Linear elasticity I

In this TA session we'll dive deep into linear elasticity theory. Linear elasticity by itself can be the topic of a year-long course, and in the TA's we'll try to convey a significant fraction of the richness of the theory.

1 Hooke's law, stiffness, and compliance

Hooke's law is

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} = \left(\lambda + \frac{2}{3}\mu\right)\mathrm{tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\left(\varepsilon_{ij} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\delta_{ij}\right) . \tag{1}$$

Let's write it explicitly, to get a better feel of what's going on

$$\begin{aligned}
\sigma_{xx} &= 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = (2\mu+\lambda)\varepsilon_{xx} + \lambda\left(\varepsilon_{yy} + \varepsilon_{zz}\right) ,\\
\sigma_{yy} &= 2\mu\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = (2\mu+\lambda)\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{zz}\right) , \qquad (2)\\
\sigma_{zz} &= 2\mu\varepsilon_{zz} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = (2\mu+\lambda)\varepsilon_{zz} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) ,
\end{aligned}$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij}, \qquad i \neq j \tag{3}$$

or in matrix form

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{zy}\\ \sigma_{zx}\\ \sigma_{xy}\end{pmatrix} = \frac{2\mu}{1-2\nu}\begin{pmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{zy}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{pmatrix} . \tag{4}$$

The shear terms $i \neq j$ have a simple dependence, while the others are a bit more complicated. This equation has the general form of $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon}$, where $\boldsymbol{C}$ is called the stiffness tensor.

Let's try to invert these relations to find $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ - that is, let's find the compliance matrix for an isotropic linear elastic material. The same considerations that we used to derive Eq. (1) (i.e. that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are related by a 4-rank isotropic tensor) allow us to write

$$\varepsilon_{ij} = a\,\sigma_{kk}\delta_{ij} + b\,\sigma_{ij} , \tag{5}$$

so finding the compliance reduces to finding a, b. If $\mathrm{tr}\,\boldsymbol{\varepsilon} = 0$, then Eq.(1) and (5)[1] reduce to, respectively,

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} , \tag{6}$$

$$\varepsilon_{ij} = b\,\sigma_{ij} , \tag{7}$$

[1] Recall that $\mathrm{tr}\,\boldsymbol{\sigma} \propto \mathrm{tr}\,\boldsymbol{\varepsilon}$

so we immediately find $b=(2\mu)^{-1}$. Taking the trace of Eq. (1) and (5) gives, respectively

$$\mathrm{tr}(\boldsymbol{\sigma}) = (3\lambda+2\mu)\,\mathrm{tr}(\boldsymbol{\varepsilon}) \ , \tag{8}$$

$$\mathrm{tr}(\boldsymbol{\varepsilon}) = (3a+b)\,\mathrm{tr}(\boldsymbol{\sigma}) \ , \tag{9}$$

which tells us that

$$\begin{aligned} 3a+b=(3\lambda+2\mu)^{-1} \quad &\Rightarrow \quad a=\frac{1}{3}\left(\frac{1}{3\lambda+2\mu}-b\right)=\frac{-\lambda}{2\mu(3\lambda+2\mu)} \Rightarrow \\ \varepsilon_{ij} &= -\frac{\lambda}{2\mu(3\lambda+2\mu)}\,\mathrm{tr}(\boldsymbol{\sigma})\delta_{ij}+\frac{1}{2\mu}\sigma_{ij} \ . \end{aligned} \tag{10}$$

Writing explicitly, we have

$$\begin{aligned} \varepsilon_{xx} &= \left(\frac{1}{2\mu}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\right)\sigma_{xx}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\left(\sigma_{yy}+\sigma_{zz}\right) \ , \\ \varepsilon_{yy} &= \left(\frac{1}{2\mu}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\right)\sigma_{yy}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\left(\sigma_{xx}+\sigma_{zz}\right) \ , \\ \varepsilon_{zz} &= \left(\frac{1}{2\mu}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\right)\sigma_{zz}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\left(\sigma_{xx}+\sigma_{yy}\right) \ , \end{aligned} \tag{11}$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} \qquad \text{for } i\neq j \ . \tag{12}$$

