Linear elasticity III and Thermo elasticity

1 Elastic waves

I remind you that you have shown in class that the Navier-Lamé Equation,

$$(\lambda+\mu)\nabla(\nabla\cdot\boldsymbol{u})+\mu\nabla^2\boldsymbol{u}+\boldsymbol{b}=\rho\partial_{tt}\boldsymbol{u}\ , \tag{1}$$

is basically two uncoupled wave equations for dilatational and shear waves. They propagate at velocities

$$c_s=\sqrt{\frac{\mu}{\rho}}\ , \qquad\qquad c_d=\sqrt{\frac{\lambda+2\mu}{\rho}}\ . \tag{2}$$

Thus, Eq. (1) can also be written as

$$(c_d^2-c_s^2)\nabla(\nabla\cdot\boldsymbol{u})+c_s^2\nabla^2\boldsymbol{u}+\boldsymbol{b}/\rho=\partial_{tt}\boldsymbol{u}\ . \tag{3}$$

The two wave speeds differ by a significant factor. c_s is always smaller than c_d and their ratio is

$$\beta\equiv\frac{c_s}{c_d}=\sqrt{\frac{\mu}{\lambda+2\mu}}=\sqrt{\frac{1-2\nu}{2(1-\nu)}}\ . \tag{4}$$

For a typical value of $\nu=1/3$, this gives a ratio of $\frac{1}{2}$. This function is plotted in Fig. 3. Note that the ratio goes to 0 for $\nu\to\frac{1}{2}$. This is because incompressible materials ($\nu=1/2$) the dilatational velocity c_d diverges (as the bulk modulus K diverges). Seismographers use the difference in propagation velocity to determine the distance to an earthquake source, as is seen in Fig. 3.

1.1 Leftovers from Eran's lecture

In class, you have discussed the polarization of these two waves by writing

$$\boldsymbol{u}=g(\boldsymbol{x}\cdot\boldsymbol{n}-c\,t)\boldsymbol{a}\ , \tag{5}$$

where $\boldsymbol{n}$ is the propagation direction, $\boldsymbol{a}$ is the direction of the displacement and $|\boldsymbol{n}|=|\boldsymbol{a}|=1$. You have shown without proof that this implies

$$(c_d^2-c_s^2)(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}+(c_s^2-c^2)\boldsymbol{a}=0\ . \tag{6}$$

Eran promised that I will show how to get from the former to the latter. This is done simply by applying the differential operators to $\boldsymbol{u}$ giving

$$\nabla g=\partial_i g=g'n_i=g'\boldsymbol{n}\ , \tag{7}$$

$$\nabla\boldsymbol{u}=\partial_j u_i=\partial_j(ga_i)=g'n_ja_i=g'\boldsymbol{a}\otimes\boldsymbol{n}\ , \tag{8}$$

$$\nabla^2\boldsymbol{u}=\nabla\cdot\nabla\boldsymbol{u}=\partial_j(g'n_ja_i)=g''n_jn_ja_i=g''\boldsymbol{a}\ , \tag{9}$$

$$\nabla\cdot\boldsymbol{u}=\mathrm{tr}\,(\nabla\boldsymbol{u})=g'\boldsymbol{a}\cdot\boldsymbol{n}\ , \tag{10}$$

$$\nabla(\nabla\cdot\boldsymbol{u})=\partial_i(g'\boldsymbol{a}\cdot\boldsymbol{n})=g''(\boldsymbol{a}\cdot\boldsymbol{n})n_i=g''(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}\ , \tag{11}$$

$$\partial_{tt}\boldsymbol{u}=c^2g''\boldsymbol{a}\ . \tag{12}$$

Plugging Eqs. (11)-(12) into (3) gives immediately Eq. (6).

The two waves are independent in the bulk. However, on the boundary of a body the traction-free condition $\sigma_{ij}n_j = 0$ couples between the two modes, and more modes arise with a distinct propagation velocity. These are called Rayleigh waves, and are very interesting.

