Linear elasticity II

1 Green's function for an infinite medium

It seems that the time is ripe to fully and completely solve a problem, with all the 2π's and everything, without resorting to hand waving and scaling arguments. While the emphasis will still be on the structure of the problem, we think it will be instructive, at least once, to write down a problem and solve it exactly.

A nice problem to consider is the response of an infinite linear isotropic homogeneous elastic medium to a localized force $\vec{f} = F_i\delta(\vec{r})$, i.e. finding the Green's function of an infinite medium.

We define the Green function (matrix) $G_{ij}(\vec{r}_1, \vec{r}_2)$ as the displacement in the i direction at the point $\vec{r}_1$ as a response to a localized force in the j direction applied at $\vec{r}_2$. For homogeneous materials we know that $G_{ij}(\vec{r}_1, \vec{r}_2) = G_{ij}(\vec{r}_1 - \vec{r}_2)$. We therefore denote $\vec{r} = \vec{r}_1 - \vec{r}_2$. You all know well that, within the linear elastic theory, this will allow us to solve the problem of an arbitrary force distribution $f(\vec{r})$ by convolving $f(\vec{r})$ with the Green function.

Conceptually, the structure is the following. We would like to find a displacement field $u_i(\vec{r})$, from which we calculate

$$u_i \quad \Rightarrow \quad \varepsilon_{ij} \quad \Rightarrow \quad \sigma_{ij} \quad \Rightarrow \quad \partial_j\sigma_{ij} = \delta(\vec{r}) \ ,$$

but of course, we will want to do the whole thing backwards. I stress (no pun intended[1]) that we already know how to express $\text{div}(\boldsymbol{\sigma})$ in terms of u_i - this is what we called the Navier-Lamé equation. But for completeness, let's do it again:

$$\sigma_{ij} = \lambda\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij} = \lambda\partial_k u_k\delta_{ij} + \mu\left(\partial_i u_j + \partial_j u_i\right) \ ,$$
$$\partial_j\sigma_{ij} = \partial_j\left(\lambda\partial_k u_k\delta_{ij} + \mu\left(\partial_i u_j + \partial_j u_i\right)\right) = (\lambda+\mu)\partial_i\partial_j u_j + \mu\partial_j\partial_j u_i \ ,$$

which is nothing but the u-dependent term of the Navier-Lamé equation. Since the equation is linear, it seems right to solve the problem by Fourier transform. We use the conventions

$$u_i(\vec{q}) = \int d^3\vec{x}\, e^{i\vec{q}\cdot\vec{r}} u_i(\vec{r}) \ , \tag{1}$$
$$u_i(\vec{r}) = \frac{1}{(2\pi)^3}\int d^3\vec{q}\, e^{-i\vec{q}\cdot\vec{r}} u_i(\vec{q}) \ . \tag{2}$$

The equation we want to transform is

$$(\lambda+\mu)\partial_j\partial_i u_j + \mu\partial_j\partial_j u_i = -F_i\delta(\vec{x}) \ , \tag{3}$$

which readily gives

$$-(\lambda+\mu)q_j q_i\, u_j - \mu\, q_j q_j\, u_i = -F_i \ . \tag{4}$$

[1] Just kidding, of course it's intended.

This is a matrix equation:

$$[(\lambda+\mu)q_jq_i + \mu\, q_kq_k\delta_{ij}]\, u_j = F_i \ . \tag{5}$$

Or even more explicitly:

$$\begin{pmatrix} (\lambda+\mu)\,q_1^2+\mu|\vec{q}|^2 & (\lambda+\mu)\,q_1q_2 & (\lambda+\mu)\,q_1q_3 \\ (\lambda+\mu)\,q_1q_2 & (\lambda+\mu)\,q_2^2+\mu|\vec{q}|^2 & (\lambda+\mu)\,q_2q_3 \\ (\lambda+\mu)\,q_1q_3 & (\lambda+\mu)\,q_2q_3 & (\lambda+\mu)\,q_3^2+\mu|\vec{q}|^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \ . \tag{6}$$

