Visco-Elasticity

1 Viscoelastic waves

An important application of visco-elastic materials is in energy absorbing devices which are used as mechanical dampers. To get a feeling how this works in principle, let us consider wave propagation through a visco-elastic material. We focus on a scalar case and write the displacement field as $u(x,t) = u^*(x,\omega)e^{i\omega t}$ and the stress field as

$$\sigma(x,t) = \sigma^*(x,\omega)e^{i\omega t} = G^*(\omega)\frac{\partial u^*}{\partial x}e^{i\omega t} \ , \tag{1}$$

where we used $\varepsilon(x,t) = \frac{\partial u(x,t)}{\partial x} = \frac{\partial u^*}{\partial x}\,e^{i\omega t}$. Substituting these expressions in the momentum balance equation

$$\frac{\partial \sigma}{\partial x} = \rho\frac{\partial^2 u}{\partial t^2} \tag{2}$$

we obtain

$$G^*\left(\omega\right)\frac{\partial^2 u^*}{\partial x^2} = -\rho\omega^2 u^* \ . \tag{3}$$

Using $u^* \sim e^{ikx}$, we obtain a propagating plane-wave solution of the form

$$u(x,t) \sim \exp\left[i\omega\left(-\sqrt{\frac{\rho}{G^*\left(\omega\right)}}x + t\right)\right] \ . \tag{4}$$

We observe that G^* plays the role of the elastic constant in an ordinary (elastic) plane-waves and that $\sqrt{G^*/\rho}$ is a complex wave-speed. Suppose we would like to transmit low frequency waves and strongly attenuate high frequency ones. What kind of a material do we need? We would like to have a strong dissipative (viscous-like) response at high frequencies and an elastic response at low frequency. Therefore, our material should be Kelvin-Voigt-like. Let us use the complex modulus of the Kelvin-Voigt model (Eq. (9.43) in the notes) as an example

$$G^* = E(1 + i\omega\tau) \ , \tag{5}$$

where $\tau = \eta/E$. We are interested in the inverse complex speed

$$\sqrt{\frac{\rho}{G^*}} = \sqrt{\frac{\rho}{E}}\frac{1}{\sqrt{1+i\omega\tau}} \equiv \frac{1}{c}\frac{1}{\sqrt{1+i\omega\tau}} \ , \tag{6}$$

shown in Fig. 1 in the limits $\omega\tau \ll 1$ and $\omega\tau \gg 1$. In the low frequency limit, $\omega\tau \ll 1$, we have

$$\sqrt{\frac{\rho}{G^*}} = \frac{1}{c}\frac{1}{\sqrt{1+i\omega\tau}} \simeq \frac{1}{c}\left(1 - \frac{i\omega\tau}{2}\right) \ . \tag{7}$$

Substituting this result into Eq. (4) we obtain

$$u \sim \exp\left[-\frac{i\omega\left(x - c\,t\right)}{c}\right]\exp\left[-\frac{x}{\ell(\omega)}\right] \ , \tag{8}$$

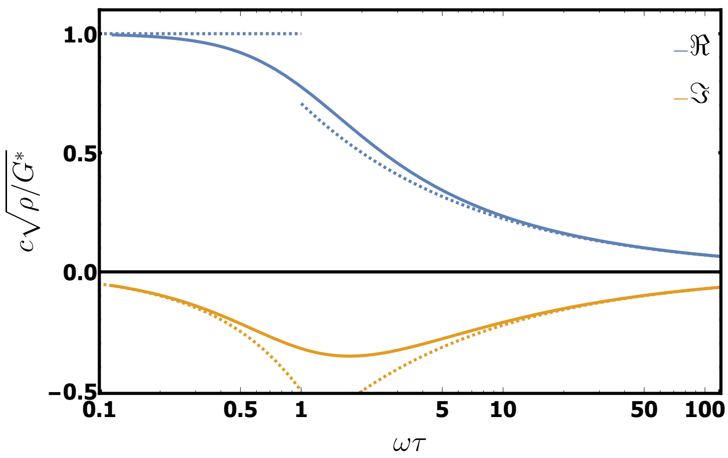

Figure 1: The real and imaginary parts of the inverse complex speed for a Kelvin-Voigt material. The low and high frequency limits are presentes in dashed lines.

where

$$\ell(\omega) \equiv \frac{2c}{\omega^2\tau} = \frac{\lambda}{\pi\omega\,\tau} \gg \lambda \qquad \text{for} \qquad \omega\,\tau \ll 1 \;, \tag{9}$$

where $\lambda = 2\pi c/\omega$ is the wavelength. Therefore, in the low frequency limit waves propagate at the ordinary wave speed with a large attenuation length scale ℓ (many wavelengths). This is expected as the Kelvin-Voigt model is predominantly elastic in the long timescales limit.

