Non-Equilibrium Continuum Physics **HW set #6**

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Due 05.08.25 Submit as: NECP_HW6_⟨FirstName⟩_⟨LastName⟩.pdf

Plasticity and spatiotemporal irreversible dynamics

Instructions: This HW set is devoted to the last topics covered in the course, i.e., plasticity and spatiotemporal irreversible dynamics. The problems you are asked to consider also refer to earlier topics covered in the course (e.g., thermo-elasticity and visco-elasticity), we kindly ask you to use this opportunity to go over the entire course materials carefully and systematically. We also encourage you, if possible, to consider the optional/bonus items.

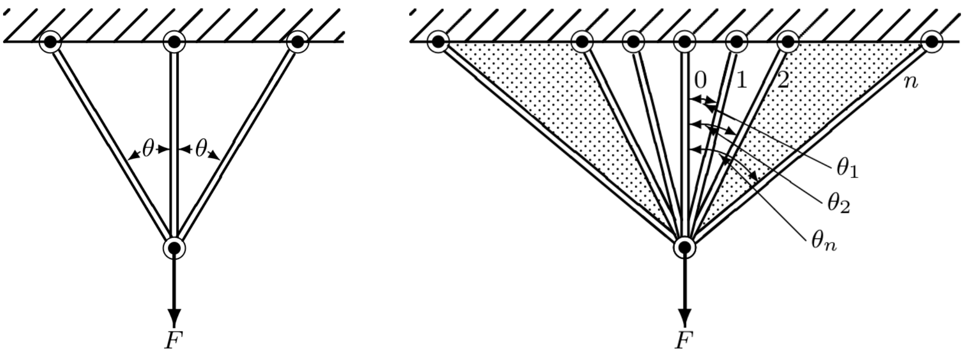
1 Multiple elasto-plastic rods

1.1 Pulled elastic-perfect-elsatic rod

An incompressible elastic-perfect-plastic cylindrical rod, of Young's modulus E, yield stress $\sigma_Y \ll E$, length L and cross section A is compressed/pulled under uniaxial stress along its axis until its length is multiplied by a factor λ. How much work did the external loading preform? How much of it was dissipated? Work in the regime that $|\lambda - 1| \ll 1$, but plastic deformation does occur.

1.2 Rods connected by pins

Consider the setting shown in Fig 1a: three elastic-perfect-plastic rods with cross sectional area A are connected with pins that can transfer only axial forces but no torques, and a vertical force F is exerted on them. The top pins are held at fixed positions to the ceiling (but not at a fixed angle). All rods have Young's modulus E and yield stress $\sigma_Y \ll E$. When $F = 0$ the system is stress-free. Assume small deformations.



(a) Three rod setup. (b) **Optional/bonus** question setup.

Figure 1: n-rods setup.

A. Denote the vertical displacement of the loading point by Δ. Calculate and plot $\Delta(F)$ (choose some values for the parameters you need). What is the maximal force F_E for which the response is elastic? What is the maximal force F_U that can be applied?

B. Calculate the residual strains and stresses if the force is removed after the displacement was Δ.

C. Suppose no force is applied, but the temperature is increased (or decreased) by ΔT. Calculate the minimal temperature difference ΔT_E that causes plastic deformation (assume α_T, σ_Y, E are T-independent).

D. **Optional/bonus**: repeat (a) for the case where there are 5 bars, or better yet, $2n+1$. The setup is shown in Figure 1b. Assume the system is symmetric with respect to horizontal reflection.

2 Spatio-temporal dynamics and irreversible processes

Consider a 1D membrane (1D thin film) that is bounded to a substrate. When the membrane is pulled away from the substrate by external forces it may peel off. Our goal here is to understand the conditions for this to happen and the spatially-extended peeling dynamics that emerge.

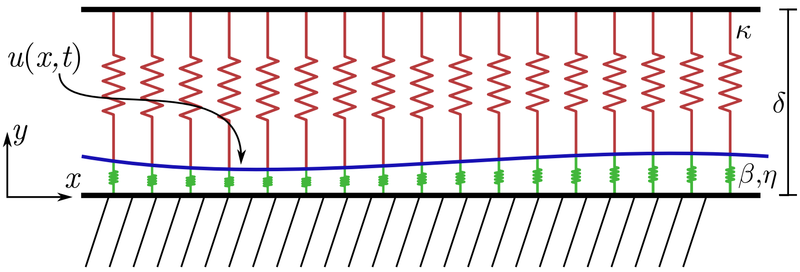


Figure 2: See text for description.

A coordinate system is shown in Fig. 2, where the x is the direction along the substrate and y is the direction perpendicular to it. The membrane is characterized by a linear mass density ρ and interacts with the substrate through a force density (i.e., force per unit length) $f(\cdot)$, which is a yet unspecified function of the dynamical fields in the problem. $f(\cdot)$ represents the adhesive interaction of the membrane with the substrate. An array of dense *linear* springs is attached to the membrane from above. When the membrane is loaded externally, the array of linear springs is moved vertically, i.e., in the y direction. The loading we consider corresponds to the array of linear springs moved vertically a distance δ from the substrate and remains fixed independently of the response of the membrane (fixed-displacement boundary condition), cf. Fig. 2. The magnitude of δ is controlled externally. As a result of the external loading, the membrane features some time-dependent displacement field $u(x,t)\hat{\boldsymbol{y}}$ that we would like to investigate.

