Non-Equilibrium Continuum Physics

Extended lecture notes by Eran Bouchbinder

(Dated: May 9, 2025)

This course is intended to introduce graduate students to the essentials of modern continuum physics, with a focus on non-equilibrium phenomena in solids and within a thermodynamic perspective. Special focus is given to emergent phenomena, where collective many-body systems reveal physical principles that cannot be inferred from the microscopic physics of a small number of degrees of freedom. General concepts and principles — such as conservation laws, symmetries, material frame-indifference, dissipation inequalities and non-equilibrium behaviors, spatiotemporal symmetry-breaking instabilities and configurational forces — are emphasized. Examples cover a wide range of physical phenomena and applications in diverse disciplines. The power of field theory as a mathematical structure that does not make direct reference to microscopic length scales well below those of the phenomenon of interest is highlighted. Some basic mathematical tools and techniques are introduced. The course highlights essential ideas and basic physical intuition. Together with courses on fluid mechanics and soft condensed matter, a broad background and understanding of continuum physics will be established.

The course will be given within a framework of 12-13 two-hour lectures and 12-13 two-hour tutorial sessions with a focus on problem-solving. No prior knowledge of the subject is assumed. Basic knowledge of statistical thermodynamics, vector calculus, partial differential equations, dynamical systems and complex analysis is required.

These extended lecture notes (book draft) are self-contained and in principle no other materials are needed.

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Tutorials: Wednesday, 09:15–11:00

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Final grade:

About 6-7 problems sets throughout the semester (40%) and final exam/assignment (60%).

Attendance:

Expected and highly encouraged (both lectures and tutorial sessions).

General Principles and Concepts

I. INTRODUCTION: BACKGROUND AND MOTIVATION

We start by considering the course's title. By 'non-equilibrium' we refer to physical phenomena that cannot be properly treated in the framework of equilibrium thermodynamics. That is, we refer to phenomena that involve irreversible processes and dissipation. We will, however, make an effort to adhere as much as possible to thermodynamic formulations (i.e., we will not focus on purely dynamical systems) and also devote time to reversible phenomena (both because they are often missing from current physics education and because they set the stage for discussing irreversible phenomena). By 'continuum' we refer to the scientific approach that treats macroscopic phenomena without making explicit reference to the discreteness of matter or more generally to microscopic length and time scales. This also implies that we focus on collective phenomena that involve spatially extended systems and a macroscopic number of degrees of freedom (atoms, molecules, grains etc.). We therefore treat materials as continua and use the language of field theory to describe the phenomena of interest. A crucial concept in this context is that of emergent phenomena, which refers to the fundamental idea that collective many-body systems reveal laws/behavior that cannot be inferred from microscopic laws of physics and a small number of degrees of freedom; that is, "More is Different", adopting the famous title of Philip W. Anderson (see Science **177**, 393 (1972)).

'Physics' is surely a bit too broad here, yet it represents the idea that the tools and concepts that will be discussed have a very broad range of applications in different branches of physics. In addition, the topics considered can be discussed from various perspectives — such as applied mathematics, engineering sciences and materials science —, but we will adopt a physicist perspective. To make 'physics' even more specific in the present context, we note that we will mainly focus on thermal and mechanical phenomena, rather than electrical, magnetic or chemical phenomena. By 'thermal' and 'mechanical' — or 'thermomechanical' we refer to material phenomena that involve deformation, material and heat flow and failure, and where the driving forces are thermal and mechanical in nature. 'Classical continuum mechanics' typically refers to 'solid mechanics' and 'fluid mechanics' from a classical (i.e., non-quantum) physics perspective. In this course we will mainly focus on solids in the broadest sense of the word.

The word 'solid' is not easily defined. The most intricate aspect of such a definition is that it involves an observation timescale (at least if we do not consider single crystals). However, for the purpose of this course, it will be sufficient to define a solid as a material that can support shear forces over sufficiently long timescales. We therefore do not focus on Newtonian fluids and very soft materials (though we certainly mention them), both of which are discussed in complementary courses. Nevertheless, we will discuss solid phenomena such as visco-elasticity and nonlinear elasticity.

Why should one study the subjects taught in this course? Well, there are many (good) reasons. Let us mention a few of them. First, macroscopic physics deals with emergent phenomena that cannot be understood from microscopic laws applied to a small number of constituent elements (degrees of freedom). That is, macroscopic systems feature new qualitative coarse-grained properties and dynamics. This is a deep conceptual, to some extent even philosophical, issue that should be systematically introduced. Second, many of the macroscopic phenomena around us are both non-equilibrium and thermomechanical in nature. This course offers tools to understand some of these phenomena. Third, continuum physics phenomena, and solid-related phenomena in particular, are ubiquitous in many branches of science and therefore understanding them may be very useful for researchers in a broad range of disciplines. Fourth, the conceptual and mathematical tools of non-equilibrium thermodynamics and field theory are extremely useful in many branches of science, and thus constitute an important part of scientific education. Finally, some of the issues discussed in this course are related to several outstanding unsolved problems. Hence, the course will expose students to the beauty and depth of a fundamental and active field of research. It would be impossible to even scratch the surface of the huge ongoing solid-related activity. Let us mention a few examples: **(i)** It has been quite recently recognized that the mechanics of living matter, cells in particular, plays a central role in biology. For example, it has been discovered that the stiffness of the substrate on which stem cells grow can significantly affect their differentiation. **(ii)** Biomimetics: researchers have realized that natural/biological systems exhibit superior mechanical properties, and hence aim at mimicking the design principles of these systems in man-made ones. For example, people have managed to build superior adhesives based on Gecko's motion on a wall. People have succeeded in synthesizing better composite materials based on the structures observed in hard tissues, such as cortical bone and dentin. **(iii)** The efforts to understand the physics of driven disordered systems (granular materials, molecular glasses, colloidal suspensions etc.) are deeply related to one of the most outstanding questions in non-equilibrium statistical physics. **(iv)** People have recently realized there are intimate relations between geometry and mechanics. For example, by controlling the intrinsic metric of materials, macroscopic shapes can be explained and designed. **(v)** The rupture of materials and interfaces has a growing influence on our understanding and control of the world around us. For example, there are exciting developments in understanding Earthquakes, the failure of interfaces between two tectonic plates in the Earth's crust **(vi)** Developments in understanding the plastic deformation of amorphous and crystalline solids offer deep new insights about strongly nonlinear and dissipative systems, and open the way to new and exciting applications.