In the opposite limit, $\omega\tau \gg 1$, we have

$$\sqrt{\frac{\rho}{G^*}} = \frac{1}{c}\frac{1}{\sqrt{1+i\omega\tau}} \simeq \frac{1-i}{c\sqrt{2\omega\tau}} \;. \tag{10}$$

Substituting this result into Eq. (4) we obtain (defining $\tilde{\ell}(\omega) \equiv c\sqrt{2\omega\tau}/\omega$)

$$u \sim \exp\left[-\frac{ix}{\tilde{\ell}(\omega)} + i\omega\,t\right]\exp\left[-\frac{x}{\tilde{\ell}(\omega)}\right] = \exp\left[-i\frac{(x-\tilde{\ell}(\omega)\,\omega\,t)}{\tilde{\ell}(\omega)}\right]\exp\left[-\frac{x}{\tilde{\ell}(\omega)}\right] \;, \tag{11}$$

which shows that both the wavelength and the decay length are determined by $\tilde{\ell}(\omega)$ (which actually means that they wavelength is ill-defined). Hence we conclude that in the high frequency limit wave propagation is completely attenuated.

2 Viscoelasticity in more than 1 dimensions

Up to now we only dealt with 1-dimensional models of viscoelasticity. These can be easily formulated because all the fields (stress, strain, etc.) are scalar and it is easy to write relations such as

$$\sigma = \int G(t-t')\dot{\varepsilon}(t')dt' \;. \tag{12}$$

How would you generalize this to 2D or 3D where $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ are tensors? Consider Hooke's law, which we write here as

$$\boldsymbol{\sigma} = \lambda\boldsymbol{I}\,\mathrm{tr}\,\boldsymbol{\varepsilon} + 2\mu\boldsymbol{\varepsilon} \;.$$

Let's look at two components of σ_{ij}, one with $i = j$ and and one with $i \neq j$:

$$\begin{aligned} \sigma_{xx} &= 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) \ , \\ \sigma_{ij} &= 2\mu\varepsilon_{ij}, \qquad i \neq j \ . \end{aligned} \tag{13}$$

The shear components of the strain and stress are simply proportional to each other. Thus, one can define the time-dependent shear modulus $\mu(t)$ and use all the formalism that you know by now. For example, one can write

$$\sigma_{ij} = 2\int \mu(t - t')\dot{\varepsilon}_{ij}(t')dt' \qquad \text{for } i \neq j \ , \tag{14}$$

and also the inverse relation etc.. What about the diagonal components which are mixed? A convenient way to deal with this is to write Hooke's law using the deviatoric and volumetric components, i.e. to define

$$\boldsymbol{\varepsilon}^{\text{dev}} \equiv \boldsymbol{\varepsilon} - \frac{1}{3}\boldsymbol{I}\,\text{tr}\,\boldsymbol{\varepsilon} \ , \qquad \boldsymbol{\sigma}^{\text{dev}} \equiv \boldsymbol{\sigma} - \frac{1}{3}\boldsymbol{I}\,\text{tr}\,\boldsymbol{\sigma} \ , \tag{15}$$

$$\boldsymbol{\varepsilon}^{\text{vol}} \equiv \frac{1}{3}\boldsymbol{I}\,\text{tr}\,\boldsymbol{\varepsilon} \ , \qquad \boldsymbol{\sigma}^{\text{vol}} \equiv \frac{1}{3}\boldsymbol{I}\,\text{tr}\,\boldsymbol{\sigma} \ . \tag{16}$$

With these, Hooke's law can be written as

$$\boldsymbol{\sigma} = \lambda\boldsymbol{I}\,\text{tr}\,\boldsymbol{\varepsilon} + 2\mu\boldsymbol{\varepsilon} = \lambda\boldsymbol{I}\,\text{tr}\,\boldsymbol{\varepsilon}^{\text{vol}} + 2\mu\left(\boldsymbol{\varepsilon}^{\text{dev}} + \boldsymbol{\varepsilon}^{\text{vol}}\right) \ . \tag{17}$$