As discussed in class, the term in parentheses in Eq.(11) is the inverse of the Young's modulus, and for uniaxial stress it reads

$$E=\sigma_{xx}/\varepsilon_{xx}=\left(\frac{1}{2\mu}-\frac{\lambda}{2\mu(3\lambda+2\mu)}\right)^{-1}=\mu\frac{3\lambda+2\mu}{\lambda+\mu} \ . \tag{13}$$

It is the microscopic analogue of the spring constant. If the uniaxial stress is in the x direction, then we have

$$\varepsilon_{yy}=-\frac{\lambda}{2\mu(3\lambda+2\mu)}\sigma_{xx}=-\frac{\lambda}{2\mu(3\lambda+2\mu)}E\varepsilon_{xx}=-\frac{\lambda}{2(\lambda+\mu)}\varepsilon_{xx} \ , \tag{14}$$

and as discussed in class, $-\varepsilon_{yy}/\varepsilon_{xx}$ is known as the Poisson ratio $\nu=\frac{\lambda}{2(\lambda+\mu)}$. Rewriting Eqs. (11) with these quantities yields a much nicer expression:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx}-\nu(\sigma_{yy}+\sigma_{zz})\right] \ , \\ \varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy}-\nu(\sigma_{xx}+\sigma_{zz})\right] \ , \\ \varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz}-\nu(\sigma_{xx}+\sigma_{yy})\right] \ , \\ \varepsilon_{ij} &= \frac{1+\nu}{E}\sigma_{ij} \quad \text{for} \quad i\neq j \ , \end{aligned} \tag{15}$$

or in matrix form

$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{zy}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{pmatrix}=\frac{1}{E}\begin{pmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{zy}\\ \sigma_{zx}\\ \sigma_{xy}\end{pmatrix} \ . \tag{16}$$

This is just an inversion of Eq. (4) into the form $\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}$, where $\boldsymbol{S} = \boldsymbol{C}^{-1}$ is the compliance tensor (if you noticed that $\boldsymbol{C}$ is called the stiffness tensor and $\boldsymbol{S}$ is called the compliance tensor and wondered about it, this is not a mistake and there is no intention to confuse you. It is a long-time convention that cannot be reverted anymore). A useful table with all the conversions is found in Wikipedia. We are now ready to perform the reduction to 2D.

2 2D elasticity

As shown in class, the field equation of elasticity is the Navier-Lamé equation

$$\rho\partial_{tt}\boldsymbol{u} = (\lambda+\mu)\nabla\left(\nabla\cdot\boldsymbol{u}\right) + \mu\nabla^2\boldsymbol{u} + \boldsymbol{b}\ . \tag{17}$$

It is notoriously difficult to solve. However, things become much simpler in 2D. There are 3 generic ways in which one can obtain an effectively 2D elastic system, by ignoring the z dimension:

1. **Plane stress**: when $\sigma_{zi} = 0$. This typically holds in very thin systems (in the z direction).

2. **Plane strain**: when the system is translationally invariant in z, and therefore ∂_z of anything vanishes. This typically holds in very thick (in the z direction) systems.

3. **Anti plane - scalar elasticity**: If the motion is only in z and does not depend on z. This is mainly a pedagogical example, though some physical examples exist, mainly thin sheets and mode-III fracture (tearing).

4. **"Flatland"**: If the world truly is two dimensional. We will not treat this case as it is a bit delicate.

The fourth case is a bit delicate, and we will not discuss it in the course. We'll now develop the theory for the first two cases, and Eran demonstrated in class the formalism of scalar elasticity (the third case). We will also see an example of scalar elasticity in the second part of the TA.