Unfortunately, due to time limitations, the course cannot follow a *historical perspective* which highlights the evolution of the developed ideas. These may provide very important scientific, sociological and psychological insights, especially for research students and young researchers. Whenever possible, historical notes will be made.

II. MATHEMATICAL PRELIMINARIES: TENSOR ANALYSIS

The fundamental assumption of continuum physics is that under a wide range of conditions we can treat materials as *continuous* in space and time, disregarding their discrete structure and time-evolution at microscopic length and time scales, respectively. Therefore, we can ascribe to each point in space-time physical properties that can be described by continuous functions, i.e., *fields*. This implies that derivatives are well defined and hence that we can use the powerful tools of differential calculus. In order to understand what kind of continuous functions, hereafter termed fields, should be used, we should bear in mind that physical laws must be independent of the position and orientation of an observer, and the time of observation (note that we restrict ourselves to classical physics, excluding the theory of relativity). We are concerned here, however, with the mathematical objects that allow us to formulate this and related principles. Most generally, we are interested in the language that naturally allows a mathematical formulation of continuum physical laws. The basic ingredients in this language are *tensor fields*, which are the major focus on the opening part of the course.

Tensor fields are characterized, among other things, by their *order* (sometimes also termed *rank*). Zero-order tensors are *scalars*, for example the temperature field $T(\boldsymbol{x}, t)$ within a body, where $\boldsymbol{x}$ is a 3-dimensional Euclidean space and t is time. First-order tensors are *vectors*, for example the velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ of a fluid. **Why do we need to consider objects that are higher-order than vectors?** The best way to answer this question is through an example. Consider a material areal element and the force acting on it (if the material areal element is a surface element, then the force is applied externally and if the material areal element is inside the bulk material, then the force is exerted by neighboring material). The point is that both the areal element and the force acting on it are basically vectors, i.e., they both have an orientation (the orientation of the areal element is usually quantified by the direction of the normal to it). Therefore, in order to characterize this physical situation one should say that a force in the ith direction is acting on a material areal element whose normal points in the jth direction. The resulting object is defined using two vectors, but it is not a vector itself. We need a higher-order tensor to describe it.

Our main interest here is second-order tensors, which play a major role in continuum physics. A second-order tensor $\boldsymbol{A}$ can be viewed as a linear operator or a linear function that maps a vector, say $\boldsymbol{u}$, to a vector, say $\boldsymbol{v}$,

$$\boldsymbol{v} = \boldsymbol{A}\boldsymbol{u} \, . \tag{2.1}$$

Linearity implies that

$$\boldsymbol{A}(\alpha\boldsymbol{u}+\boldsymbol{v}) = \alpha\boldsymbol{A}\boldsymbol{u}+\boldsymbol{A}\boldsymbol{v} \ , \tag{2.2}$$

for every scalar α and vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. For brevity, second-order tensors will be usually referred to simply as tensors (zero-order tensors will be termed scalars, first-order tensors will be termed vectors and higher than second-order tensors will be explicitly referred to according to their order).

The most natural way to define (or express) tensors in terms of vectors is through the *dyadic* (or *tensor*) product of orthonormal base vectors $\{\boldsymbol{e}_i\}$

$$\boldsymbol{A} = A_{ij}\,\boldsymbol{e}_i\otimes\boldsymbol{e}_j \ , \tag{2.3}$$

where Einstein summation convention is adopted, $\{A_{ij}\}$ is a set of numbers and $\{i,j\}$ run over space dimensions. For those who feel more comfortable with Dirac's Bra-Ket notation, the dyadic product above can be also written as $\boldsymbol{A} = A_{ij}\,|\boldsymbol{e}_i\!><\!\boldsymbol{e}_j|$. In general, the dyad $\boldsymbol{u}\otimes\boldsymbol{v}$ is defined as

$$\boldsymbol{u}\otimes\boldsymbol{v} = \boldsymbol{u}\boldsymbol{v}^T \ , \tag{2.4}$$

where vectors are assumed to be represented by column vectors and the superscript T denotes the transpose operation. If $\{\boldsymbol{e}_i\}$ is an orthonormal set of Cartesian base vectors, we have (for example)

$$\boldsymbol{e}_2\otimes\boldsymbol{e}_3 = \boldsymbol{e}_2\boldsymbol{e}_3^T = \begin{pmatrix}0\\1\\0\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix} \ . \tag{2.5}$$

Therefore, second-order tensors can be directly *represented* by matrices. Thus, tensor algebra essentially reduces to matrix algebra. It is useful to note that for every three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ we have

$$\boldsymbol{u}\otimes\boldsymbol{v}\,\boldsymbol{w} = (\boldsymbol{v}\cdot\boldsymbol{w})\,\boldsymbol{u} \ . \tag{2.6}$$

where $\cdot$ is the usual inner (dot) product of vectors. In the Bra-Ket notation the above simply reads $|\boldsymbol{u}\!><\!\boldsymbol{v}|\boldsymbol{w}\!>$. This immediately allows us to rewrite Eq. (2.1) as

$$v_i\boldsymbol{e}_i = \boldsymbol{v} = \boldsymbol{A}\boldsymbol{u} = (A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j)(u_k\boldsymbol{e}_k) = A_{ij}u_k(\boldsymbol{e}_j\cdot\boldsymbol{e}_k)\,\boldsymbol{e}_i = A_{ij}u_j\boldsymbol{e}_i \ , \tag{2.7}$$

which shows that the matrix representation preserves known properties of matrix algebra ($v_i = A_{ij}u_j$). The matrix representation allows us to define additional tensorial operators. For example, we can define

$$\mathrm{tr}(\boldsymbol{A}) \equiv \boldsymbol{e}_k \cdot (A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j)\,\boldsymbol{e}_k = A_{ij} < \boldsymbol{e}_k|\boldsymbol{e}_i > < \boldsymbol{e}_j|\boldsymbol{e}_k > = A_{ij}\delta_{ik}\delta_{jk} = A_{kk} \ , \tag{2.8}$$

$$\boldsymbol{A}^T = (A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j)^T = A_{ij}\boldsymbol{e}_j \otimes \boldsymbol{e}_i = A_{ji}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \ , \tag{2.9}$$

$$\boldsymbol{AB} = (A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j)(B_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l) = A_{ij}B_{kl}\delta_{jk}\boldsymbol{e}_i \otimes \boldsymbol{e}_l = A_{ij}B_{jl}\boldsymbol{e}_i \otimes \boldsymbol{e}_l \ . \tag{2.10}$$