This matrix can be inverted by using any of your favorite methods, giving

$$u_i = \frac{1}{\mu}\left[\frac{\delta_{ij}}{q_kq_k} - \frac{1}{2(1-\nu)}\frac{q_iq_j}{(q_kq_k)^2}\right] F_j \ . \tag{7}$$

Where we used $\nu = \frac{\lambda}{2(\lambda+\mu)}$. In other words, we have found the Fourier representation of the Green function:

$$G_{ij}(\vec{q}) = \frac{1}{\mu}\left[\frac{\delta_{ij}}{q_kq_k} - \frac{1}{2(1-\nu)}\frac{q_iq_j}{(q_kq_k)^2}\right] \ . \tag{8}$$

We now need to preform the inverse Fourier transform. We'll begin with the first term, and do it in spherical coordinates with the q_z direction parallel to $\vec{r}$:

$$\begin{aligned} \frac{1}{(2\pi)^3}\int d^3\vec{q}\,\frac{e^{-i\vec{q}\cdot\vec{x}}}{q^2} &= \frac{1}{(2\pi)^3}\int \frac{e^{-iqr\cos\theta}}{q^2} q^2\sin\theta\, d\theta\, dq\, d\phi = \frac{1}{(2\pi)^2}\int e^{-iqr\cos\theta}\sin\theta\, d\theta\, dq \\ &= \frac{1}{(2\pi)^2}\int \frac{e^{iqr}-e^{-iqr}}{iqr}dq = \frac{2}{(2\pi)^2}\int_0^\infty \frac{\sin(qr)}{qr}dq = \frac{1}{4\pi r} \ . \end{aligned}$$

If this result surprises you, maybe you should remind yourself of the first linear field theory that you met in your life - Poisson's equation for a point charge $\nabla^2\phi = \delta(\vec{r})$. I'll let you complete the analogy by yourselves.

For the second term, we use a dirty trick:

$$\begin{aligned} \mathcal{F}^{-1}\left\{\frac{q_iq_j}{(q_kq_k)^2}\right\} &= -\frac{1}{2}\mathcal{F}^{-1}\left\{q_i\,\frac{\partial}{\partial q_j}\left(\frac{1}{q_kq_k}\right)\right\} = \frac{i}{2}\frac{\partial}{\partial x_i}\mathcal{F}^{-1}\left\{\frac{\partial}{\partial q_j}\left(\frac{1}{q_kq_k}\right)\right\} \\ &= \frac{i}{2}\frac{\partial}{\partial x_i}\left(-ix_j\mathcal{F}^{-1}\left\{\frac{1}{q_kq_k}\right\}\right) = \frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{x_j}{4\pi r}\right) \\ &= \frac{1}{2}\left(\frac{\delta_{ij}}{4\pi r} - \frac{x_ix_j}{4\pi r^3}\right) \ . \end{aligned} \tag{9}$$

Plugging in (8) we get

$$G_{ij}(\vec{r}) = \frac{1}{16(1-\nu)\pi\mu r}\left[(3-4\nu)\delta_{ij} + \frac{x_ix_j}{r^2}\right] \ . \tag{10}$$

A more elegant way to go, is to write G_{ij} as gradients of r (I mean $|r|$, not $\vec{r}$):

$$G_{ij}(\vec{r}) = \frac{1}{8\pi\mu}\left[\partial_k\partial_k r\,\delta_{ij} - \frac{1}{2(1-\nu)}\partial_i\partial_j r\right] = \frac{1}{8\pi\mu}\left[\boldsymbol{I}\,\nabla^2 r - \frac{\nabla\nabla r}{2(1-\nu)}\right] \ . \tag{11}$$