A. **Equation of motion**. The equation of motion of the membrane in the continuum limit is

$$\rho\partial_{tt}u = \partial_x\sigma + \kappa(\delta - u) - f(u, \partial_t u) \ . \tag{1}$$

Explain the physical meaning of all of the symbols appearing in the equation, explicitly state the dimensions (=units) of the physical quantities they represent and interpret the physical significance of the 4 terms appearing (1 on the left-hand-side and 3 on the right-hand-side), including the signs.

B. **Constitutive relations**. The membrane is described by the following linear elastic constitutive relation $\sigma = \mu\,\partial_x u$. The adhesive interaction of the membrane with the substrate is linear visco-elastic, as long as the separation does not exceed u_0. When it does, irreversible rupture occurs and the interaction vanishes, cf. Fig. 3. This is described by

$$f(u, \partial_t u) = (\beta\, u + \eta\, \partial_t u)\, H(u_0 - u) \;, \tag{2}$$

where $H(\cdot)$ is Heaviside's step function (the springs do not reconnect after rupture under any condition). Explain the physical meaning of β and η, state their dimensions and write down the equation of motion for $u(x,t)$.

C. **Non-dimensionalization**. Write a nondimensional version of Eq. (1), by measuring length in units of u_0, velocities in units of the wave-speed c (define it!) and forces in units of μ. State explicitly the expressions for the non-dimensional quantities (say, dimensionless time is $\tilde{t}$, the dimensionless space coordinate is $\tilde{x}$ etc.) and write down the dimensionless equation of motion (for the ease of notation, you may drop the tildes in the final expression and onwards).

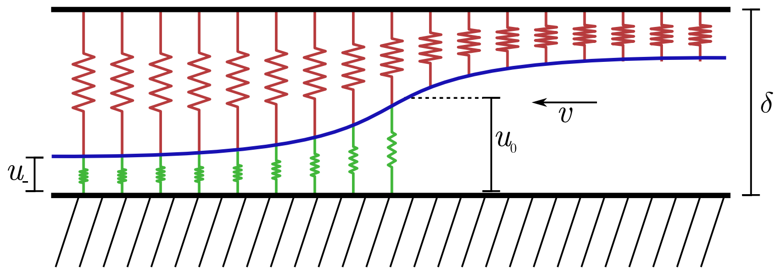


Figure 3: See text for description.

D. **Decohesion (peeling) fronts**.

(a) What is the maximal loading δ_{max} such that the membrane can support a static, homogeneous (= space-independent) solution?

(b) As we discussed several times in the course (e.g., dislocation-mediated plastic flow or crack-mediated failure), the separation ("decohesion/peeling") process does not take place homogeneously in space, but is generically mediated by the propagation of localized spatiotemporal objects ("defects"), which allow rupture to occur at loading below δ_{max}. In this system, these are decohesion (peeling) fronts that mark the transition between the region in space where the adhesion interaction f is finite and the region in space where it vanishes. Assume that the decohesion front travels at a steady speed v (already dimensionless) in the *negative* x direction, cf. Fig. 3. Define a coordinate system moving with the tip of the decohesion (peeling) front, say ξ, and write down the equation of motion for $u(\xi)$. Define $\xi = 0$ as the point where rupture occurs.

(c) Calculate $u(\xi \to -\infty)$ and denote it by u_-.

(d) Calculate the shape of the front $u(\xi)$ for $\xi > 0$ and $\xi < 0$. In particular, show that the solution for $\xi < 0$ is

$$u(\xi) = u_- + (1 - u_-) \exp\left[\left(\frac{\eta v + \sqrt{\eta^2 v^2 + 4(\beta + \kappa)(1 - v^2)}}{2(1 - v^2)}\right)\xi\right] . \tag{3}$$

(e) **Load-velocity relation**. The velocity of front propagation v is determined by the loading δ. Derive an implicit equation that defines the load-velocity relation $v(\delta)$. That is, find a function G that satisfies $G(\delta, v; \kappa, \beta, \eta) = 0$. What is the minimal load δ_c that allows for a decohension front to exist, and what is the front velocity in this case? What is the maximal possible front velocity? Discuss the results.

Optional/bonus items (you are encouraged to consider them):

F. **Dynamic energy balance**. Write down the dynamic energy balance involved in decohesion (peeling) front propagation (hint: multiply your equation of motion by $\partial_\xi u$ and integrate over space). In your result, use u_- and do **NOT** calculate the viscous dissipation contribution explicitly (i.e., leave it as an integral). Explain in detail the physical meaning/significance of the different terms in the equation you obtain (preferably in relation to concepts taught in the course). What happens in the limit $\eta \to 0$?

G. **Validity of the linear constitutive relation**. The membrane constitutive relation was assumed to be linear elastic. Discuss the validity of this assumption in light of your solution (hint: you may want to look at the largest gradient).