log\frac{b^2}{a^2} \ , \tag{9}$$

and c satisfies $\sigma_{rr}\,(r\!=\!a) = -p$, that is

$$\frac{p}{\sigma_y} = 1 - \frac{c^2}{b^2} + \log\frac{c^2}{a^2} \ . \tag{10}$$

Now, what happens if we remove the internal pressure? How do we deal with this kind of (un)loading? What is the constitutive law that one should use?

This is a tricky subject and there are many subtleties in the general case. In our case of perfect plasticity you can think about it in the following manner: The perfect plastic constitutive law makes sure that the stress state at any given point in the material will always be inside the yield surface (in the elastic case) or strictly on it (in the plastic case). In other words, every point which is in a plastic state is also exactly on the threshold of yielding. Thus, the unloading dynamics is governed by elasticity. Or more precisely, as we'll soon see, at least the first part of it is governed by elasticity.

So we conclude that to get the unloaded state we need to subtract the fully elastic solution from the elasto-plastic solution. That is, we need to subtract Eq. (7) with $c \to a$

and $\sigma_y \to p/(1-\frac{a^2}{b^2})$ from Eqs. (7)-(8). The result is

$$\begin{aligned} \sigma_{rr} &= -\sigma_y \left(\frac{c^2}{a^2} - \frac{p}{p_E}\right)\left(\frac{a^2}{r^2} - \frac{a^2}{b^2}\right) \\ \sigma_{\theta\theta} &= \quad \sigma_y \left(\frac{c^2}{a^2} - \frac{p}{p_E}\right)\left(\frac{a^2}{r^2} + \frac{a^2}{b^2}\right) \end{aligned} \qquad c < r < b\ , \tag{11}$$

$$\begin{aligned} \sigma_{rr} &= -\sigma_y \left[\frac{p}{p_E}\left(1 - \frac{a^2}{r^2}\right) - \log\frac{r^2}{a^2}\right] \\ \sigma_{\theta\theta} &= -\sigma_y \left[\frac{p}{p_E}\left(1 + \frac{a^2}{r^2}\right) - \log\frac{r^2}{a^2} - 2\right] \end{aligned} \qquad a < r < c\ . \tag{12}$$

Note that the system has no tractions at the boundaries but the stress field does not vanish! These stresses are called residual stresses. The largest value of $|\sigma_{\theta\theta} - \sigma_{rr}|/2$ is at $r = a$, where it is $\sigma_y(p/p_E - 1)$. Unloading is thus purely elastic if $p/p_E \le 2$. This is surely the case if $p_U < 2p_E$. That is, if

$$\sigma_y \log\frac{b^2}{a^2} < 2\sigma_y \left(1 - \frac{a^2}{b^2}\right)\ . \tag{13}$$

The condition (13) is satisfied if $b/a \le 2.218$. If, on the other hand, $p > 2p_E$ then the unloading itself will create a new plastic zone at $a < r < c'$.

We can therefore define $p_s = \min(2p_E, p_U)$ (s for shakedown). If $p_E < p < p_s$ then unloading is elastic although the loading was elastic-plastic, and *every subsequent loading/unloading with pressure up to p is also elastic!*.

Physically, the portions of the cylinder that have underwent plastic deformation are now providing additional hoop stresses to the cylinder, making it stronger than it was before the plastic flow. In the context of reinforcing metal cylinders so that they can withstand high internal pressures (you can guess what is the technological motivation for that) this is called auto-frettage ("frettage" is French for the process of putting hoops). In a more general context this is called "shakedown". A similar concept is used in prestressed concrete. To sum it up, we have four regimes

$0 < p < p_E$ System is fully elastic.

$p_E < p < p_s$ Elastic-plastic loading, elastic unloading.

$p_s < p < p_U$ Elastic-plastic loading and unloading, if exists.

$p_U < p$ No static axiosymetric solution exists.

2 Plastic cavitation

Earlier in the course, we considered the problem of elastic cavitation in soft solids. Can we analyze a similar problem for hard solids?

