Non-Equilibrium Continuum Physics **HW set #3**

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Due 28.05.25 Submit as: NECP_HW3_⟨FirstName⟩_⟨LastName⟩.pdf

Static linear elasticity and thermo-elasticity

1 A thick-walled spherical shell under internal and external pressure

Consider a thick-walled spherical shell of inner radius a and outer radius b made of an isotropic and homogeneous linear elastic material, see Fig. 1. The inner part of the shell is filled with a fluid

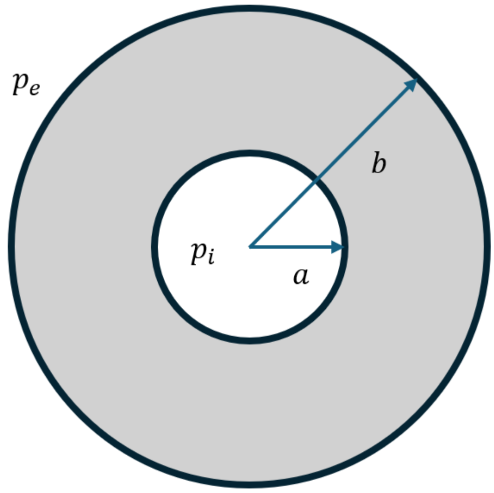

Figure 1: A planar cut of the 3D problem under consideration, see text.

of hydrostatic pressure p_i, while the outer part of the shell is surrounded by a fluid of hydrostatic pressure p_e, see Fig. 1. Our goal is to derive the static deformation and stress in the shell (i.e., inertia can be neglected), and understand their properties.

(i) In view of the symmetry of the problem (consider spherical coordinates (r, ϕ, θ), see Fig. 1), rationalize the statement that the static Navier-Lamé equation takes the form

$$\partial_{rr} u_r + \frac{2\partial_r u_r}{r} - \frac{2u_r}{r^2} = 0 \ . \tag{1}$$

(ii) Write down the boundary conditions for the problem. Pay attention to signs.

(iii) The compatibility condition for the problem at hand takes the form

$$\varepsilon_{rr} = \frac{\partial}{\partial r}\left(r\,\varepsilon_{\theta\theta}\right) \ . \tag{2}$$

Use it to obtain the relation between the strain components ε_{rr}, $\varepsilon_{\phi\phi}$ and $\varepsilon_{\theta\theta}$ in terms of the displacement field u_r.

(iv) Find the parametric solution of Eq. (1). Hint: Consider power-law solutions and ask yourself how many independent solutions you expect to have.

(v) Use your parametric solution, together with Hooke's law

$$\varepsilon_{rr} = \frac{1}{E}\left(\sigma_{rr} - 2\,\nu\,\sigma_{\theta\theta}\right), \qquad \varepsilon_{\theta\theta} = \frac{1}{E}\left[(1-\nu)\,\sigma_{\theta\theta} - \nu\,\sigma_{rr}\right] \tag{3}$$

and the boundary conditions to show that the stress components take the form

$$\sigma_{rr}(r) = -\frac{1}{2}(p_i + p_e) + \frac{p_i - p_e}{2\left[1-(a/b)^3\right]}\left[1 + \left(\frac{a}{b}\right)^3 - 2\left(\frac{a}{r}\right)^3\right] \tag{4}$$

$$\sigma_{\phi\phi}(r) = \sigma_{\theta\theta}(r) = -\frac{1}{2}(p_i + p_e) + \frac{p_i - p_e}{2[1-(a/b)^3]}\left[1 + \left(\frac{a}{b}\right)^3 + \left(\frac{a}{r}\right)^3\right] . \tag{5}$$

(vi) Discuss the physical properties of the solution (e.g., consider various limits such as $p_i \gg p_e$, $p_e \gg p_i$, $p_i \to p_e$, $b \gg a$).

(vii) Consider the limit $p_i \gg p_e$, i.e., set $p_i = p$ and $p_e = 0$, and discuss the properties of the displacement field u_r (e.g., is $u_r(r)$ monotonic? if not, discuss the properties of its stationary points. Moreover, what happens in the incompressible limit, $\nu \to 1/2$?).

(viii) The problem you solved above is related to the cylindrical cavity discussed in class. Discuss the differences and similarities between the two problems.

2 Heated rod

A rod of radius R and length L is pointed along the z direction, and is held between two rigid walls at $z = 0$ and $z = L$. It is free of constraints in the other two dimensions. The rod is then uniformly heated by some amount ΔT. Without solving the problem completely (although you may do it, if you have nothing better to do) estimate how the

(i) stresses in the rod,

(ii) strains in the rod,

(iii) forces on the walls,

(iv) elastic energy stored in the rod

scale with R, L and ΔT.

3 Heating an infinite material

Consider an infinite 2D material, from which a circular hole is taken out. The material is now heated by some amount. Will the hole shrink or expand?

4 Temperature distribution in a 3D material

Consider a static infinite 3D material with a given arbitrary distribution of temperature $T(x, y, z)$, that decays at infinity: $T(\vec{r}) \to T_\infty$, as $|\vec{r}| \to \infty$. Before reading further it might by nice to try to estimate: if the temperature variation is localized, how does the displacement field decay at large $\boldsymbol{r}$? And the strain field?

Here's a nice way to gain intuition as to what temperature gradients do in thermo-elasticity: Prove that the displacement field is curl-free, i.e. is of the form $\vec{u} = \nabla\phi$, and that ϕ satisfies Poisson's equation $\nabla^2\phi = T$. Assuming you already have some intuition about electrostatics, this should help you gain intuition about thermo-elasticity.

Guidance: Begin with Navier-Lamé equation $(\lambda + \mu)\nabla(\nabla \cdot \boldsymbol{u}) + \mu\nabla^2\boldsymbol{u} = K\alpha\nabla T$. You can guess the correct form for u, and if it works then you're done because the solution is unique. Some vector-analysis identities might prove useful.