Taking the deviatoric and volumetric parts of this expression, one immediately finds that

$$\boldsymbol{\sigma}^{\text{vol}} = 3K\boldsymbol{\varepsilon}^{\text{vol}} \ , \qquad \boldsymbol{\sigma}^{\text{dev}} = 2\mu\boldsymbol{\varepsilon}^{\text{dev}} \ . \tag{18}$$

where $K = \lambda + \frac{2}{3}\mu$ is the bulk modulus. Thus, also the volumetric and deviatoric parts of $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are proportional to each other and the same logic of Eq. (14) can be applied here, using the time-dependent bulk and shear moduli,

$$\boldsymbol{\sigma}^{\text{vol}} = 3\int K(t - t')\dot{\boldsymbol{\varepsilon}}^{\text{vol}}(t')dt' \ , \qquad \boldsymbol{\sigma}^{\text{dev}} = 2\int \mu(t - t')\dot{\boldsymbol{\varepsilon}}^{\text{dev}}(t')dt' \ . \tag{19}$$

2.1 The Correspondence Principle

With these formulæ, we are ready to present the powerful "Correspondence Principle" which offers a generic way to think about (and often to actually solve) viscoelastic problems by considering the corresponding elastic problems. To see exactly how this goes, write (on the left) what exactly is an elastic problem, and the corresponding viscoelastic problem (on the right):

Elasticity

An elastic problem is the PDE

$$\rho \ddot{u}_i(\boldsymbol{x},t) = \partial_j \sigma_{ij}(\boldsymbol{x},t) + b_i(\boldsymbol{x},t) \ , \qquad (20)$$

with the definitions

$$\begin{aligned} \boldsymbol{\sigma}^{\text{vol}} &= 3K\boldsymbol{\varepsilon}^{\text{vol}} \ , \\ \boldsymbol{\sigma}^{\text{dev}} &= 2\mu\boldsymbol{\varepsilon}^{\text{dev}} \ , \\ \varepsilon_{ij} &= \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \ , \end{aligned} \qquad (21)$$

and with the boundary conditions

$$\sigma_{ij}n_j = T_i \qquad \text{or} \qquad u_i = d_i \ . \qquad (22)$$

Viscoelasticity

A viscoelastic problem is the PDE

$$\rho \ddot{u}_i(\boldsymbol{x},t) = \partial_j \sigma_{ij}(\boldsymbol{x},t) + b_i(\boldsymbol{x},t) \ , \qquad (23)$$

with the definitions

$$\begin{aligned} \boldsymbol{\sigma}^{\text{vol}} &= 3\int K(t-t')\dot{\boldsymbol{\varepsilon}}^{\text{vol}}(t') \ , \\ \boldsymbol{\sigma}^{\text{dev}} &= 2\int \mu(t-t')\dot{\boldsymbol{\varepsilon}}^{\text{dev}} \ , \\ \varepsilon_{ij} &= \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \ , \end{aligned} \qquad (24)$$

and with the boundary conditions

$$\sigma_{ij}n_j = T_i \qquad \text{or} \qquad u_i = d_i \ . \qquad (25)$$

The similarity is obvious. How do we take advantage of it? Taking the Laplace transform of the viscoelastic relations and assuming quasi-static conditions (i.e. neglecting the inertial term in the force-balance equations) we get

Viscoelasticity in s-domain

A viscoelastic problem in the in s-domain is the PDE

$$0 = \partial_j \sigma_{ij}(\boldsymbol{x},s) + b_i(\boldsymbol{x},s) \ , \qquad (26)$$

with the definitions

$$\begin{aligned} \boldsymbol{\sigma}^{\text{vol}} &= 3sK(s)\,\boldsymbol{\varepsilon}^{\text{vol}} \ , \\ \boldsymbol{\sigma}^{\text{dev}} &= 2s\mu(s)\,\boldsymbol{\varepsilon}^{\text{dev}} \ , \\ \varepsilon_{ij} &= \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \ , \end{aligned} \qquad (27)$$

and with the boundary conditions

$$\sigma_{ij}n_j = T_i \qquad \text{or} \qquad u_i = d_i \ . \qquad (28)$$

So you see that if you have a viscoelastic problem, you can use whatever you know to solve the corresponding elastic problem with the recipe:

1. Take a viscoelastic boundary value problem,
2. Replace all time dependent variables in all the governing equations by their Laplace transform,
3. Replace all material properties by s times their Laplace transform,
4. Solve the elastic problem in the s-domain,
5. Perform an inverse Laplace transform to get the viscoelastic solution in the t domain.