To see how one reduces elasticity to 2 dimensions, let us explicitly write Hooke's law (15) in terms of μ and ν. We note that although the stiffness matrix is a 4-rank tensor, it can be represented as a 6 by 6 matrix by rearranging the entries as in Eq. (4).

2.1 Plane-stress

We first consider objects that are thin in one dimension, say z, and are deformed in the xy-plane. What happens in the z-direction? Since the two planes $z = 0$ and $z = h$ (where h is the thickness which is much smaller than any other lengthscale in the problem) are traction-free, we approximate $\sigma_{zz} = 0$ everywhere (an approximation that becomes better and better as $h \to 0$). Similarly, we have $\sigma_{zy} = \sigma_{zx} = 0$. We can therefore set $\sigma_{zz} = \sigma_{zy} = \sigma_{zx} = 0$ in Eq. (16) to obtain

$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix} = \frac{1}{E}\begin{pmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 1+\nu\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix}\ , \tag{18}$$

and

$$\varepsilon_{zz} = -\frac{\nu}{E}\left(\sigma_{xx} + \sigma_{yy}\right) \ . \tag{19}$$

To obtain the plane-stress analog of Eq. (17), the Navier-Lamé equation, we need to invert Eq. (18), obtaining

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} = \frac{E}{1-\nu^2}\begin{pmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & 1-\nu\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix} \ . \tag{20}$$

Note that the last equation *can not be obtained* from Eq. (4) by simply removing columns and rows. We can now substitute the last relation in the 2D momentum balance equation $\nabla\cdot\boldsymbol{\sigma} = \rho\partial_{tt}\boldsymbol{u}$ (we stress again that $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are already 2D here). The resulting 2D equation reads

$$\left[\frac{\nu E}{1-\nu^2} + \mu\right]\nabla\left(\nabla\cdot\boldsymbol{u}\right) + \mu\nabla^2\boldsymbol{u} = \rho\partial_{tt}\boldsymbol{u} \ , \tag{21}$$

which is identical in form to Eq. (17) simply with a renormalized λ

$$\lambda \to \tilde{\lambda} = \frac{\nu E}{1-\nu^2} = \frac{2\nu\mu}{1-\nu} = \frac{2\lambda\mu}{\lambda+2\mu} \ . \tag{22}$$

The shear modulus μ remains unchanged

$$\tilde{\mu} = \mu = \frac{E}{2(1+\nu)} \ . \tag{23}$$

Finally, we can substitute $\sigma_{xx}(x,y)$ and $\sigma_{yy}(x,y)$ inside Eq. (19) to obtain $\varepsilon_{zz}(x,y)$. Note that $u_z(x,y,z) = \varepsilon_{zz}(x,y)z$ is linear in z.

2.2 Plane-strain

We now consider objects that are very thick in one dimension, say z, and are deformed in the xy-plane with no z dependence. These physical conditions are termed plane-strain and are characterized by $\varepsilon_{zx} = \varepsilon_{zy} = \varepsilon_{zz} = 0$. Eliminating these components from Eq. (4) we obtain

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} = \frac{2\mu}{1-2\nu}\begin{pmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & 1-2\nu\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix} \ , \tag{24}$$

and

$$\sigma_{zz}(x,y) = \frac{2\mu\nu}{1-2\nu}\left[\varepsilon_{xx}(x,y) + \varepsilon_{yy}(x,y)\right] \ . \tag{25}$$

We can now substitute Eq. (24) in the 2D momentum balance equation $\nabla\cdot\boldsymbol{\sigma} = \rho\partial_{tt}\boldsymbol{u}$ (where again $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are 2D). The resulting 2D equation is identical to Eq. (17), both in form and in the elastic constants. With the solution at hand, we can use Eq. (25) to calculate $\sigma_{zz}(x,y)$. Finally, we note that Eq. (24) can be inverted to