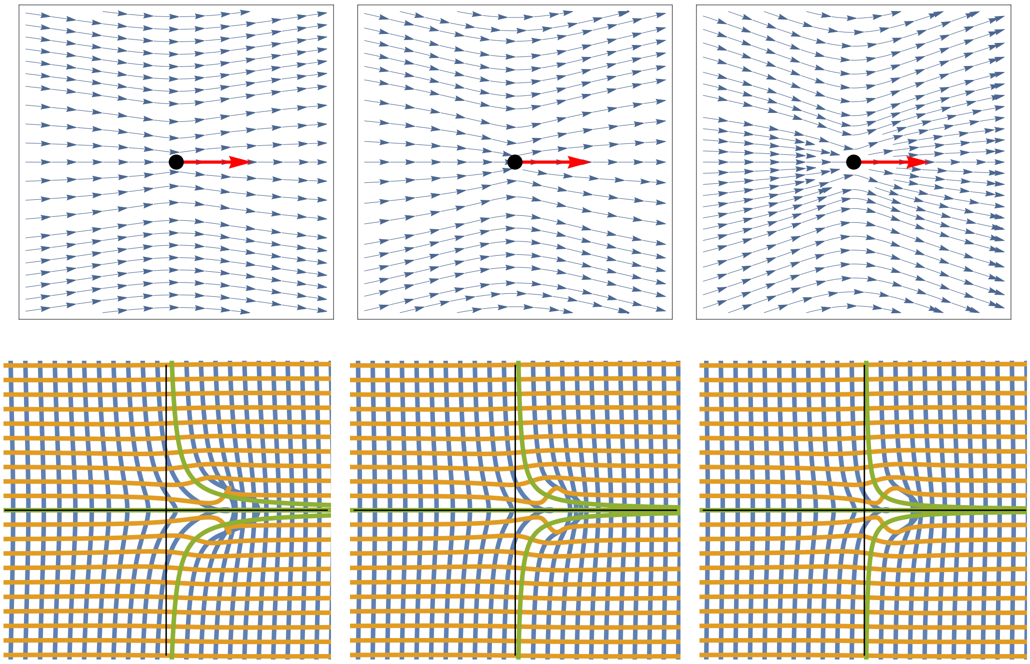

Figure 1: Top: Displacement field lines in the $x-y$ plane for point $\vec{F} = F\hat{x}$ in the horizontal direction. From left to right, with $\nu = 0,\ 0.33,\ 0.5$. Bottom: deformation of a regular mesh under this motion. Note that crossing of two lines of the same color is physically forbidden (why?).

1.1 Notes about the solution

1. The displacements go as $1/r$, which means that the strain/stress go as $1/r^2$. Therefore the elastic energy density, which goes like ε^2, goes like $1/r^4$ and *its integral diverges*. This is much like the case of electrostatics, where the total energy of the electrical field of a point charge diverges.

2. The scaling $u \sim 1/r$ could also have been obtained from simple dimensional analysis. It is quite common that dimensional considerations in elasticity take the "dimension" from the shear modulus μ, and then there's an unknown (and usually uninteresting) function of ν.

3. You might have noticed that G_{ij} is a symmetric matrix. This might look at first glance as a trivial property that stems from the translational symmetry or rotational symmetry (=isotropy), but this is not the case. This symmetry property does not stem from any simple argument (that I can think of). Instead, this symmetry is a special case of a more general property that is called *reciprocity*. For a general linear elastic solid, and by general I mean that $C_{ijkl}(\boldsymbol{r})$ can have any symmetry and can even depend on space, the static Green function satisfies

$$G_{ij}(\boldsymbol{r}, \boldsymbol{r}') = G_{ji}(\boldsymbol{r}', \boldsymbol{r}) \ . \tag{12}$$

2 Clapeyron's & Betti's theorems and reciprocity

Clapeyron's theorem is a nice result in static elasticity that says that "the total elastic energy stored in a body is equal to half the work done by the external forces computed assuming these forces had remained constant from the initial state to the final state". You already know a trivial version of this theorem, in the context of the energy stored in a simple spring. Assume you load a given spring with a force F. The energy stored in the spring is

$$\mathcal{U} = \tfrac{1}{2}kx^2 = \tfrac{1}{2}(kx)x = \tfrac{1}{2}Fx \ . \tag{13}$$

If you assume that during the elongation of the spring the force had the constant value F (although the force clearly started from zero and ramped up to F) you get that the total work done was Fx. The real energy, however, is exactly one half that value.