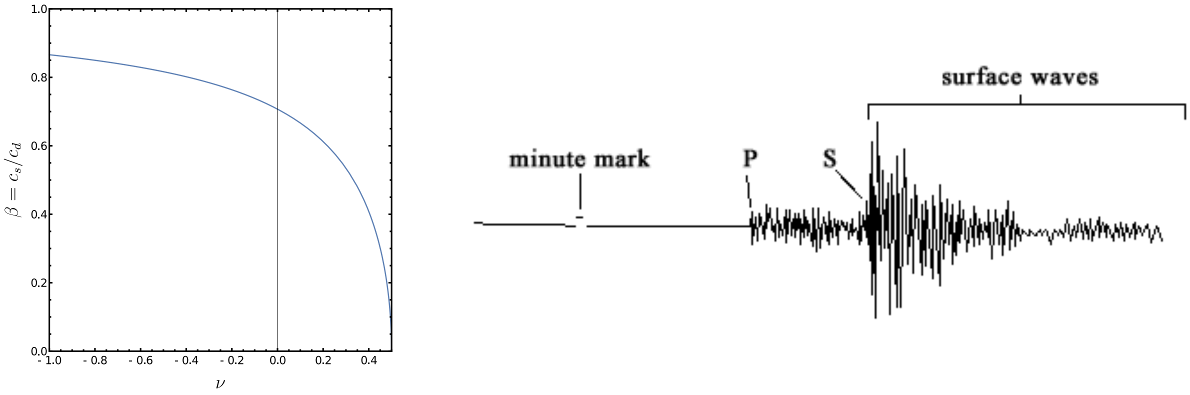


Figure 1: Left: c_s/c_d as a function of Poisson's ratio (Eq. (4)). Right: Seismograph reading of an earthquake. One can clearly see a P-wave (longitudinal) and an S-wave (transverse) arriving at different times. Later, surface waves are visible. The time difference can be used to obtain the distance from the earthquake source.

1.2 Rayleigh waves

So let's see exactly how this works. We want to look at surface waves which propagate, say, in the x-direction. To this end, consider a material that fills the lower half-space $z < 0$, and assume that

$$\boldsymbol{u} = \boldsymbol{f}(z)e^{ikx-i\omega t} \ . \tag{13}$$

If $\boldsymbol{u}$ satisfies the wave equation $(\frac{1}{c_i^2}\partial_{tt} - \nabla^2)\boldsymbol{u} = 0$, with $c_i = c_s$ or c_d, we have

$$\partial_{zz}\boldsymbol{f} = \left(k^2 - \frac{\omega^2}{c_i^2}\right)\boldsymbol{f} \ .$$

If $k^2 > \frac{\omega^2}{c_i^2}$ this gives a damped wave in the bulk. We denote

$$\boldsymbol{f}(z) = \boldsymbol{\gamma}^{(i)}e^{\eta_i z}, \qquad \eta_i = \sqrt{k^2 - \frac{\omega^2}{c_i^2}}, \quad i = s, d \ .$$

As stated above, Rayleigh waves are modes which mix dilational and shear waves. We therefore guess the ansatz

$$\boldsymbol{u} = \boldsymbol{u}^{(d)} + \boldsymbol{u}^{(s)} \ , \tag{14}$$

$$\boldsymbol{u}^{(i)} = \left(\gamma_x^{(i)}\hat{\boldsymbol{x}} + \gamma_z^{(i)}\hat{\boldsymbol{z}}\right)e^{\eta_i z + ikx - i\omega t} \ , \tag{15}$$

where $\boldsymbol{u}^{(d)}, \boldsymbol{u}^{(s)}$ are dilatational and shear waves, and $\gamma_j^{(i)}$ are constants. That is, each of $\boldsymbol{u}^{(d)}, \boldsymbol{u}^{(s)}$ satisfies its own wave equation,

$$(\partial_{tt} - c_s^2 \nabla^2)\boldsymbol{u}^{(s)} = 0 \qquad (\partial_{tt} - c_d^2 \nabla^2)\boldsymbol{u}^{(d)} = 0 \ . \tag{16}$$

They both oscillate with the same frequency ω (the ω of Eq. (13)). Of course, the $\boldsymbol{u}^{(i)}$ are not exactly bulk modes, because they decay exponentially with z, each over over a different length-scale η_i.