$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{xy}\end{pmatrix} = \frac{1}{2\mu}\begin{pmatrix}1-\nu & -\nu & 0\\ -\nu & 1-\nu & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{xy}\end{pmatrix} \ , \tag{26}$$

which can not be simply obtained from Eq. (16) by eliminating columns and rows. Using the last relation we can rewrite Eq. (25) as

$$\sigma_{zz}(x,y) = \nu\left[\sigma_{xx}(x,y) + \sigma_{yy}(x,y)\right] \ . \tag{27}$$

In summary, we see that in both plane-stress and plane-strain cases we can work with 2D objects instead of their 3D counterparts, which is a significant simplification. This allows to use the heavy mathematical tools available in 2D: complex analysis and conformal mapping. One has, though, to be careful with the elastic constants as explained above.

3 Complex representation of scalar elasticity

We study a case of scalar elasticity, where $\boldsymbol{u} = u_{x_3}(x_1,x_2)\boldsymbol{e}_{x_3}$. The strains are

$$\varepsilon_{x_2,x_3} = \frac{1}{2}(\partial_{x_2}u_{x_3} + \partial_{x_3}u_{x_2}) = \frac{1}{2}\partial_{x_2}u_{x_3} \ , \tag{28}$$

$$\varepsilon_{x_1,x_3} = \frac{1}{2}(\partial_{x_1}u_{x_3} + \partial_{x_3}u_{x_1}) = \frac{1}{2}\partial_{x_1}u_{x_3} \ , \tag{29}$$

$$\varepsilon_{x_1,x_1} = \varepsilon_{x_2,x_2} = \varepsilon_{x_3,x_3} = \varepsilon_{x_1,x_2} = 0 \ . \tag{30}$$

We have seen in class that $\nabla^2 u_{x_3} = 0$, that is, u_{x_3} is a harmonic function. This means that we can write u_{x_3} as

$$u_{x_3} = 2\Re(\phi) = \phi(z) + \overline{\phi(z)}, \qquad z = x_1 + ix_2 \ , \tag{31}$$

where ϕ is an analytic complex function. We will use the Cauchy-Riemman equations, that tell us that

$$\partial_{x_1}\phi = -i\partial_{x_2}\phi = \phi' \ , \tag{32}$$

$$\partial_{x_1}\overline{\phi} = \overline{\partial_{x_1}\phi} = i\partial_{x_2}\overline{\phi} = \overline{\phi}' \ , \tag{33}$$

and therefore the stresses are

$$\begin{aligned}
\sigma_{x_1,x_3} &= \mu\partial_{x_1}u_{x_3} = \mu\left(\partial_{x_1}\phi + \partial_{x_1}\overline{\phi}\right) = \mu\left(\phi' + \overline{\phi}'\right) = 2\mu\Re(\phi') \ , \\
\sigma_{x_2,x_3} &= \mu\partial_{x_2}u_{x_3} = \mu\left(\partial_{x_2}\phi + \partial_{x_2}\overline{\phi}\right) = i\mu\left(\phi' - \overline{\phi}'\right) = -2\mu\Im(\phi') \ , \\
\Rightarrow \qquad & 2\mu\phi' = \sigma_{x_1,x_3} - i\sigma_{x_2,x_3} \ ,
\end{aligned} \tag{34}$$

and all other components vanish.

If our domain contains a free boundary, given by a curve that is parameterized by $x_1(s), x_2(s)$ with s being arc-length parametrization, then the normal to the boundary is given by $\boldsymbol{n} = (\partial_s x_2, -\partial_s x_1)$. On the boundary we thus have