We can define the *double dot product* (or the *contraction*) of two tensors as

$$\begin{aligned}\boldsymbol{A} : \boldsymbol{B} &= (A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (B_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l) \equiv A_{ij}B_{kl}(\boldsymbol{e}_i \cdot \boldsymbol{e}_k)(\boldsymbol{e}_j \cdot \boldsymbol{e}_l) \\ &= A_{ij}B_{kl}\delta_{ik}\delta_{jl} = A_{ij}B_{ij} = \mathrm{tr}(\boldsymbol{A}\,\boldsymbol{B}^T) \ .\end{aligned} \tag{2.11}$$

This is a natural way of generating a scalar out of two tensors, which is the tensorial generalization of the usual vectorial dot product (hence the name). It plays an important role in the thermodynamics of deforming bodies. Furthermore, it allows us to project a tensor on a base dyad

$$(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : \boldsymbol{A} = (\boldsymbol{e}_i \otimes \boldsymbol{e}_j) : (A_{kl}\boldsymbol{e}_k \otimes \boldsymbol{e}_l) = A_{kl}(\boldsymbol{e}_i \cdot \boldsymbol{e}_k)(\boldsymbol{e}_j \cdot \boldsymbol{e}_l) = A_{kl}\delta_{ik}\delta_{jl} = A_{ij} \ , \tag{2.12}$$

i.e., to extract a component of a tensor.

We can now define the identity tensor as

$$\boldsymbol{I} = \delta_{ij}(\boldsymbol{e}_i \otimes \boldsymbol{e}_j) \ , \tag{2.13}$$

which immediately allows to define the inverse of a tensor (when it exists) following

$$\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I} \ . \tag{2.14}$$

The existence of the inverse is guaranteed when $\det \boldsymbol{A} \neq 0$, where the determinant of a tensor is defined using the determinant of its matrix representation. Note also that one can decompose any second-order tensor to a sum of symmetric and skew-symmetric (antisymmetric) parts as

$$\boldsymbol{A} = \boldsymbol{A}_{sym} + \boldsymbol{A}_{skew} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T) + \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T) \ . \tag{2.15}$$

Occasionally, physical constraints render the tensors of interest symmetric, i.e., $\boldsymbol{A} = \boldsymbol{A}^T$. In this case, we can diagonalize the tensor by formulating the eigenvalue problem

$$\boldsymbol{A}\boldsymbol{a}_i = \lambda_i \boldsymbol{a}_i \ , \tag{2.16}$$

where $\{\lambda_i\}$ and $\{\boldsymbol{a}_i\}$ are the eigenvalues (principal values) and the orthonormal eigenvectors (principal directions), respectively. This problem is analogous to finding the roots of

$$\det(\boldsymbol{A} - \lambda \boldsymbol{I}) = -\lambda^3 + \lambda^2 I_1(\boldsymbol{A}) - \lambda I_2(\boldsymbol{A}) + I_3(\boldsymbol{A}) = 0 \ , \tag{2.17}$$

where the *principal invariants* $\{I_i(\boldsymbol{A})\}$ are given by

$$I_1(\boldsymbol{A}) = \mathrm{tr}(\boldsymbol{A}), \quad I_2(\boldsymbol{A}) = \frac{1}{2}\left[\mathrm{tr}^2(\boldsymbol{A}) - \mathrm{tr}(\boldsymbol{A}^2)\right], \quad I_3(\boldsymbol{A}) = \det(\boldsymbol{A}) \ . \tag{2.18}$$

Note that the symmetry of $\boldsymbol{A}$ ensures that the eigenvalues are real and that an orthonormal set of eigenvectors can be constructed. Therefore, we can represent any symmetric tensor as

$$\boldsymbol{A} = \lambda_i \, \boldsymbol{a}_i \otimes \boldsymbol{a}_i \ , \tag{2.19}$$

assuming no degeneracy. This is called the *spectral decomposition* of a symmetric tensor $\boldsymbol{A}$. It is very useful because it represents a tensor by 3 real numbers and 3 unit vectors. It also allows us to define functions of tensors. For example, for positive definite tensors ($\lambda_i > 0$), we can define

$$\ln(\boldsymbol{A}) = \ln(\lambda_i)\, \boldsymbol{a}_i \otimes \boldsymbol{a}_i \ , \tag{2.20}$$

$$\sqrt{\boldsymbol{A}} = \sqrt{\lambda_i}\, \boldsymbol{a}_i \otimes \boldsymbol{a}_i \ . \tag{2.21}$$

In general, one can define functions of tensors that are themselves scalars, vectors or tensors. Consider, for example, a scalar function of a tensor $f(\boldsymbol{A})$ (e.g., the energy density of a deforming solid). Consequently, we need to consider *tensor calculus*. For example, the derivative of $f(\boldsymbol{A})$ with respect to $\boldsymbol{A}$ is a tensor which takes the form

$$\frac{\partial f}{\partial \boldsymbol{A}} = \frac{\partial f}{\partial A_{ij}} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \ . \tag{2.22}$$

The differential of $f(\boldsymbol{A})$ is a scalar and reads

$$df = \frac{\partial f}{\partial \boldsymbol{A}} : d\boldsymbol{A} = \frac{\partial f}{\partial A_{ij}} dA_{ij} \ . \tag{2.23}$$

Consider then a tensorial function of a tensor $\boldsymbol{F}(\boldsymbol{A})$, which is encountered quite regularly in continuum physics. Its derivative $\boldsymbol{D}$ is defined as

$$\begin{aligned} \boldsymbol{D} = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{A}} = \frac{\partial \boldsymbol{F}}{\partial A_{ij}} \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j = \frac{\partial F_{kl}}{\partial A_{ij}} \boldsymbol{e}_k \otimes \boldsymbol{e}_l \otimes \boldsymbol{e}_i \otimes \boldsymbol{e}_j \ , \\ \Longrightarrow D_{klij} = \frac{\partial F_{kl}}{\partial A_{ij}} \ , \end{aligned} \tag{2.24}$$

which is a fourth-order tensor.

We will now define some differential operators that either produce tensors or act on tensors. First, consider a vector field $\boldsymbol{v}(\boldsymbol{x})$ and define its gradient as

$$\nabla \boldsymbol{v} = \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{v}}{\partial x_j} \otimes \boldsymbol{e}_j = \frac{\partial v_i}{\partial x_j} \boldsymbol{e}_i \otimes \boldsymbol{e}_j \ , \tag{2.25}$$

which is a second-order tensor. Then, consider the divergence of a tensor

$$\nabla\cdot\boldsymbol{A} = \frac{\partial \boldsymbol{A}}{\partial x_k}\boldsymbol{e}_k = \frac{\partial A_{ij}}{\partial x_k}\boldsymbol{e}_i \otimes \boldsymbol{e}_j \boldsymbol{e}_k = \frac{\partial A_{ij}}{\partial x_j}\boldsymbol{e}_i \ , \tag{2.26}$$

which is a vector. The last two objects are extensively used in continuum physics.