Following the discussion about the different polarizations of the different types of waves, note that we should demand

$$\vec{\nabla} \cdot \boldsymbol{u}^{(s)} = \vec{\nabla} \times \boldsymbol{u}^{(d)} = 0 \ . \tag{17}$$

Plugging the ansatz into equation (17) yields

$$\frac{\partial u_x^{(s)}}{\partial x} + \frac{\partial u_z^{(s)}}{\partial z} = \left(ik\gamma_x^{(s)} + \eta_s \gamma_z^{(s)} \right) e^{\cdots} = 0 \qquad \Rightarrow \qquad \frac{\gamma_z^{(s)}}{\gamma_x^{(s)}} = -i\frac{k}{\eta_s} \ , \tag{18}$$

$$\frac{\partial u_x^{(d)}}{\partial z} - \frac{\partial u_z^{(d)}}{\partial x} = \left(\eta_d \gamma_x^{(d)} - ik\gamma_z^{(d)} \right) e^{\cdots} = 0 \qquad \Rightarrow \qquad \frac{\gamma_z^{(d)}}{\gamma_x^{(d)}} = -i\frac{\eta_d}{k} \ . \tag{19}$$

So we write

$$\boldsymbol{u}^{(s)} = A\left(\eta_s \hat{\boldsymbol{x}} - ik\hat{\boldsymbol{z}}\right) e^{\eta_s z + ikx - i\omega t} \qquad A \in \mathbb{C} \ , \tag{20}$$

$$\boldsymbol{u}^{(d)} = B\left(ik\hat{\boldsymbol{x}} + \eta_d \hat{\boldsymbol{z}}\right) e^{\eta_d z + ikx - i\omega t} \qquad B \in \mathbb{C} \ , \tag{21}$$

We now want to demand that the boundary is traction-free. That is, we want to impose $\sigma_{ij}\big|_{z=0} n_j = 0$, where n_j is the local normal to the deformed surface. In principle, $\hat{n}$ also changes because the surface deforms. However, since $\boldsymbol{\sigma}$ is already first-order in the deformation, we are allowed to take the zeroth order of $\hat{n}$, that is, we can take $\hat{n} = \hat{z}$. Therefore, imposing the traction-free boundary conditions means $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$ on $z = 0$. This translates via Hooke's law to

$$\sigma_{xz} = 2\mu\varepsilon_{xz} = \mu\left(\partial_z u_x + \partial_x u_z\right) = 0 \ , \tag{22}$$

$$\sigma_{zz} = (2\mu + \lambda)\varepsilon_{zz} + \lambda\varepsilon_{xx} = (2\mu + \lambda)\partial_z u_z + \lambda\partial_x u_x = \frac{c_d^2 \partial_z u_z + \left(c_d^2 - 2c_s^2\right)\partial_x u_x}{\rho} = 0 \ . \tag{23}$$

We now plug Eqs. (20)-(21) into (22)-(23). This is some uninteresting but necessary algebra. Eq. (22) is relatively simple:

$$0 = \partial_z u_x + \partial_x u_z = \left(\eta_s^2 A + i\eta_d kB\right) + \left(k^2 A + i\eta_d kB\right) = \left(\eta_s^2 + k^2\right)A + 2i\eta_d kB \ , \tag{24}$$

Eq. (23) requires some simplification in order to be sensible:

$$\begin{aligned} 0 &= c_d^2 \partial_z u_z + \left(c_d^2 - 2c_s^2\right)\partial_x u_x \\ &= c_d^2\left(\eta_d^2 B - ik\eta_s A\right) - \left(c_d^2 - 2c_s^2\right)\left(ik\eta_s A - k^2 B\right) \\ &= \left(\frac{c_d^2}{c_s^2}\left(\eta_d^2 - k^2\right) + 2k^2\right)B - 2ik\eta_s A \end{aligned} \tag{25}$$

But since $(k^2 - \eta_i^2)c_i^2 = \omega^2$ is the same for both i's, we can replace $(\eta_d^2 - k^2)\, c_d^2$ in the last equation by $(\eta_s^2 - k^2)\, c_s^2$ and get

$$\left(\eta_s^2 + k^2\right) B - 2ik\eta_s A = 0 \ , \tag{26}$$

Eq. (24) together with (26) form a linear set of equations:

$$\begin{pmatrix} k^2 + \eta_s^2 & 2ik\eta_d \\ -2i\eta_s k & k^2 + \eta_s^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