$$\begin{aligned}
0 &= \sigma_{x_3,x_1}n_{x_1} + \sigma_{x_3,x_2}n_{x_2} \\
&= \mu\left[\left(\partial_{x_1}\phi + \partial_{x_1}\overline{\phi}\right)\partial_s x_2 - \left(\partial_{x_2}\phi + \partial_{x_2}\overline{\phi}\right)\partial_s x_1\right] \\
&= \mu\left[\left(-i\partial_{x_2}\phi + i\partial_{x_2}\overline{\phi}\right)\partial_s x_2 - \left(i\partial_{x_1}\phi - i\partial_{x_1}\overline{\phi}\right)\partial_s x_1\right] \\
&= -i\mu\left[\left(\frac{\partial\phi}{\partial x_2}\frac{\partial x_2}{\partial s} + \frac{\partial\phi}{\partial x_1}\frac{\partial x_1}{\partial s}\right) - \left(\frac{\partial\overline{\phi}}{\partial x_2}\frac{\partial x_2}{\partial s} + \frac{\partial\overline{\phi}}{\partial x_1}\frac{\partial x_1}{\partial s}\right)\right] \\
&= -\mu\left(\frac{\partial\phi}{\partial s} - \frac{\partial\overline{\phi}}{\partial s}\right) = 2\mu\frac{\partial\Im(\phi)}{\partial s} \ ,
\end{aligned} \tag{35}$$

so on the boundary $\Im(\phi)$ is constant. Since ϕ is only given up to an additive constant, we can choose $\Im(\phi) = 0$, or, in other words, $\phi = \overline{\phi}$ on the boundary. We see that solving for the displacement field is equivalent to finding an analytic function whose imaginary part is constant on the boundary.

4 Conformal mapping: Inglis crack
(Refernce: Marder & Fineberg, Physics Reports 1999)

Complex treatment of 2D elasticity is very useful because Laplace's equation is conformally invariant, so one can use conformal mappings to deform the region over which we need to solve the equation into a more convenient geometry. Here we'll see an application of this method, which is called the Inglis (mode III) problem. In 1913 Charles Inglis solved the general problem of an elliptic hole in an infinite plate subject to distant loading. His solution turned out to be one of the cornerstones of fracture mechanics, and was later used and generalized by the works of Griffith, Irwin, and others.

So let's look at an infinite plane with an elliptic hole, subject to antiplane shear $\sigma_{x_2,x_3} = \sigma_\infty$ at $x_2 \to \pm\infty$. As working with ellipses is unpleasant, we want to find a conformal mapping that maps the region outside the ellipse to a region outside a circle. Luckily, such a mapping is well known, and is given by

$$z = f(\omega) = R\left(\omega + \frac{\rho}{\omega}\right) \ , \tag{36}$$

$$\omega = f^{-1}(z) = \frac{z}{2R} + \sqrt{\left(\frac{z}{2R}\right)^2 - \rho} \ . \tag{37}$$

f maps the unit circle in the ω-plane to an ellipse with axes $R(1 \pm \rho)$ in the z-plane. $0 \le \rho \le 1$ is a parameter that measures the ellipse's eccentricity[2] - when $\rho = 0$ the ellipse is a circle, while for $\rho = 1$ it is a 1D crack of length $4R$. The conformal mapping is shown in Fig. (1).

The crux of the conformal mapping technique is that while in the real coordinates the geometry is elliptic (and thus complicated), in the ω-plane the domain is a circle (simple!), and therefore we want to reformulate the problem in terms of ω. That is, we want to describe ϕ as a function of ω, by the mapping $\phi(\omega) = \phi(\omega(z))$.

On the hole's boundary, which is the unit circle in ω-plane, we have

$$\phi(\omega) = \overline{\phi(\omega)} = \overline{\phi}(\overline{\omega}) = \overline{\phi}(1/\omega) \ , \tag{38}$$

because on the unit circle $\overline{\omega} = 1/\omega$. The property (38) can be analytically extended to all the ω-plane.