The tensorial version of Gauss' theorem for relating volume integrals to surface integrals reads

$$\int_V \nabla\cdot\boldsymbol{A}\, dV = \oint_S \boldsymbol{A}\boldsymbol{n}\, dS \ , \tag{2.27}$$

where V and S are the volume and the enclosing surface, respectively, and $\boldsymbol{n}$ is the outward unit normal to the surface. Obviously, the theorem is satisfied for scalars and vectors as well. It would be useful to recall also Stokes' theorem for relating line integrals to surface integrals

$$\int_S (\nabla\times\boldsymbol{v})\cdot\boldsymbol{n} dS = \oint_l \boldsymbol{v}\cdot d\boldsymbol{l} \ , \tag{2.28}$$

where S and l are the surface and its bounding curve, respectively, and $\boldsymbol{n}$ is the outward unit normal to the surface.

Finally, we should ask ourselves how do tensors transform under a coordinate transformation (from $\boldsymbol{x}$ to $\boldsymbol{x}'$)

$$\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x} \ , \tag{2.29}$$

where $\boldsymbol{Q}$ is a proper ($\det\boldsymbol{Q}=1$) orthogonal transformation matrix $\boldsymbol{Q}^T=\boldsymbol{Q}^{-1}$ (note that it is not a tensor). In order to understand the transformation properties of the orthonormal base vectors $\{\boldsymbol{e}_i\}$ we first note that

$$\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x} \Longrightarrow \boldsymbol{x} = \boldsymbol{Q}^T\boldsymbol{x}' \Longrightarrow x_i = Q^T_{ij}x'_j = Q_{ji}x'_j \ . \tag{2.30}$$

A vector is an object that retains its (geometric) identity under a coordinate transformation. For example, a general position vector $\boldsymbol{r}$ can be *represented* using two different base vectors sets $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{e}'_i\}$ as

$$\boldsymbol{r} = x_i\boldsymbol{e}_i = x'_j\boldsymbol{e}'_j \ . \tag{2.31}$$

Using Eq. (2.30) we obtain

$$x_i\boldsymbol{e}_i = (Q_{ji}x'_j)\boldsymbol{e}_i = x'_j(Q_{ji}\boldsymbol{e}_i) = x'_j\boldsymbol{e}'_j \ , \tag{2.32}$$

which implies

$$\boldsymbol{e}'_i = Q_{ij}\boldsymbol{e}_j \ . \tag{2.33}$$

In order to derive the transformation law for tensors representation we first note that tensors, like vectors, are objects that retain their (geometric) identity under a coordinate transformation and therefore we must have

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i \otimes \boldsymbol{e}_j = A'_{ij}\boldsymbol{e}'_i \otimes \boldsymbol{e}'_j \ . \tag{2.34}$$

Using Eq. (2.33) we obtain

$$\boldsymbol{A} = A'_{ij}\boldsymbol{e}'_i \otimes \boldsymbol{e}'_j = A'_{ij}Q_{ik}\boldsymbol{e}_k \otimes Q_{jl}\boldsymbol{e}_l = (A'_{ij}Q_{ik}Q_{jl})\boldsymbol{e}_k \otimes \boldsymbol{e}_l \ . \tag{2.35}$$

which implies

$$A_{kl} = A'_{ij}Q_{ik}Q_{jl} \ . \tag{2.36}$$

This is the transformation law for the components of a tensor and in many textbooks it serves as a definition of a tensor. Eq. (2.36) can be written in terms of matrix representation as

$$[\boldsymbol{A}] = \boldsymbol{Q}^T[\boldsymbol{A}]'\boldsymbol{Q} \Longrightarrow [\boldsymbol{A}]' = \boldsymbol{Q}[\boldsymbol{A}]\boldsymbol{Q}^T \ , \tag{2.37}$$

where $[\cdot]$ is the matrix representation of a tensor with respect to a set of base vectors. Though we did not make the explicit distinction between a tensor and its matrix representation earlier, it is important in the present context; $[\boldsymbol{A}]$ and $[\boldsymbol{A}]'$ are different representations of the same object, the tensor $\boldsymbol{A}$, but **not** different tensors. An isotropic tensor is a tensor whose representation is independent of the coordinate system, i.e.,

$$A_{ij} = A'_{ij} \quad \text{or} \quad [\boldsymbol{A}] = [\boldsymbol{A}]' \ . \tag{2.38}$$

We note in passing that in the present context we do not distinguish between covariant and contravariant tensors, a distinction that is relevant for non-Cartesian tensors (a *Cartesian tensor* is a tensor in three-dimensional Euclidean space for which a coordinate transformation $\boldsymbol{x}' = \boldsymbol{Q}\boldsymbol{x}$ satisfies $\partial x'_i/\partial x_j = \partial x_j/\partial x'_i$).

III. MOTION, DEFORMATION AND STRESS

Solid materials are deformed under applied driving forces. In order to describe the deformation of solids, consider a body at a given time, typically in the absence of external driving forces, and assign to each material point a position vector $\boldsymbol{X}$ with respect to some fixed coordinate system (i.e., we already use the continuum assumption). For simplicity, set $t\!=\!0$. You can think of $\boldsymbol{X}$ as the label of each point in the body.

At $t\!>\!0$ the body experiences some external forcing that deforms it to a state in which each material point is described by a position vector $\boldsymbol{x}$. We then define the **motion** as the following mapping

$$\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{X}, t) = \boldsymbol{\varphi}(\boldsymbol{X}, t) \ . \tag{3.1}$$

The vector function $\boldsymbol{\varphi}(\cdot)$ maps each point in the initial state $\boldsymbol{X}$ to a point in the current state $\boldsymbol{x}$ at $t\!>\!0$. This immediately implies that $\boldsymbol{X} = \boldsymbol{\varphi}(\boldsymbol{X}, t=0)$, i.e., at time $t=0$ $\boldsymbol{\varphi}(\cdot)$ is the identity vector. The initial state $\boldsymbol{X}$ is usually termed the *reference/undeformed configuration* and the current state is termed the *current/deformed configuration*. We assume that $\boldsymbol{\varphi}(\cdot)$ is a one-to-one mapping, i.e., that it can be inverted

$$\boldsymbol{X} = \boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t) \ . \tag{3.2}$$

The inverse mapping $\boldsymbol{\varphi}^{-1}(\cdot)$ tells us where a material point, that is currently at $\boldsymbol{x}$, was at time $t=0$.