The proof is pretty simple. Assume an elastic body is subject to volume forces $\boldsymbol{b}(\boldsymbol{r})$ and surface tractions $\boldsymbol{t}(\boldsymbol{r})$. The total elastic energy in the body is

$$\begin{aligned}\mathcal{U} &= \int_\Omega \tfrac{1}{2}\sigma_{ij}(\boldsymbol{r})\varepsilon_{ij}(\boldsymbol{r})d^3\boldsymbol{r} \overset{(1)}{=} \tfrac{1}{2}\int_\Omega \sigma_{ij}\partial_i u_j \, d^3\boldsymbol{r} = \tfrac{1}{2}\int_\Omega \Big[\partial_i(\sigma_{ij}u_j) - u_j\partial_i\sigma_{ij}\Big] d^3\boldsymbol{r} \\ &\overset{(2)}{=} \tfrac{1}{2}\left[\int_{\partial\Omega} \sigma_{ij}u_j n_i \, d^2\boldsymbol{r} + \int_\Omega b_j u_j \, d^3\boldsymbol{r}\right] = \tfrac{1}{2}\left[\int_{\partial\Omega} t_j u_j \, d^2\boldsymbol{r} + \int_\Omega b_j u_j \, d^3\boldsymbol{r}\right] \ ,\end{aligned} \tag{14}$$

where in the transition (1) we used the symmetry of $\boldsymbol{\sigma}$ and in (2) we used Gauss' theorem and the equilibrium condition $\partial_j\sigma_{ij} + b_i = 0$.

Betti's theorem (sometimes called "Betti's reciprocal theorem") is a general important result about the energetics in static elasticity. It is useful in theoretical analyses (e.g. for proving the uniqueness of solutions to the static Navier-Lamè equation) and is also for practical use if average quantities are calculated (you might have an exercise about that in the HW).

Suppose that when a set of body and traction forces $\boldsymbol{b}^{(1)}$ and $\boldsymbol{t}^{(1)}$ is applied to a body, the resulting deformation field is $\boldsymbol{u}^{(1)}$, and the stress and strain fields are $\boldsymbol{\sigma}^{(1)}$ and $\boldsymbol{\varepsilon}^{(1)}$. Suppose also that when a different set of body and traction forces, $\boldsymbol{b}^{(2)}$ and $\boldsymbol{t}^{(2)}$, is applied then the resulting deformation field is $\boldsymbol{u}^{(2)}$, and the stress and strain fields are $\boldsymbol{\sigma}^{(2)}$ and $\boldsymbol{\varepsilon}^{(2)}$. Betti's theorem states that

$$\int_{\partial\Omega} t_i^{(1)}u_i^{(2)}d^2\boldsymbol{r} + \int_\Omega b_i^{(1)}u_i^{(2)}d^3\boldsymbol{r} = \int_{\partial\Omega} t_i^{(2)}u_i^{(1)}d^2\boldsymbol{r} + \int_\Omega b_i^{(2)}u_i^{(1)}d^3\boldsymbol{r} \ . \tag{15}$$

That is, the work done by the set (1) through the displacements produced by the set (2) is equal to the work done by the set (2) through the displacements produced by the set (1).