What are the singularities of ϕ? Outside the hole, it must be completely regular, except at infinity where it diverges as $\phi \sim z$. This is because Eq. (34) tells us that far from the hole we have $\partial_z \phi \propto \sigma/\mu$, and therefore we conclude that

$$\phi \approx -i\frac{\sigma_\infty}{\mu} z \approx -i\frac{\sigma_\infty}{\mu} R\omega, \qquad \text{for } \omega, z \to \infty \ . \tag{39}$$

[2] Note that ρ isn't the eccentricity as usually defined in geometry, which is $e = \frac{2\sqrt{\rho}}{\rho+1}$.

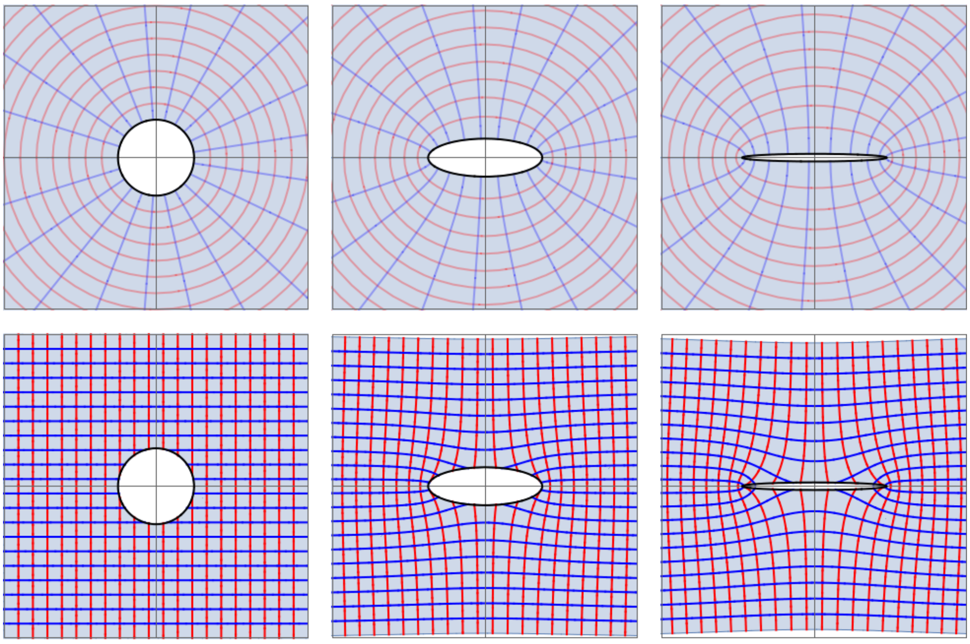

Figure 1: The conformal mapping. Polar lines (top row) and the Cartesian lines (bottom row) are shown. Note that, after the mapping, the lines remain perpendicular.

Using the analytical continuation of Eq. (38), we get that

$$\overline{\phi}(1/\omega) \approx -i\frac{\sigma_\infty}{\mu}R\omega, \qquad \text{for } \omega \to \infty \ , \tag{40}$$

or equivalently,

$$\phi(\omega) \approx i\frac{\sigma_\infty R}{\mu\omega}, \qquad \text{for } \omega \to 0 \ , \tag{41}$$

and there are no other singularities inside the unit circle. Having determined all the possible singularities of ϕ, it is determined up to an additive constant. It must be

$$\phi(\omega) = i\frac{\sigma_\infty R}{\mu}\left(\frac{1}{\omega} - \omega\right) \ . \tag{42}$$

As discussed above, another way of finding ϕ is to find a function whose imaginary part vanishes on the boundary on the hole, i.e. on the unit circle. The function $i\,(1/\omega - \omega)$ fits this requirement, therefore, it is exactly the function we're looking for, up to a multiplicative factor which we have obtained from the external BC.