Obviously, our goal is to describe the properties and spatiotemporal dynamics of the current state of the material at $t>0$. This can be done either using the $\boldsymbol{X}$ labeling, which is called the *material (Lagrangian) description*, or the $\boldsymbol{x}$ positions, which is called the *spatial (Eulerian) description*. The choice between these descriptions is a matter of convenience. For a given physical phenomenon under consideration, one description may be more convenient than the other. We will discuss this issue later in the course.

A quantity of fundamental importance is the *displacement field* defined as

$$\boldsymbol{U}(\boldsymbol{X}, t) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X} \ . \tag{3.3}$$

This material description can be converted into a spatial description following

$$\boldsymbol{U}(\boldsymbol{X}, t) = \boldsymbol{x}(\boldsymbol{X}, t) - \boldsymbol{X} = \boldsymbol{x} - \boldsymbol{X}(\boldsymbol{x}, t) = \boldsymbol{U}(\boldsymbol{\varphi}^{-1}(\boldsymbol{x}, t), t) = \boldsymbol{u}(\boldsymbol{x}, t) \ . \tag{3.4}$$

Note that $\boldsymbol{U}$ and $\boldsymbol{u}$ are different functions of different arguments, though their values are the same. The velocity and acceleration fields are defined as

$$\boldsymbol{V}(\boldsymbol{X},t) = \partial_t \boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{v}(\boldsymbol{x},t) \quad \text{and} \quad \boldsymbol{A}(\boldsymbol{X},t) = \partial_{tt} \boldsymbol{U}(\boldsymbol{X},t) = \boldsymbol{a}(\boldsymbol{x},t)\ . \tag{3.5}$$

The corresponding spatial descriptions can be easily obtained using $\boldsymbol{\varphi}(\cdot)$.

The material time derivative D/Dt, which we abbreviate by D_t, is defined as the partial derivative with respect to time, keeping the Lagrangian coordinate $\boldsymbol{X}$ fixed. For a material field $\mathcal{F}(\boldsymbol{X},t)$ (scalar or vector. For a tensor, see the discussion of objectivity/frame-indifference later in the course) we have

$$D_t \mathcal{F}(\boldsymbol{X},t) \equiv \left(\partial_t \mathcal{F}(\boldsymbol{X},t)\right)_{\boldsymbol{X}}\ , \tag{3.6}$$

where we stress that $\boldsymbol{X}$ is held fixed here. This derivative represents the time rate of change of a field $\mathcal{F}$, as seen by an observer moving with a particle that was at $\boldsymbol{X}$ at time $t=0$. We can then ask ourselves what happens when we operate with the material derivative on an Eulerian field $f(\boldsymbol{x},t)$. Using the definition in Eq. (3.6), we obtain

$$\begin{aligned} \frac{Df(\boldsymbol{x},t)}{Dt} &= \left(\frac{\partial f(\boldsymbol{\varphi}(\boldsymbol{X},t),t)}{\partial t}\right)_{\boldsymbol{X}=\boldsymbol{\varphi}^{-1}(\boldsymbol{x},t)} \\ &= \left(\frac{\partial f(\boldsymbol{x},t)}{\partial t}\right)_{\boldsymbol{x}} + \left(\frac{\partial f(\boldsymbol{x},t)}{\partial \boldsymbol{x}}\right)_t \left(\frac{\partial \boldsymbol{\varphi}(\boldsymbol{X},t)}{\partial t}\right)_{\boldsymbol{X}=\boldsymbol{\varphi}^{-1}(\boldsymbol{x},t)}\ . \end{aligned} \tag{3.7}$$

The last term in the above expression is the velocity field, cf. Eq. (3.5), implying that

$$\frac{D(\cdots)}{Dt} = \frac{\partial(\cdots)}{\partial t} + v_k \frac{\partial(\cdots)}{\partial x_k}\ . \tag{3.8}$$

The second contribution on the right hand side of the above equation is termed the *convective rate of change* and hence the material derivative of an Eulerian field is sometimes called the *convective derivative*. Finally, note that since the material derivative of an Eulerian field is just the total time derivative of the Eulerian field, viewing $\boldsymbol{x}(t)$ as a function of time, it is sometimes denoted by a superimposed dot, i.e., $\dot{f}(\boldsymbol{x},t) = D_t f(\boldsymbol{x},t)$. If $f(\boldsymbol{x},t)$ is the velocity field we obtain

$$D_t \boldsymbol{v}(\boldsymbol{x},t) = \partial_t \boldsymbol{v}(\boldsymbol{x},t) + \boldsymbol{v}(\boldsymbol{x},t) \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v}(\boldsymbol{x},t)\ . \tag{3.9}$$

The latter nonlinearity is very important in fluid mechanics, though it appears also in the context of elasto-plasticity. Note that we distinguish between the *spatial gradient* $\nabla_{\boldsymbol{x}}$ and the *material gradient* $\nabla_{\boldsymbol{X}}$, which are different differential operators. Fluid flows are usually described using an Eulerian description. Nevertheless, Lagrangian formulations can be revealing, see for example the Lagrangian turbulence simulation at: http://www.youtube.com/watch?v=LHIIn72dRPk

In order to discuss the physics of deformation we need to know how material line elements change their length and orientation. Therefore, we define the *deformation gradient tensor* $\boldsymbol{F}$ that maps an infinitesimal line element in the reference configuration $d\boldsymbol{X}$ to an infinitesimal line element in the deformed configuration

$$d\boldsymbol{x} = \boldsymbol{F}(\boldsymbol{X}, t) d\boldsymbol{X} \ . \tag{3.10}$$

Hence,

$$\boldsymbol{F}(\boldsymbol{X}, t) = \nabla_{\boldsymbol{X}} \varphi(\boldsymbol{X}, t) \ . \tag{3.11}$$