The answer is definitely yes, such an analogous phenomenon exists for hard solids, though the physical processes is different; while for soft solids elastic deformation can be very large and lead to cavitation, hard solids show a limited range of elastic response and the origin of cavitation is plastic deformation.

We follow the kinematic analysis of elastic cavitation (starting in Eq.(7.50) in Eran's lecture notes), which is reproduced here

$$\lambda_r = \left(1 + \frac{L^3 - \ell^3}{r^3}\right)^{2/3} \ , \tag{14}$$

where λ_r is the radial stretch, L is the radius of the undeformed cavity and ℓ is the radius of the deformed one. The logarithmic strain ϵ_r reads

$$\epsilon_r = \log \lambda_r = \frac{2}{3} \log\left(1 + \frac{L^3 - \ell^3}{r^3}\right) \ . \tag{15}$$

Note that we use the logarithmic strain because it is thermodynamically-conjugated to the Cauchy stress that we use next. The force balance equation in terms of the Cauchy stress is given by $\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0$, and the boundary conditions are

$$\sigma_r(r = \ell) = 0 \quad \text{and} \quad \sigma_r(r \to \infty) = \sigma^\infty \ . \tag{16}$$

Since symmetry implies $\sigma_\theta = \sigma_\phi$ and we assume incompressibility, the stress state is essentially uniaxial and we can write down a general constitutive law as

$$\sigma_r - \sigma_\theta = \sigma_y f(\epsilon_r) \ . \tag{17}$$

We then have

$$\begin{aligned} \int_\ell^\infty d\sigma_r &= -2\sigma_y \int_\ell^\infty \frac{f(\epsilon_r) dr}{r} = -2\sigma_y \int_\ell^\infty f\left[\frac{2}{3}\log\left(1 + \frac{(L/\ell)^3 - 1}{(r/\ell)^3}\right)\right] \frac{d(r/\ell)}{(r/\ell)} \\ \Longrightarrow \sigma^\infty &= -2\sigma_y \int_1^\infty f\left[\frac{2}{3}\log\left(1 + \frac{(L/\ell)^3 - 1}{x^3}\right)\right] \frac{dx}{x} \ . \end{aligned} \tag{18}$$

The cavitation threshold σ_c is defined as the stress needed to grow the cavity indefinitely, i.e. $\ell \gg L$. This leads to

$$\sigma_c = \lim_{\ell \to \infty} \sigma^\infty = -2\sigma_y \int_1^\infty f\left[\frac{2}{3}\log\left(1 - x^{-3}\right)\right] x^{-1} dx \ . \tag{19}$$

The constitutive law $(\sigma_r - \sigma_\theta)/\sigma_y = f(\epsilon_r)$ we adopt is that of perfect plastic material, which we interpret here as pertaining to the logarithmic strain and also allow all quantities to be signed,

$$\begin{aligned} f(\epsilon_r) &= \frac{\epsilon_r}{\epsilon_y} \quad \text{for} \quad |\epsilon_r| < \epsilon_y \\ f(\epsilon_r) &= \text{sign}(\epsilon_r) \quad \text{for} \quad |\epsilon_r| \geq \epsilon_y \ , \end{aligned} \tag{20}$$

where $\epsilon_y \equiv \sigma_y/E$. With this law at hand, after a few rather simple mathematical manipulations, we obtain a nice analytic result. First, we use the yield strain ϵ_y inside the argument of $f(\cdot)$ in the above integral

$$-\epsilon_y = \frac{2}{3}\log\left(1 - x_y^{-3}\right) \qquad \Longrightarrow \qquad x_y = \left[1 - \exp(-3\epsilon_y/2)\right]^{-1/3} \ . \tag{21}$$

This allows us to use the constitutive law in order to split the integral into its elastic and plastic contributions as

$$\frac{\sigma_c}{\sigma_y} = \underbrace{2\int_1^{x_y} x^{-1}dx}_{\text{Plastic domain}} - \underbrace{\frac{4}{3\epsilon_y}\int_{x_y}^{\infty} \log\left(1-x^{-3}\right)x^{-1}dx}_{\text{Elastic domain}} \ . \tag{22}$$