2.2 Example: Reinforced thick wall cylinder

As an example, consider a cylinder whose inner part is filled with a viscoelastic material as shown in Fig. 2. This geometry is relevant for solid propellant rockets, because the fuel can be reasonably well described as a linear viscoelastic material. Of course, the real problem has many other complications (for example, the inner boundary is not a circle but rather a star shape to help with thrust and ablation), but this is already interesting enough. The outer cylinder shell is much stiffer than the fuel and for our purposes can be taken to be infinitely rigid (=undeformable). The inner and outer radii are denoted a and b, respectively. In the center of the cylinder an internal pressure p_0 is applied. What is the deformation of the solid fuel?

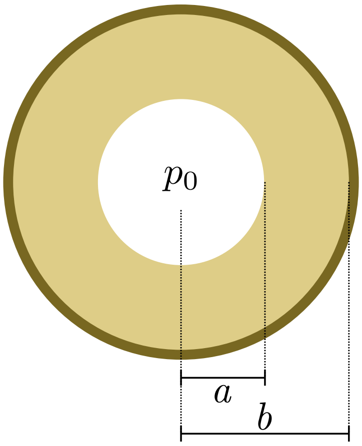

Figure 2: Reinforced thick Wall cylinder

Some variants of the corresponding elastic problem were solved in the lectures (the heated annulus, cylindrical cavity, etc.). Therefore, I will not go through the solution of the elastic problem in much detail. I will just remind you that it can readily be solved by using Airy's stress function χ, which satisfies the biharmonic equation. Using polar coordinates and assuming azimuthal symmetry, it reads

$$\nabla^2\nabla^2\chi(r,\theta) = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\partial_r\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\partial_r\right)\chi(r) = 0\ . \tag{29}$$

Imposing the boundary conditions $\sigma_{rr}(r\!=\!a)\!=\!-p_0$ and $u(r\!=\!b)\!=\!0$ one can solve to get

$$\sigma_{rr} = -p_0\frac{1+(1-2\nu)\left(\frac{b}{r}\right)^2}{1+(1-2\nu)\left(\frac{b}{a}\right)^2}\ , \qquad \sigma_{\theta\theta} = -p_0\frac{1-(1-2\nu)\left(\frac{b}{r}\right)^2}{1+(1-2\nu)\left(\frac{b}{a}\right)^2}\ , \tag{30}$$

and all other components of $\boldsymbol{\sigma}$ vanish. Note that the only difference between the two is a change of sign in the numerator so in what follows we write both Equations (30) as

$$\sigma_{r,\theta} = -p_0\frac{1\pm(1-2\nu)\left(\frac{b}{r}\right)^2}{1+(1-2\nu)\left(\frac{b}{a}\right)^2}\ . \tag{31}$$

where $+$ corresponds to σ_{rr} and $-$ to $\sigma_{\theta\theta}$.

Now we are ready to calculate the time-dependent response of the viscoelastic fuel. To be concrete, we need to assume something about its viscoelastic properties. A common approximation is that the material is "elastic in compression", i.e. that its bulk modulus is purely elastic and does not have viscous properties. This is a reasonable quantitative assumption. For the shear part, let's take a Maxwell model. In addition, for simplicity let's assume that the pressure is applied infinitely fast, i.e. $p(t) = p_0 H(t)$. Thus,

$$\begin{aligned} \mu(t) &= H(t)\mu_0 e^{-t/\tau} &\Rightarrow\qquad \mu(s) &= \frac{\mu_0}{s+\tau^{-1}} \ , \\ K(t) &= H(t)K_0 &\Rightarrow\qquad K(s) &= \frac{K_0}{s} \ , \\ p(t) &= H(t)p_0 &\Rightarrow\qquad p(s) &= \frac{p_0}{s} \ . \end{aligned} \tag{32}$$

We use the conversion $\nu = \frac{3K-2\mu}{2(3K+\mu)}$ to calculate $(1-2\nu) = \frac{1}{\frac{1}{3}+\frac{K}{\mu}}$. Thus, the stress is

$$\sigma_{r,\theta}(r,s) = -p(s)\frac{1 \pm \frac{1}{\frac{1}{3}+\frac{K(s)}{\mu(s)}}\left(\frac{b}{r}\right)^2}{1 + \frac{1}{\frac{1}{3}+\frac{K(s)}{\mu(s)}}\left(\frac{b}{a}\right)^2} \ . \tag{33}$$