As will become apparent later in the course, $\boldsymbol{F}$ is not a proper tensor, but rather a two-point tensor, i.e., a tensor that relates two configurations. We can further define the *displacement gradient tensor* as

$$\boldsymbol{H}(\boldsymbol{X}, t) = \nabla_{\boldsymbol{X}} \boldsymbol{U}(\boldsymbol{X}, t) \ , \tag{3.12}$$

which implies

$$\boldsymbol{F} = \boldsymbol{I} + \boldsymbol{H} \ . \tag{3.13}$$

Here, and elsewhere, $\boldsymbol{I}$ is the identity tensor. The deformation gradient tensor $\boldsymbol{F}$ describes both the rotation and the stretching of a material line element, which also implies that it is not symmetric. From a basic physics perspective, it is clear that interaction potentials are sensitive to the relative distance between particles, but not to local rigid rotations. Consequently, we are interested in separating rotations from stretching, where the latter quantifies the change in length of material elements. We can, therefore, decompose $\boldsymbol{F}$ as

$$\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R} \ , \tag{3.14}$$

where $\boldsymbol{R}$ is a proper rotation tensor, $\det \boldsymbol{R} = +1$, and $\boldsymbol{U}$ (should not be confused with the displacement field) and $\boldsymbol{V}$ are the right and left stretch tensors, respectively (which are of course symmetric). This is the so-called *polar decomposition.* Note that

$$\boldsymbol{R}\boldsymbol{R}^T = \boldsymbol{I}, \quad \boldsymbol{U} = \boldsymbol{U}^T, \quad \boldsymbol{V} = \boldsymbol{V}^T, \quad \boldsymbol{V} = \boldsymbol{R}\boldsymbol{U}\boldsymbol{R}^T \ . \tag{3.15}$$

Therefore, $\boldsymbol{U}$ and $\boldsymbol{V}$ have the same eigenvalues (principal stretches), but different eigenvectors (principal directions). Hence, we can write the spectral decomposition as

$$\boldsymbol{V} = \lambda_i \, \boldsymbol{N}_i \otimes \boldsymbol{N}_i, \tag{3.16}$$

$$\boldsymbol{U} = \lambda_i \, \boldsymbol{M}_i \otimes \boldsymbol{M}_i, \tag{3.17}$$

with

$$\lambda_i > 0, \quad \boldsymbol{N}_i \otimes \boldsymbol{N}_i = \boldsymbol{R}\boldsymbol{M}_i \otimes \boldsymbol{R}\boldsymbol{M}_i \ . \tag{3.18}$$

A. Strain measures

At this stage, we are interested in constructing quantities that are based on the stretch tensors discussed above in order to be able, eventually, to define the energy of deformation. For this aim, we need to discuss strain measures. Unlike displacements and stretches, which are directly measurable quantities (whether it always make physical sense and over which timescales, will be discussed later), *strain measures* are *concepts* that are defined as function of the stretches, and may be conveniently chosen differently in different physical situations. The basic idea is simple; we would like to come up with a measure of the relative change in length of material line elements. Consider first the scalar (one-dimensional) case. If the reference length of a material element is ℓ_0 and its deformed length is $\ell = \lambda\ell_0$, then a simple strain measure is constructed by

$$g(\lambda) = \frac{\ell - \ell_0}{\ell_0} = \lambda - 1 \ . \tag{3.19}$$

This definition follows our intuitive notion of strain, i.e., (i) It is a monotonically increasing function of the stretch λ (ii) It vanishes when $\lambda = 1$. It is, however, by no means unique. In fact, every monotonically increasing function of λ which reduces to the above definition when λ is close to unity, i.e., satisfies $g(1) = 0$ and $g'(1) = 1$, would qualify. These conditions ensure that upon linearization, all strain measures agree. For example,

$$g(\lambda) = \int_{\ell_0}^{\ell} \frac{d\ell'}{\ell'} = \ln\left(\frac{\ell}{\ell_0}\right) = \ln\lambda \ , \tag{3.20}$$

$$g(\lambda) = \frac{\ell^2 - \ell_0^2}{2\ell_0^2} = \frac{1}{2}\left(\lambda^2 - 1\right) \ . \tag{3.21}$$

Obviously, there are infinitely many more. The three possibilities we presented above, however, are well-motivated from a physical point of view. Before explaining this, we note that the scalar (one-dimensional) definitions adopted above can be easily generalized to rotationally invariant tensorial forms as

$$\boldsymbol{E}_B = (\lambda_i - 1)\, \boldsymbol{M}_i \otimes \boldsymbol{M}_i = \boldsymbol{U} - \boldsymbol{I}, \tag{3.22}$$

$$\boldsymbol{E}_H = (\ln\lambda_i)\, \boldsymbol{M}_i \otimes \boldsymbol{M}_i = \ln\boldsymbol{U}, \tag{3.23}$$

$$\boldsymbol{E} = \frac{1}{2}(\lambda_i^2 - 1)\, \boldsymbol{M}_i \otimes \boldsymbol{M}_i = \frac{1}{2}\left(\boldsymbol{U}^2 - \boldsymbol{I}\right) = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}\right) \ . \tag{3.24}$$

$\boldsymbol{E}_B$ is the Biot (extensional) strain tensor. It is the most intuitive strain measure. Its main disadvantage is that it cannot be directly expressed in terms of the deformation gradient tensor $\boldsymbol{F}$, but rather has to be calculated from it by a polar decomposition. $\boldsymbol{E}_H$ is the Hencky (logarithmic) strain (which is also not expressible in terms of $\boldsymbol{F}$ alone). Its one-dimensional form, Eq. (3.20), clearly demonstrates that $d\boldsymbol{E}_H$ is an incremental strain that measures incremental changes in the length of material line elements relative to their *current* length. Finally, $\boldsymbol{E}$ is the Green-Lagrange (metric) strain. While it is difficult to motivate its one-dimensional form, Eq. (3.21), its tensorial form has a clear physical meaning. To see this, consider infinitesimal line elements of size $d\ell$ and $d\ell'$ in the reference and deformed configurations respectively and construct the following measure of the change in their length

$$(d\ell')^2-(d\ell)^2 = dx_i dx_i - dX_i dX_i = F_{ij}dX_j F_{ik}dX_k - dX_j\delta_{jk}dX_k = \\ 2dX_j\left[\frac{1}{2}\left(F_{ij}F_{ik}-\delta_{jk}\right)\right]dX_k = 2dX_j\left[\frac{1}{2}\left(F^T_{ji}F_{ik}-\delta_{jk}\right)\right]dX_k \equiv 2dX_j E_{jk} dX_k \ . \tag{3.25}$$