To see this, consider the total energy of the system if both sets of forces are applied:

$$\mathcal{U} = \frac{1}{2}\int_\Omega \left(\sigma_{ij}^{(1)} + \sigma_{ij}^{(2)}\right)\left(\varepsilon_{ij}^{(1)} + \varepsilon_{ij}^{(2)}\right) d^3\boldsymbol{r} \tag{16}$$

$$= \frac{1}{2}\int_\Omega \left[\sigma_{ij}^{(1)}\varepsilon_{ij}^{(1)} + \sigma_{ij}^{(2)}\varepsilon_{ij}^{(2)} + \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)} + \sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)}\right] d^3\boldsymbol{r} \tag{17}$$

$$= \mathcal{U}^{(1)} + \mathcal{U}^{(2)} + \frac{1}{2}\int_\Omega \Big[\underbrace{\sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)}}_{A} + \underbrace{\sigma_{ij}^{(2)}\varepsilon_{ij}^{(1)}}_{B}\Big] d^3\boldsymbol{r} \ . \tag{18}$$

The first two terms are the energies of each of the "pure modes" and the integral is the "interaction energy". Using exactly the same arguments as we used in Eq. (14) it is possible (and easy) to show that term A equals the left-hand-side of Eq. (15) and term B equals the right-hand-side. However, terms A and B are equal since

$$A = \sigma_{ij}^{(1)}\varepsilon_{ij}^{(2)} = C_{ijkl}\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)} = C_{klij}\varepsilon_{kl}^{(1)}\varepsilon_{ij}^{(2)} = \varepsilon_{kl}^{(1)}\sigma_{kl}^{(2)} = B \ , \tag{19}$$

where the symmetry $C_{ijkl} = C_{klij}$ was used, and the theorem is proven. Note that we did not invoke isotropy or translational invariance. The symmetry $C_{ijkl} = C_{klij}$ is a thermodynamic one that holds for the most general linear elastic case.

With Betti's theorem it is very easy to prove the reciprocity property, Eq. (12). If the set (1) is a point force $\boldsymbol{F}^{(1)}$ applied at $\boldsymbol{r}^{(1)}$ and set (2) is a point force $\boldsymbol{F}^{(2)}$ applied at $\boldsymbol{r}^{(2)}$ then

$$\begin{aligned} b_i^{(1)} &= F_i^{(1)}\delta(\boldsymbol{r}-\boldsymbol{r}^{(1)}) & \qquad b_i^{(2)} &= F_i^{(2)}\delta(\boldsymbol{r}-\boldsymbol{r}^{(2)}) \\ u_i^{(1)}(\boldsymbol{r}) &= G_{ij}(\boldsymbol{r},\boldsymbol{r}^{(1)})F_j^{(1)} & \qquad u_i^{(2)}(\boldsymbol{r}) &= G_{ij}(\boldsymbol{r},\boldsymbol{r}^{(2)})F_j^{(2)} \ . \end{aligned} \tag{20}$$

Homogeneous boundary conditions on $\partial\Omega$ mean either $\boldsymbol{u} = 0$ or $\boldsymbol{t} = 0$, or that some portions of the surface is subject to that and other portions to the other. In any case, this means that the boundary integrals in Eq. (15) vanish identically. Then, using the properties of the delta function, we immediately get

$$F_i^{(1)}F_j^{(2)}G_{ij}(\boldsymbol{r}^{(1)},\boldsymbol{r}^{(2)}) = F_i^{(2)}F_j^{(1)}G_{ij}(\boldsymbol{r}^{(2)},\boldsymbol{r}^{(1)}) \ , \tag{21}$$

and since $\boldsymbol{F}_1$, $\boldsymbol{F}_2$, $\boldsymbol{r}^{(1)}$ and $\boldsymbol{r}^{(2)}$ were arbitrary the reciprocal theorem of Eq. (12) is obtained.

3 Equations of elasticity as a consequence of minimizing an action

In this section we will show how the equations of linear elasticity arise from the minimization of an action S through the Euler-Lagrange equations. As we all know, the size to be minimized is the *action*, which is the time integral of the *Lagrangian*

$$S = \int_{t_0}^{t_f} L\,dt \ , \tag{22}$$

where $L = T - U$ is the difference between the kinetic and potential energies.