The condition for a non-trivial solution to exist is det $= 0$, that is $\left(k^2 + \eta_s^2\right)^2 = 4k^2\eta_s\eta_d$. Plugging in $\eta_i^2 = k^2 - \left(\frac{\omega}{c_i}\right)^2$ and squaring, this gives

$$\left(2k^2 - \frac{\omega^2}{c_s^2}\right)^4 = 16\, k^4 \left(k^2 - \frac{\omega^2}{c_s^2}\right)\left(k^2 - \frac{\omega^2}{c_d^2}\right) \ . \tag{27}$$

This is the dispersion relation for Rayleigh waves (sometimes this equation is called the Rayleigh equation). It is a very nice and simple dispersion relation because...it is linear! Huh! you didn't see that coming now, did you? Divide both sides by k^8 to get

$$\left(2 - \left(\frac{\omega}{kc_s}\right)^2\right)^4 = 16\left(1 - \left(\frac{\omega}{kc_s}\right)^2\right)\left(1 - \left(\frac{\omega}{kc_s}\right)^2\left(\frac{c_s}{c_d}\right)^2\right) \ .$$

Denoting the dimensionless phase velocity $z = \frac{\omega}{kc_s} = \frac{c_{ph}}{c_s}$ and remembering our definition $\beta = \frac{c_s}{c_d}$ (cf. Eq. (4)), this turns to

$$\left(2 - z^2\right)^4 - 16\left(1 - z^2\right)\left(1 - \beta^2 z^2\right) = 0 \ ,$$

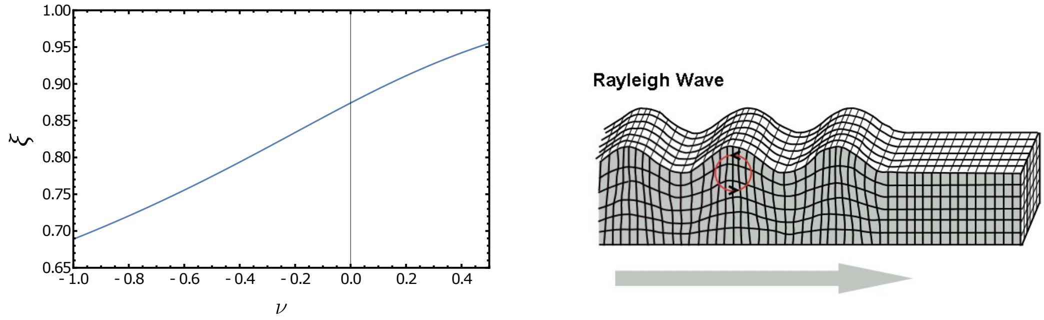
or,

$$z^6 - 8z^4 + 8\left(3 - 2\beta^2\right) z^2 + 16\left(\beta^2 - 1\right) = 0$$

So knowing β, which is a material parameter that equals $\sqrt{\frac{1-2\nu}{2(1-\nu)}}$ gives the (physically unique) solution for z and thus completely defines the **linear** dispersion relation $\omega = zc_s k$. the solution is shown in Fig. 2a, and it is seen that the wave speed, zc_s, is somewhat slower than c_s.

1.2.1 Some remarks regarding Rayleigh waves

- Dilational and shear waves travel at two different speeds. Nevertheless, Rayleigh waves couple the two (!) to create a different mode that travels at a third speed (!!), and all this is within a linear theory (!!!).
- The coupling comes from the traction-free boundary condition.
- A single Rayleigh mode with k, ω is a combination of two evanescent bulk modes with the same ω, but different k.



(a) Numerical solution for z. (b) A Rayleigh wave (from wikipedia).

Figure 2: Stuff about Rayleigh waves.

- The bulk modes are evanescent because the velocity of the Rayleigh mode is slower than c_s and c_d. This makes η_s, η_d real. Otherwise, the modes will not be localized on the surface.
- Rayleigh waves are surface waves.Therefore, their magnitude decreases only as $1/\sqrt{r}$ rather than the bulk $1/r$. In large earthquakes, some Rayleigh waves circle the earth a few times before dissipating!
- They are confined to propagate on the surface and decay exponentially with depth. Therefore, the amplitude of earthquake-generated Rayleigh waves is generally a decreasing function of the depth of the earthquake's hypocenter (origin/focus).
- The particle trajectories in a Rayleigh wave are elliptic, much like in ocean surface waves.