Therefore,

$$\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{F}^T\boldsymbol{F}-\boldsymbol{I}\right) = \frac{1}{2}\left(\boldsymbol{C}-\boldsymbol{I}\right) = \frac{1}{2}\left(\boldsymbol{U}^2-\boldsymbol{I}\right) = \frac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^T+\boldsymbol{H}^T\boldsymbol{H}) \ , \tag{3.26}$$

where $\mathbf{C}\equiv\boldsymbol{F}^T\boldsymbol{F}$ is the right Cauchy-Green deformation tensor. So $\boldsymbol{E}$ is indeed a *material* metric strain tensor. Further note that $\boldsymbol{E}$ is quadratically nonlinear in the displacement gradient $\boldsymbol{H}$. The linear part of $\boldsymbol{E}$

$$\boldsymbol{\varepsilon} \equiv \frac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^T) \tag{3.27}$$

is the linear (infinitesimal) strain tensor, which is not a true strain measure (as it is *not* rotationally invariant), but nevertheless is the basic object in the linearized field theory of elasticity (to be discussed later in the course). We can easily derive the *spatial* counterpart of $\boldsymbol{E}$, by having $(d\ell')^2-(d\ell)^2\equiv 2dx_j e_{jk} dx_k$, with (prove)

$$\boldsymbol{e} = \frac{1}{2}\left(\boldsymbol{I}-\boldsymbol{F}^{-T}\boldsymbol{F}^{-1}\right) = \frac{1}{2}\left(\boldsymbol{I}-\boldsymbol{b}^{-1}\right) \ . \tag{3.28}$$

$\boldsymbol{b}\equiv\boldsymbol{F}\boldsymbol{F}^T$ is the left Cauchy-Green deformation tensor (also termed the Finger tensor, which is sometimes denoted by $\boldsymbol{B}$). $\boldsymbol{e}$, known as the Euler-Almansi strain tensor, is a *spatial* metric strain tensor.

The deformation gradient tensor $\boldsymbol{F}$ maps objects from the undeformed to the deformed configuration. For example, consider a volume element in the deformed configuration (assume $\boldsymbol{F}$ has already been diagonalized)

$$d\boldsymbol{x}^3 = dx_1dx_2dx_3 = F_{11}dX_1F_{22}dX_2F_{33}dX_3 = J(\boldsymbol{X},t)d\boldsymbol{X}^3 \ , \tag{3.29}$$

where

$$J(\boldsymbol{X},t) = \det \boldsymbol{F}(\boldsymbol{X},t) \ . \tag{3.30}$$

Consider then a surface element in the undeformed configuration $d\boldsymbol{S} = dS\,\boldsymbol{N}$, where dS an infinitesimal area and $\boldsymbol{N}$ is a unit normal. The corresponding surface element in the deformed configuration is $d\boldsymbol{s} = ds\,\boldsymbol{n}$. To relate these quantities, we consider an arbitrary line element $d\boldsymbol{X}$ going through $d\boldsymbol{S}$ and express the spanned volume element by a dot product $d\boldsymbol{X}^3 = d\boldsymbol{S}\cdot d\boldsymbol{X}$. $d\boldsymbol{X}$ maps to $d\boldsymbol{x}$, which spans a corresponding volume element in the deformed configuration $d\boldsymbol{x}^3 = d\boldsymbol{s}\cdot d\boldsymbol{x}$. Using Eq. (3.29), the relation $d\boldsymbol{s}\cdot \boldsymbol{F}d\boldsymbol{X} = \boldsymbol{F}^T d\boldsymbol{s}\cdot d\boldsymbol{X}$ (i.e., $ds_i F_{ij} dX_j = F^T_{ji} ds_i dX_j$) and the fact that $d\boldsymbol{X}$ is an arbitrary line element, we obtain

$$d\boldsymbol{S} = J^{-1}\boldsymbol{F}^T d\boldsymbol{s} \ . \tag{3.31}$$

The spatial velocity gradient $\boldsymbol{L}(\boldsymbol{x},t)$ is defined as

$$\boldsymbol{L} \equiv \frac{\partial \boldsymbol{v}(\boldsymbol{x},t)}{\partial \boldsymbol{x}} = \dot{\boldsymbol{F}}\boldsymbol{F}^{-1} \ . \tag{3.32}$$

The symmetric part of $\boldsymbol{L}$, $\boldsymbol{D} = \frac{1}{2}(\boldsymbol{L} + \boldsymbol{L}^T)$, is an important quantity called the *rate of deformation* tensor. The anti-symmetric part of $\boldsymbol{L}$, $\boldsymbol{W} = \frac{1}{2}(\boldsymbol{L} - \boldsymbol{L}^T)$, is called the *spin (vorticity)* tensor.

B. The concept of stress

As was mentioned at the beginning of this Chapter, material deformation is induced by forces. In order to describe and quantify forces at the continuum level we need the concept of *stress* (sketched earlier in Chapter II to motivate the need for tensors). Consider a surface element $d\boldsymbol{s}$ in the deformed configuration. It is characterized by an outward normal $\boldsymbol{n}$ and a unit area ds. The surface element can be a part of the external boundary of the body or a part of an imaginary internal surface. The force acting on it, either by external agents in the former case or by neighboring material in the latter case, is denoted by $d\boldsymbol{f}$. We postulate, following Cauchy, that we can define a *traction vector* $\boldsymbol{t}$ such that

$$d\boldsymbol{f} = \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})\,ds \ . \tag{3.33}$$

Cauchy proved that there exists a unique symmetric second-order tensor $\boldsymbol{\sigma}(\boldsymbol{x},t)$ (i.e., $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, the physical meaning of which will be discussed later) such that

$$\boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n}) = \boldsymbol{\sigma}(\boldsymbol{x},t)\,\boldsymbol{n} \ . \tag{3.34}$$

The spatial tensor $\boldsymbol{\sigma}$ is called the *Cauchy stress*. Its physical meaning becomes clear when we write Eq. (3.34) in components form, $t_i = \sigma_{ij} n_j$. Therefore, σ_{ij} is the force per unit area in the ith direction, acting on a surface element whose outward normal has a component n_j in the jth direction. A corollary of Eq. (3.34)

$$\boldsymbol{t}(\boldsymbol{x}, t, -\boldsymbol{n}) = -\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}) \ , \tag{3.35}$$

is nothing but *Newton's third law (action and reaction).*

As $\boldsymbol{\sigma}$ is defined in terms of the deformed configuration, which is not known a priori (one should solve for it using the stresses themselves), $\boldsymbol{\sigma}$ is not always a useful quantity (it is the only relevant quantity in the linearized field theory of elasticity, where we do not distinguish between the deformed and undeformed configurations). To overcome this difficulty, we can define alternative stress measures that are useful for calculations. In general, we will show later that thermodynamics allows us to define for any strain measure a work-conjugate stress measure. Here, we define one such mechanically-motivated stress measure. Let us define a (fictitious) reference configuration traction vector $\boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N})$ as

$$d\boldsymbol{f} = \boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n})\, ds = \boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N})\, dS \ , \tag{3.36}$$

where $\boldsymbol{N}$ and dS are the reference outward normal and unit area, respectively, whose images in the deformed configuration are $\boldsymbol{n}$ and ds, respectively. Following Cauchy, there exists a tensor $\boldsymbol{P}(\boldsymbol{X}, t)$ such that

$$\boldsymbol{T}(\boldsymbol{X}, t, \boldsymbol{N}) = \boldsymbol{P}(\boldsymbol{X}, t)\, \boldsymbol{N} \ . \tag{3.37}$$