We now recall that there exists a small parameter in the problem, $\epsilon_y \ll 1$ (since for ordinary hard solids the yield stress is much smaller than the elastic modulus). Therefore, we have

$$x_y \simeq \left(\frac{2}{3\epsilon_y}\right)^{1/3} \gg 1 \quad \text{and} \quad \log\left(1-x^{-3}\right) \simeq -x^{-3} \ . \tag{23}$$

This immediately yields

$$\frac{\sigma_c}{\sigma_y} \simeq 2\log(x)\Big|_1^{\left(\frac{2}{3\epsilon_y}\right)^{1/3}} - \frac{4}{3\epsilon_y} \times \frac{x^{-3}}{3}\Big|_{\left(\frac{2}{3\epsilon_y}\right)^{1/3}}^{\infty} = \frac{2}{3}\log\left(\frac{2}{3\epsilon_y}\right) + \frac{2}{3} \ . \tag{24}$$

Therefore,

$$\frac{\sigma_c}{E} \simeq \frac{2\epsilon_y}{3}\left[1+\log\left(\frac{2}{3\epsilon_y}\right)\right] \ . \tag{25}$$

As expected, σ_c is an increasing function of σ_y (for a fixed E), but the dependence is not trivial and could not have been guessed to begin with (note, though, that the elastic term is related to the elasticity limit presented in the spherical shell example when in the $b/a \to \infty$ limit). This is an example of unlimited plastic flow under a fixed applied stress ("plastic collapse").

3 Kramers-Kronig Relation

The KK relation is a fundamental relation between the real and imaginary parts of a response function $\hat{G}(\omega)$. In our case, it relates the storage and loss moduli $\hat{G}'$ and $\hat{G}''$ but it is very general and has applications in experimental and theoretical physics, as well as in signal processing and electrical engineering. The essence of these relations lies in the fact that the imaginary and real parts of an analytic function are not independent, and are related via the Cauchy-Riemann Equations, which in turn imply Cauchy's integral formula (residue calculus).

We will give two proofs of the KK relations. The standard residue-calculus one, and another one that singles out the effect of causality. The actual theorem is almost misleadingly simple: ***Theorem***: *Let $\hat{G}(\omega) = \hat{G}'(\omega) + i\hat{G}''(\omega)$ be an analytic function in the upper half plane that decays at infinity faster than $|\omega|^{-1}$. Then*

$$\hat{G}'(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\hat{G}''(\omega')}{\omega'-\omega}\,d\omega' \ , \qquad \hat{G}''(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\hat{G}'(\omega')}{\omega'-\omega}\,d\omega' \ , \tag{26}$$

where $\mathcal{P}$ denotes the principal value.

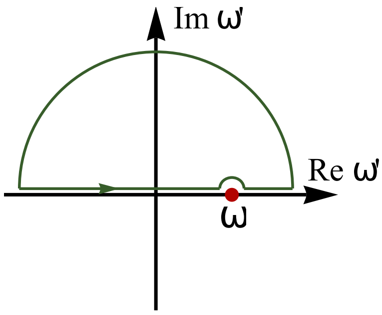

Figure 1: Contour of integration in the upper half plane. Source: wiki commons.

Proof I: The integral of $\frac{\hat{G}(\omega)}{\omega'-\omega}$ over the contour described in Fig. 1 is clearly 0, because the integrand is analytic. The integral over the half circle vanishes (because $\hat{G}(\omega)$ decays fast enough), and the integral over the bump is $-i\pi\hat{G}(\omega)$ (minus one half of the residue). We therefore have

$$\hat{G}(\omega) = \frac{1}{i\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{\hat{G}(\omega')}{\omega'-\omega}d\omega' \ . \tag{27}$$

Writing this equation in its real and imaginary parts gives exactly Eqs. (26) ■ .

This proof is completely trivial in terms of residue calculus, but as a physicist I am not quite satisfied by this proof. It leaves me with a feeling that I accept the theorem, but I don't *understand* it. Furthermore, we must ask (a) how do we know that $\hat{G}(\omega)$ decays at infinity? and (b) how do we know that $\hat{G}$ is analytic in the upper half plane?.