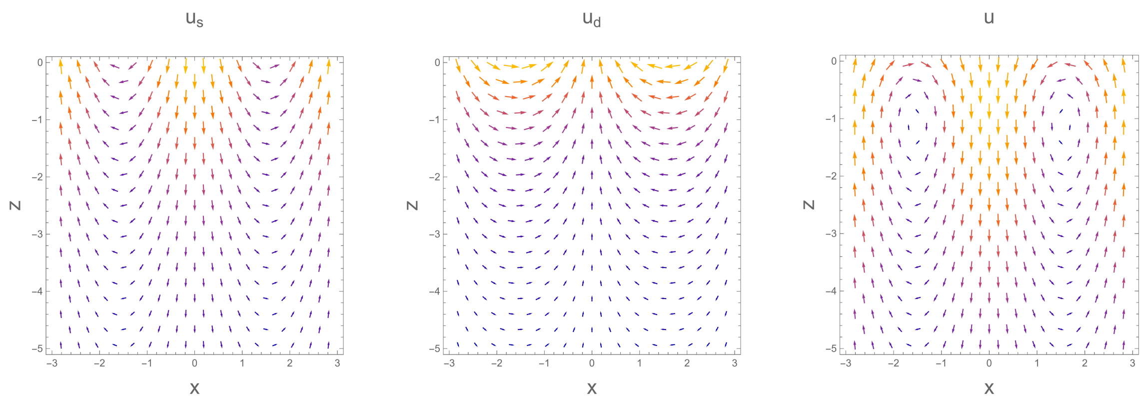


Figure 3: The real parts of u_s, u_d and $u = u_s + u_d$ of the obtained solutions to the Rayleigh derivation. This was produced with $\rho = 1000\text{kg/m}^3$, $\mu = 50\text{GPa}$, and $\nu = \frac{1}{3}$ (resulting in $\beta = \frac{1}{2}$).

2 Thermoelasticity

You have seen in class how a scalar field — the temperature — could couple to the displacement field in the energy functional, and how this coupling gives rise to interesting physics. Here we will consider two examples: the first case will aim at giving some intuition about the thermal effects on stresses, strains and the like. The second example will demonstrate how to satisfy boundary conditions for one field while we already know the solution of a related field, and will demonstrate the nontrivial effect of the temperature field.

2.1 Warm-up — heated rod

Consider a rod of radius R and length L pointing along the z direction, and being held between two rigid walls at $z = 0$ and $z = L$. It is free of constraints in the other two dimensions. The rod is then uniformly heated by some amount ΔT. Lets estimate the stresses and strains in the rod, forces on the walls, and elastic energy stored in the rod scale with the lengths R and L and ΔT (without solving the problem completely).

Warming up a material by ΔT is equivalent to increasing all its length scales by $\sim \alpha_T \Delta T$. Therefore, the situation is equivalent to putting a rod of length $L(1+\frac{1}{3}\alpha_T\Delta T)$ between walls which are only L apart. The strain is therefore $\Delta L/L \sim \alpha_T \Delta T$ and is independent of R and L. The stress is linear in the strain and therefore has the same scaling. The forces on the walls go like

$$F \sim \boldsymbol{\sigma} R^2 = \alpha_T \Delta T R^2$$

and the elastic energy stored in the rod goes like

$$u \sim \int_V \boldsymbol{\epsilon}^2 d^3x \sim R^2 L (\alpha_T \Delta T)^2$$

2.2 Heated annulus — boundary conditions

Recall the problem Eran discussed in class: a thin annulus of internal radius R_1 and external radius R_2 heated according to a nonuniform, purely radial, temperature field $T(r)$. In class you have derived the resulting displacement fields. Here is a brief review.

Due to the angular symmetry the only non-vanishing displacement component is $u_r(r,\theta) = u(r)$. We then have to solve

$$\partial_{rr} u + \frac{\partial_r u}{r} - \frac{u}{r^2} = \frac{\alpha_T K}{\lambda + 2\mu} \partial_r T \ . \tag{28}$$

As both the inner and outer surfaces of the annulus are traction-free, we have the following boundary conditions

$$\sigma_{rr}(r = R_1) = \sigma_{rr}(r = R_2) = 0 \ . \tag{29}$$

By integrating twice Eq. (28) you have obtained

$$u(r) = \frac{\alpha_T K}{\lambda + 2\mu} \frac{1}{r} \int_{R_1}^{r} T(r') r' dr' + \frac{c_1 r}{2} + \frac{c_2}{r} \ , \tag{30}$$

where c_1 and c_2 are two integration constants. But our solution is not complete yet, as we need to determine c_1 and c_2 as to satisfy the boundary conditions Eq.(29).