Plugging Eqs. (32) into this one gets

$$\sigma_{r,\theta}(r,s) = p_0\left(\frac{\tilde{\mu}\tau\left(\frac{b^2}{a^2} \mp \frac{b^2}{r^2}\right)}{1+\left(1+\tilde{\mu}\left(\frac{b^2}{a^2}+\frac{1}{3}\right)\right)\tau s} - \frac{1}{s}\right) \ , \tag{34}$$

with the definition $\tilde{\mu} \equiv \mu_0/K_0$. We can now use the the formula

$$\mathcal{L}^{-1}\left\{\frac{1}{a+bs}\right\} = \frac{1}{b}e^{-at/b} \ , \tag{35}$$

to get

$$\sigma_{r,\theta}(r,t) = p_0\left(\frac{\left(\frac{b^2}{a^2} \mp \frac{b^2}{r^2}\right)\tilde{\mu}}{\left(\frac{1}{3}+\frac{b^2}{a^2}\right)\tilde{\mu}+1}\exp\left[-\frac{t/\tau}{\left(\frac{1}{3}+\frac{b^2}{a^2}\right)\tilde{\mu}+1}\right] - 1\right) \ . \tag{36}$$

A few notes are in place regarding this result:

1. First, note that for any t we have $\sigma_{rr}(r=a) = -p_0$, so the boundary conditions are satisfied at all times (also the no-displacement boundary condition at $r = b$ is satisfied at all times, but this is less immediate to see).

2. In the short time limit $t \to 0$ the exponent is unity and the solution coincides with the elastic solution of Eq. (30). This is not surprising because at short times a Maxwell material is simply elastic.

3. In the long time limit $t \to \infty$ the exponent vanishes such that *both* σ_{rr} and $\sigma_{\theta\theta}$ equal $-p_0$, i.e. the solution becomes that of a hydrostatic liquid. This is not surprising because at long times a Maxwell material is simply a fluid.

4. Note that the relaxation time is $\tau\left(\left(\frac{1}{3}+\frac{b^2}{a^2}\right)\tilde{\mu}+1\right)$. Dimensionally, it scales of course with the rheological timescale τ, but it also has a correction due to the system geometry (a,b) and the system's elastic constants (μ_0, K_0).

3 Kramers-Kronig Relation

The KK relation is a fundamental relation between the real and imaginary parts of a response function $\hat{G}(\omega)$. In our case, it relates the storage and loss moduli $\hat{G}'$ and $\hat{G}''$ but it is very general and has applications in experimental and theoretical physics, as well as in signal processing and electrical engineering. The essence of these relations lies in the fact that the imaginary and real parts of an analytic function are not independent, and are related via the Cauchy-Riemann Equations, which in turn imply Cauchy's integral formula (residue calculus).

We will give two proofs of the KK relations. The standard residue-calculus one, and another one that singles out the effect of causality. The actual theorem is almost misleadingly simple:

***Theorem**: Let $\hat{G}(\omega) = \hat{G}'(\omega) + i\hat{G}''(\omega)$ be an analytic function in the upper half plane that decays at infinity faster than $|\omega|^{-1}$. Then*

$$\hat{G}'(\omega) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\hat{G}''(\omega')}{\omega' - \omega}\, d\omega' \ , \qquad \hat{G}''(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\hat{G}'(\omega')}{\omega' - \omega}\, d\omega' \ , \tag{37}$$

where $\mathcal{P}$ denotes the principal value.

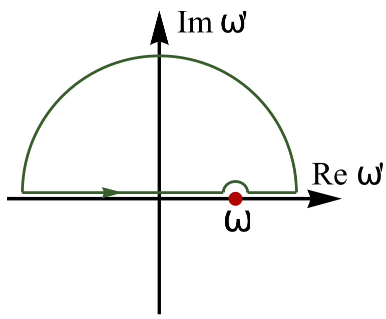

Figure 3: Contour of integration in the upper half plane. Source: wiki commons.

Proof I: The integral of $\frac{\hat{G}(\omega)}{\omega' - \omega}$ over the contour described in Fig. 3 is clearly 0, because the integrand is analytic. The integral over the half circle vanishes (because $\hat{G}(\omega)$ decays fast enough), and the integral over the bump is $-i\pi\hat{G}(\omega)$ (minus one half of the residue). We therefore have

$$\hat{G}(\omega) = \frac{1}{i\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{\hat{G}(\omega')}{\omega' - \omega} d\omega' \ . \tag{38}$$

Writing this equation in its real and imaginary parts gives exactly Eqs. (37) ■ .