We can now calculate the displacement field in the "real" coordinate z by joining Eqs. (42) and (37):

$$u_z = 2\Re\left\{-i\frac{\sigma_\infty R}{\mu}\left(\zeta + \sqrt{\zeta^2 - \rho} - \frac{1}{\zeta + \sqrt{\zeta^2 - \rho}}\right)\right\} \ , \qquad \text{where } \zeta \equiv \tfrac{z}{2R} \ . \tag{43}$$

What is the stress at the tip of the ellipse? We can differentiate $u_z(z)$ of Eq. (43) explicitly, but this gives a nasty expression that is very difficult to understand. It is simpler to use the conformal property of the mapping:

$$\begin{aligned}
\partial_z \phi(z) &= \partial_z \phi(\omega(z)) = \phi'(\omega)\frac{\partial \omega}{\partial z} \ , \\
\phi'(\omega) &= -i\frac{\sigma_\infty R}{\mu}\left(1+\frac{1}{\omega^2}\right) \ , \\
\frac{\partial \omega}{\partial z} &= \left(\frac{\partial z}{\partial w}\right)^{-1} = \frac{1}{f'(\omega)} \ , \\
f'(\omega) &= R(1-\frac{\rho}{\omega^2}) \ , \\
\phi'(z) &= \frac{-i\frac{\sigma_\infty R}{\mu}\left(1+\frac{1}{\omega^2}\right)}{R(1-\frac{\rho}{\omega^2})} = -\frac{i\sigma_\infty}{\mu}\frac{\omega(z)^2+1}{\omega(z)^2-\rho} \ .
\end{aligned} \tag{44}$$

Note that in the last equation ω is a function of z.

Now let's examine the solution. One thing we would like to know is where in space is the stress maximal. Clearly, ϕ' diverges for $w = \pm\sqrt{\rho}$, but remember that $\rho < 1$ and our domain is outside the unit circle, so this point is inside the hole. Some trivial algebra shows that the ϕ' is maximal for $\omega = \pm 1$, which are, not surprisingly, the closest points outside the unit circle to $\pm\rho$. When $\omega = \pm 1$ we have $z = \pm R(1+\rho)$ - these are the horizontal tips of the ellipse. The stresses there are

$$\begin{aligned}
\sigma_{x_1,x_3} - i\sigma_{x_2,x_3} &= \mu\phi' = -\sigma_\infty\frac{2i}{1-\rho} \Rightarrow \\
\sigma_{x_1,x_3} = 0 \qquad , \qquad \sigma_{x_2,x_3} &= \frac{2\sigma_\infty}{1-\rho} \ .
\end{aligned} \tag{45}$$

The case $\rho = 0$ gives $\sigma_{x_2,x_3} = 2\sigma_\infty$, in accordance with what was done in class. In the opposite extremity, $\rho \to 1$, the stress field diverges (but the displacement doesn't). We see that the stress at the tip decreases with the radius there. An interesting consequence of this is that in order to arrest a crack from propagating, one can drill a hole at its tip (!). This will reduce the radius of curvature at the tip and weaken the singularity.

The limiting case $\rho \to 1$ is of particular interest, as it describes a 1-dimensional cut in the material. It is known in the literature as Mode III crack. The power with which σ_{x_3,x_2} diverges in the case $\rho = 1$ can be easily obtained. In this case we have

$$\phi = -\frac{iR\sigma_\infty}{\mu}\sqrt{\frac{z^2}{R^2}-4} \ . \tag{46}$$

Plugging in $z = 2R\,(1+\delta)$ (where $\delta \in \mathbb{C}$) and keeping the leading order in δ gives

$$\begin{aligned}
\phi &= -i\frac{2\sqrt{2}R\sigma_\infty}{\mu}\sqrt{\delta} + O\left(\delta^{3/2}\right) \Rightarrow \\
\sigma_{x_2,x_3} &\sim \frac{\sigma_\infty}{\sqrt{2}\sqrt{\delta}} \ .
\end{aligned} \tag{47}$$

The fact that near the crack tip the stress field diverges as the square root of the distance from the crack tip, and that the displacement field is regular, is of general applicability, and is true for static cracks in all loading configurations. The square-root divergence is a consequence of the branch-cut at the crack surface.