The answer to (a) is that we don't, in general, but it is very reasonable to assume that systems that are driven at frequencies much higher than their natural frequencies do not respond (and therefore $\hat{G} \to 0$).

The answer to (b) is less trivial. Note that in general the Fourier transform of a "nice" function is not analytic in the upper half plane. For example, the FT of a Lorenzian $\frac{1}{1+(t/\tau)^2}$ is $\sim \exp\left(-|\omega\tau|\right)$ which is not analytic anywhere; conversely, the FT of $\exp\left(-|\omega_0\tau|\right)$ is a Lorenzian, and thus is analytic almost everywhere but has a pole at $\omega = i\omega_0$; the FT of a Gaussian is also a Gaussian, which has an essential singularity at infinity.

The fact that $\hat{G}(\omega)$ is analytic for $\Im(\omega) > 0$ stems from **causality**. In fact, one can show that $\hat{G}(\omega)$ is analytic in the upper half plane if and only if $G(t) = 0$ for $t < 0$ (this is called Titchmarsh's theorem). To see exactly what is the role that causality takes, we'll examine a different proof.

Proof II: We first remind ourselves of the trivial fact that the FT of an even function is purely real, and that of an odd function is purely imaginary. Now, for any function $G(t)$ we can define

$$G^{\text{even}}(t) \equiv \frac{G(t)+G(-t)}{2} \ , \qquad G^{\text{odd}}(t) \equiv \frac{G(t)-G(-t)}{2} \ , \tag{28}$$

such that $G(t)$ can be written as $G(t) = G^{\text{even}}(t) + G^{\text{odd}}(t)$. Therefore,

$$\hat{G}(\omega) = \mathcal{F}\left\{G^{\text{even}}(t) + G^{\text{odd}}(t)\right\} = \mathcal{F}\left\{G^{\text{even}}(t)\right\} + \mathcal{F}\left\{G^{\text{odd}}(t)\right\} = \hat{G}'(\omega) + i\hat{G}''(\omega) \ . \tag{29}$$

We can thus conclude that

$$\hat{G}' = \mathcal{F}\{G^{\text{even}}\} \qquad , \qquad \hat{G}'' = \frac{1}{i}\mathcal{F}\{G^{\text{odd}}\} \ . \tag{30}$$

In general, the odd and even parts are independent, but for a casual response function we have $G(t) = 0$ for $t < 0$ and therefore

$$\begin{aligned} G^{\text{odd}}(t) &= \frac{1}{2}\begin{cases} -G(-t) & t < 0 \\ G(t) & t > 0 \end{cases} = \frac{1}{2}G(|t|)(t) \ , \\ G^{\text{even}}(t) &= \frac{1}{2}\begin{cases} G(-t) & t < 0 \\ G(t) & t > 0 \end{cases} = \frac{1}{2}G(|t|) \ , \end{aligned} \tag{31}$$

where $(t) = \frac{t}{|t|}$ is the signum function. Thus, for $t > 0$ we have $G^{\text{odd}}(t) = G^{\text{even}}(t)$ and for $t < 0$ we have $G^{\text{odd}}(t) = -G^{\text{even}}(t)$. This can be compactly written as

$$G^{\text{odd}}(t) = (t)G^{\text{even}}(t) \qquad , \qquad G^{\text{even}}(t) = (t)G^{\text{odd}}(t) \ . \tag{32}$$

Thus, we have

$$G'(\omega) = \mathcal{F}\{G^{\text{even}}\} = \frac{1}{2\pi}\mathcal{F}\{\} * \mathcal{F}\{G^{\text{odd}}\} = \frac{i}{2\pi}\mathcal{F}\{\} * G'' \ , \tag{33}$$

$$G''(\omega) = \frac{1}{i}\mathcal{F}\{G^{\text{odd}}\} = \frac{1}{2\pi i}\mathcal{F}\{\} * \mathcal{F}\{G^{\text{even}}\} = \frac{1}{2\pi i}\mathcal{F}\{\} * G' \ , \tag{34}$$

where $*$ denote convolution. The FT of the signum function is $\frac{2i}{\omega}$. Substituting this result into the above equations gives (26).