This proof is completely trivial in terms of residue calculus, but as a physicist I am not quite satisfied by this proof. It leaves me with a feeling that I accept the theorem, but I don't *understand* it. Furthermore, we must ask (a) how do we know that $\hat{G}(\omega)$ decays at infinity? and (b) how do we know that $\hat{G}$ is analytic in the upper half plane?.

The answer to (a) is that we don't, in general, but it is very reasonable to assume that systems that are driven at frequencies much higher than their natural frequencies do not respond (and therefore $\hat{G} \to 0$).

The answer to (b) is less trivial. Note that in general the Fourier transform of a "nice" function is not analytic in the upper half plane. For example, the FT of a Lorenzian $\frac{1}{1+(t/\tau)^2}$ is $\sim \exp(-|\omega\tau|)$ which is not analytic anywhere; conversely, the FT of $\exp(-|\omega_0\tau|)$ is a Lorenzian, and thus is analytic almost everywhere but has a pole at $\omega = i\omega_0$; the FT of a Gaussian is also a Gaussian, which has an essential singularity at infinity.

The fact that $\hat{G}(\omega)$ is analytic for $\Im(\omega) > 0$ stems from **causality**. In fact, one can show that $\hat{G}(\omega)$ is analytic in the upper half plane if and only if $G(t) = 0$ for $t < 0$ (this is called Titchmarsh's theorem). To see exactly what is the role that causality takes, we'll examine a different proof.

Proof II: We first remind ourselves of the trivial fact that the FT of an even function is purely real, and that of an odd function is purely imaginary. Now, for any function $G(t)$ we can define

$$G^{\text{even}}(t) \equiv \frac{G(t)+G(-t)}{2} \ , \qquad G^{\text{odd}}(t) \equiv \frac{G(t)-G(-t)}{2} \ , \tag{39}$$

such that $G(t)$ can be written as $G(t) = G^{\text{even}}(t) + G^{\text{odd}}(t)$. Therefore,

$$\hat{G}(\omega) = \mathcal{F}\left\{G^{\text{even}}(t) + G^{\text{odd}}(t)\right\} = \mathcal{F}\left\{G^{\text{even}}(t)\right\} + \mathcal{F}\left\{G^{\text{odd}}(t)\right\} = \hat{G}'(\omega) + i\hat{G}''(\omega) \ . \tag{40}$$

We can thus conclude that

$$\hat{G}' = \mathcal{F}\{G^{\text{even}}\} \qquad , \qquad \hat{G}'' = \frac{1}{i}\mathcal{F}\{G^{\text{odd}}\} \ . \tag{41}$$

In general, the odd and even parts are independent, but for a casual response function we have $G(t) = 0$ for $t < 0$ and therefore

$$\begin{aligned} G^{\text{odd}}(t) &= \frac{1}{2}\begin{cases} -G(-t) & t<0 \\ G(t) & t>0 \end{cases} = \frac{1}{2}G(|t|)\,\text{sign}(t) \ , \\ G^{\text{even}}(t) &= \frac{1}{2}\begin{cases} G(-t) & t<0 \\ G(t) & t>0 \end{cases} = \frac{1}{2}G(|t|) \ , \end{aligned} \tag{42}$$

where $\text{sign}(t) = \frac{t}{|t|}$ is the signum function. Thus, for $t > 0$ we have $G^{\text{odd}}(t) = G^{\text{even}}(t)$ and for $t < 0$ we have $G^{\text{odd}}(t) = -G^{\text{even}}(t)$. This can be compactly written as

$$G^{\text{odd}}(t) = \text{sign}(t) G^{\text{even}}(t) \qquad , \qquad G^{\text{even}}(t) = \text{sign}(t) G^{\text{odd}}(t) \ . \tag{43}$$

Thus, we have

$$G'(\omega) = \mathcal{F}\{G^{\text{even}}\} = \frac{1}{2\pi}\mathcal{F}\{\text{sign}\} * \mathcal{F}\{G^{\text{odd}}\} = \frac{i}{2\pi}\mathcal{F}\{\text{sign}\} * G'' \ , \tag{44}$$

$$G''(\omega) = \frac{1}{i}\mathcal{F}\{G^{\text{odd}}\} = \frac{1}{2\pi i}\mathcal{F}\{\text{sign}\} * \mathcal{F}\{G^{\text{even}}\} = \frac{1}{2\pi i}\mathcal{F}\{\text{sign}\} * G' \ , \tag{45}$$

where $*$ denote convolution. The FT of the signum function is $\frac{2i}{\omega}$. Substituting this result into the above equations gives (37).