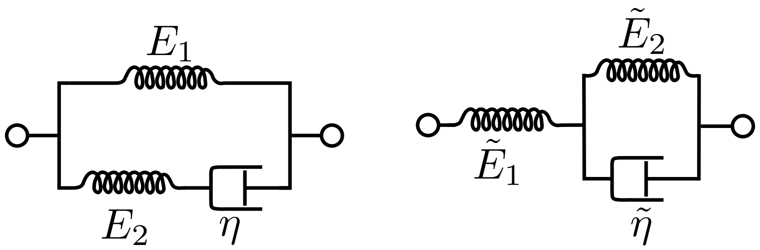
Viscoelasticity

1 Standard linear solid

Consider the one-dimensional Standard-Linear-Solid (SLS) model, described in the left panel of the Fig. below. In this exercise we will explore its visco-elastic properties in much the same way that we did in class for the Maxwell and Kelvin-Voigt models.



(a) The SLS model (b) The alternative model

Figure 1:

(i) Calculate $G^{\text{SLS}}(t)$ and $J^{\text{SLS}}(t)$.

(ii) Calculate $G^*(\omega)$. Find the effective Young's modulus for very short and very long time scales. Denote these quantities by E_0 and E_∞, respectively. Check that your finding agrees with the proper limits of the results of Q1(i). (writing J and G in terms of E_0 and E_∞ might prove to be more elegant than with E_1 and E_2).

(iii) `QUALITATIVE QUESTION:` Plot the loss and storage moduli $G'(\omega)$ and $G''(\omega)$ on a logarithmic ω scale for the case $E_1 = E_2$. If you were to transfer waves through an SLS material, which frequencies would be transmitted and which attenuated?

(iv) Calculate $\varepsilon(t)$ when the stress increases from 0 to σ_0 over a time scale T. To make things concrete, take the stress to be

$$\sigma(t) = \begin{cases} 0 & t < 0 \\ \sigma_0 \left(1 - e^{-t/T}\right) & t > 0 \end{cases} .$$

For simplicity, also set $E_1 = E_2$. Plot $\varepsilon(t)$ for the cases that T is (i) much larger than, (ii) much smaller than, and (iii) roughly equal to the relevant internal time-scale of the system (for the two extreme cases you should know the answer without any further algebra!).

(v) Consider an alternative model, defined in Fig. 1(b). Show that it is exactly equivalent to the SLS model, but with renormalized visco-elastic constants. That is, show that one can choose $\tilde{E}_1, \tilde{E}_2, \tilde{\eta}$ so that this model will have the same rheological properties as the SLS model with parameters E_1, E_2, η.

(vi) `QUALITATIVE QUESTION:` The name "SLS" model suggests that it describes a solid. What is the basic property of solids that the SLS model possess but that Maxwell model doesn't? And what is the problem with Kelvin-Voigt model?

2 Creep compliance and stress relaxation moduli

Prove the general identity

$$\int_0^t G(t-t')J(t')dt' = t \ , \tag{1}$$

and verify it explicitly for the KV and M models. Hint: Laplace transform.

3 Stored elastic energy

Define W_{sto} as the elastic energy stored in a quarter of an oscillatory cycle. Recall the definition of the phase δ (Eqs. (9.39)-(9.40) in Eran's lecture notes) and show that

$$\frac{W_{dis}}{W_{sto}} \sim \tan\delta \ . \tag{2}$$

4 Distribution of relaxation times

For some systems which exhibit a wide distribution of relaxation times the stress relaxation modulus can be written as a weighted sum of exponential decays with different decay rates:

$$G(t) = \int_\tau f(\tau) e^{-\frac{t}{\tau}} d\tau \ , \tag{3}$$

where $f(\tau)$ is the relaxation times distribution function. If the relaxation times depend on some energy barrier Δ, this can be written as

$$G(t) = G_0 \int_\Delta P(\Delta) e^{-\frac{t}{\tau(\Delta)}} d\Delta \ , \tag{4}$$

where P is the energy barrier distribution function. When the transitions are thermally–activated and the energy barriers are much larger than the thermal energy scale ($\Delta \gg k_B T$) the rate of escape times varies exponentially with the barrier height:

$$\tau(\Delta) \simeq \tau_0 e^{\frac{\Delta}{k_B T}} \ . \tag{5}$$

This is a fundamental result, derived by Arrhenius (1889, for chemical reactions) Eyring (1935) and Kramers (1940, for Brownian motion under an external potential) and the factor $e^{\frac{\Delta}{k_B T}}$ is usually termed "Arrhenius factor" or "rate factor".

(i) Assume that Δ is uniformly distributed between Δ_{min} and Δ_{max}, and calculate $G(t)$. Express your answer using rate variable $\nu(\Delta) \equiv 1/\tau(\Delta)$. You might find that the exponential integral function, $E_1(x) \equiv \int_1^\infty \frac{e^{-xy}}{y} dy$, is useful.

(ii) Choose $\nu_{min} \equiv \nu(\Delta_{min})$ and $\nu_{max} \equiv \nu(\Delta_{max})$ to be well separated and plot $G(t)$. What is special about this result? How does it differ from standard relaxation?

(iii) Obtain an analytic expression (you are allowed to be wrong by an additive constant) for $G(t)$ for an intermediate asymptotic regime, $\nu_{min}^{-1} \ll t \ll \nu_{max}^{-1}$. This is a simple mathematical model for slow/glassy relaxation emerging from a broad distribution of relaxation times (activation barriers in this case).