So? Where do we start? Lets think first about the dimensionality of the problem — the annulus is "thin" — lets think about this whole problem simply in 2D (it will be much simpler this way).

We first need to express the stress tensor using u_r. To do that, recall that $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$, with $\varepsilon_{kl} \equiv \frac{1}{2}\left(\partial_k u_l + \partial_l u_k\right)$. Lets start with $\boldsymbol{\varepsilon}$ — as $\boldsymbol{\varepsilon}$ is expressed in *tensor notation*, it implies its definition could be used in whichever coordinate system we want, as long as we are consistently transforming our differential operators appropriately.

I assume no one really remembers how the gradient looks in cylindrical coordinates — here we use your favorite Wikipedia page, Landau-Lifshitz No. 7, or your favorite resource for getting these kinds of identities. The strain is given by

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \partial_r u_r & \frac{1}{2}\left[\partial_r u_\theta - \frac{1}{r}\left(u_\theta + \partial_\theta u_r\right)\right] \\ \frac{1}{2}\left[\partial_r u_\theta - \frac{1}{r}\left(u_\theta + \partial_\theta u_r\right)\right] & \frac{1}{r}\left(\partial_\theta u_\theta + u_r\right) \end{pmatrix} . \tag{31}$$

We are really lucky, huh? Almost all the terms vanish for our problem, and we are left with

$$\boldsymbol{\varepsilon} = \begin{pmatrix} \partial_r u_r & 0 \\ 0 & \frac{1}{r} u_r \end{pmatrix} . \tag{32}$$

Now lets use our plane-stress relations to obtain the stresses. The resulting relations are

$$\sigma_{ij} = -K\alpha_T\left(T - T_0\right)\delta_{ij} + K \operatorname{tr}\boldsymbol{\varepsilon}\delta_{ij} + 2\mu(\varepsilon_{ij} - \frac{1}{3}\operatorname{tr}\boldsymbol{\varepsilon}\delta_{ij}) . \tag{33}$$

We are interested in the behavior of σ_{rr} at $r = R_1$ and R_2, so we may look at

$$\sigma_{rr} = -K\alpha_T\left(T - T_0\right) + K\left(\partial_r u_r + \frac{1}{r}u_r\right) + 2\mu\left[\partial_r u_r - \frac{1}{3}\left(\partial_r u_r + \frac{1}{r}u_r\right)\right] . \tag{34}$$

So we see that in order to evaluate the constants we will have to use both u_r and its derivative. We already have u_r in Eq. (30), and its derivative is

$$\partial_r u_r = \frac{\alpha_T K}{\lambda + 2\mu}\left(T(r) - \frac{1}{r^2}\int_{R_1}^{r} T(r')r'dr'\right) + \frac{c_1}{2} - \frac{c_2}{r^2} . \tag{35}$$

Now we use these expressions to obtain the stresses at both $r = R_1$ and R_2. The inner circle $r = R_1$ is easier, and we get

$$\sigma_{rr}\left(r = R_1\right) = K\alpha_T(T - T_0) + K\alpha_T T(R_1) + c_1(K + \frac{\mu}{3}) - c_2\frac{2\mu}{R_1^2} . \tag{36}$$

The other end is a bit nastier, but still manageable:

$$\sigma_{rr}\left(r = R_2\right) = K\alpha_T(T - T_0) + K\alpha_T\left[T(R_2) - \frac{6\mu}{R_2^2\left(3K + 4\mu\right)}\int_{R_1}^{R_2} T(r')r'dr'\right] + c_1(K + \frac{\mu}{3}) - c_2\frac{2\mu}{R_2^2} . \tag{37}$$

These are two equations with two independent constants c_1 and c_2. Solving these is trivial, but you can see already that these constants will depend on the geometry R_1 and R_2 as well as on the integral of the temperature field $\int_{R_1}^{R_2} T(r')r'dr'$ — as advertised in class.