In a continuous system we know that the kinetic and potential energies are

$$T = \int_\Omega \frac{\rho\dot{u}^2}{2}d^3x = \int_\Omega \frac{\rho}{2}\dot{u}_j\dot{u}_j d^3x \ , \tag{23}$$

and

$$U = \int_\Omega \frac{1}{2}C_{ijkl}\varepsilon_{ij}\varepsilon_{kl}d^3x = \int_\Omega \frac{1}{2}C_{ijkl}\partial_i u_j\partial_k u_l d^3x \ , \tag{24}$$

where in the last transition we used the symmetry of $\boldsymbol{C}$. Together, we see that we can write down L as a spatial integral over a *Lagrangian density* $\mathcal{L}$

$$\mathcal{L} = \frac{\rho}{2}\partial_t u_j \partial_t u_j - \frac{1}{2} C_{ijkl} \partial_i u_j \partial_k u_l \ , \tag{25}$$

and the dynamical equations are the *Euler-Lagrange* equations,

$$\frac{\partial \mathcal{L}}{\partial u_\alpha} = \partial_t \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_t u_\alpha \right)} \right) + \partial_\beta \frac{\partial \mathcal{L}}{\partial \left(\partial_\beta u_\alpha \right)} \ . \tag{26}$$

Let's look at Eq. (26) term by term:

1. $\frac{\partial \mathcal{L}}{\partial u_\alpha}$ - As $\mathcal{L}$ does not depend on u but just on its derivatives, this term is zero.

2. $\frac{\partial \mathcal{L}}{\partial(\partial_t u_\alpha)} = \frac{1}{2}\rho \frac{\partial_t u_j \partial_t u_j}{\partial(\partial_t u_\alpha)} = \rho \partial_t u_j \frac{\partial(\partial_t u_j)}{\partial(\partial_t u_\alpha)} = \rho \partial_t u_j \delta_{\alpha j} = \rho \partial_t u_\alpha$. Therefore $\partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t u_\alpha)} \right) = \rho \partial_{tt} u_\alpha$.

3. $\partial_\beta \frac{\partial \mathcal{L}}{\partial \left(\partial_\beta u_\alpha \right)}$ is a bit more delicate, due to the many more indices, so I'll do it carefully

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \left(\partial_\beta u_\alpha \right)} = & -\frac{1}{2} C_{ijkl} \left(\delta_{i\beta} \delta_{j\alpha} \partial_k u_l + \delta_{k\beta} \delta_{l\alpha} \partial_i u_j \right) = \\ & -\frac{1}{2} \left(C_{ij\beta\alpha} \partial_i u_j + C_{\beta\alpha kl} \partial_k u_l \right) = -C_{ij\beta\alpha} \partial_i u_j \ , \end{aligned} \tag{27}$$

   where we have used all the symmetries of $\boldsymbol{C}$. Taking another divergence we find

$$\partial_\beta \frac{\partial \mathcal{L}}{\partial \left(\partial_\beta u_\alpha \right)} = -C_{ij\beta\alpha} \partial_\beta \partial_i u_j \ . \tag{28}$$

Putting it all together, we find the equations of motion are

$$\rho \partial_{tt} u_\alpha = C_{ij\beta\alpha} \partial_\beta \partial_i u_j \ , \tag{29}$$

which are correct for any homogeneous linear elastic material. By setting the form for an isotropic material,

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl}) \ , \tag{30}$$

one recovers the standard Navier-Lamé equation.

Possible generalization to this derivation are:

- An non homogeneous material can also be considered. The derivation of the equations should be along the same line, except for the fact that we can't interchange ∂ and $\boldsymbol{C}$ as we have done, so the equations will be much more complicated.

- Non-linear elasticity. In general it is doable, but much more difficult. Since we want the domain of integration to stay the same for all times, you'd better try performing the calculation in the Lagrangian[2] coordinates, and write the potential U in terms of the lagrangian displacement.

[2] It's annoying having to use the same word to describe different things, I know. What can you do.