$\boldsymbol{P}(\boldsymbol{X}, t)$ is called the *first Piola-Kirchhoff stress tensor*. In fact, it is not a true tensor (it relates quantities from the deformed and undeformed configuration and hence, like $\boldsymbol{F}$, is a two-point tensor) and is not symmetric. Using the above properties, it is straightforward to show that it is related to the Cauchy stress $\boldsymbol{\sigma}$ by

$$\boldsymbol{P} = J \boldsymbol{\sigma} \boldsymbol{F}^{-T} \ . \tag{3.38}$$

The concepts of strain and stress will allow us to formulate physical laws, such as conservation laws and the laws of thermodynamics, and constitutive laws which describe material behaviors, in the rest of this course.

IV. EQUATIONS OF MOTION, THE LAWS OF THERMODYNAMICS AND OBJECTIVITY

A. Conservation laws

We first consider the mass density in the reference configuration $\rho_0(\boldsymbol{X},t)$. The conservation of mass simply implies that

$$M = \int_{\Omega_0} \rho_0(\boldsymbol{X},t) d\boldsymbol{X}^3 \tag{4.1}$$

is time-independent (Ω_0 is the region occupied by the body in the reference configuration), i.e.,

$$\frac{DM}{Dt} = \frac{D}{Dt}\int_{\Omega_0} \rho_0(\boldsymbol{X},t) d\boldsymbol{X}^3 = \frac{D}{Dt}\int_{\Omega} \rho(\boldsymbol{x},t) d\boldsymbol{x}^3 = 0 \ , \tag{4.2}$$

where Ω is the region occupied by the body in the deformed configuration. The integral form can be easily transformed into a local form. In the reference (Lagrangian) configuration it simply reads

$$\frac{D\rho_0}{Dt} = \frac{\partial \rho_0(\boldsymbol{X},t)}{\partial t} = 0 \quad \Longrightarrow \quad \rho_0(\boldsymbol{X},t) = \rho_0(\boldsymbol{X}) \ . \tag{4.3}$$

To obtain the local form in the Eulerian description, note that (by the definition of J, cf., Eq. (3.29)) $\rho_0(\boldsymbol{X})\!=\!\rho(\boldsymbol{x},t)J(\boldsymbol{X},t)$ and $\dot{J}\!=\!J\,\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v}$ (prove). Therefore,

$$\frac{D\rho_0}{Dt} = \frac{D}{Dt}\Big[\rho(\boldsymbol{x},t)J(\boldsymbol{X},t)\Big] = J\frac{D\rho}{Dt} + J\rho\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v} = 0 \ , \tag{4.4}$$

which implies

$$\frac{D\rho(\boldsymbol{x},t)}{Dt} + \rho(\boldsymbol{x},t)\nabla_{\boldsymbol{x}}\cdot\boldsymbol{v}(\boldsymbol{x},t) = \frac{\partial\rho(\boldsymbol{x},t)}{\partial t} + \nabla_{\boldsymbol{x}}\cdot\Big(\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\Big) = 0 \ . \tag{4.5}$$

This expression of local mass conservation (continuity equation) takes the general form of a local conservation law

$$\frac{\partial(\text{field})}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\text{field flux}) = \text{source/sink} \ . \tag{4.6}$$

Let us now discuss a theorem that will be very useful in formulating and manipulating other conservation laws. Consider the following 1D integral involving an Eulerian scalar field $\psi(x,t)$

$$I(t) = \int_{x_1=\varphi(X_1,t)}^{x_2=\varphi(X_2,t)} \psi(x,t)dx \ . \tag{4.7}$$

Note that $X_{1,2}$ are fixed here. Taking the time derivative of $I(t)$ (Leibnitz's rule) we obtain

$$\dot{I}(t) = \int_{\varphi(X_1,t)}^{\varphi(X_2,t)} \partial_t\psi(x,t)dx + \psi\Big(\varphi(X_2,t),t\Big)\partial_t\varphi(X_2,t) - \psi\Big(\varphi(X_1,t),t\Big)\partial_t\varphi(X_1,t) \ . \tag{4.8}$$

First recall that (generally in 3D)

$$\boldsymbol{V}(\boldsymbol{X},t) = \partial_t \boldsymbol{\varphi}(\boldsymbol{X},t) = \boldsymbol{v}(\boldsymbol{x},t) \ . \tag{4.9}$$

Then note that since X_1 and X_2 are fixed in the integral we can interpret the time derivative as a material time derivative D/Dt. Therefore, we can rewrite Eq. (4.8) as

$$\frac{D}{Dt}\int_{\varphi(X_1,t)}^{\varphi(X_2,t)} \psi(x,t)dx = \int_{\varphi(X_1,t)}^{\varphi(X_2,t)} \Big[\partial_t\psi(x,t) + \partial_x\Big(\psi(x,t)v(x,t)\Big)\Big]\, dx \ . \tag{4.10}$$

The immediate generalization of this result to volume integrals over a time dependent domain Ω reads

$$\frac{D}{Dt}\int_{\Omega} \psi(\boldsymbol{x},t)d\boldsymbol{x}^3 = \int_{\Omega} \Big[\partial_t\psi(\boldsymbol{x},t) + \nabla_{\boldsymbol{x}}\cdot\Big(\psi(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\Big)\Big]\, d\boldsymbol{x}^3 \ . \tag{4.11}$$

This is the Reynolds' transport theorem which is very useful in the context of formulating conservation laws. This is the same Osborne Reynolds (1842-1912), who is known for his studies of the transition from laminar to turbulent fluid flows, and who gave the Reynolds number its name.

Using mass conservation, we obtain (prove)

$$\frac{D}{Dt}\int_{\Omega} \rho(\boldsymbol{x},t)\psi(\boldsymbol{x},t)d\boldsymbol{x}^3 = \int_{\Omega} \rho(\boldsymbol{x},t)\frac{D\psi(\boldsymbol{x},t)}{Dt} d\boldsymbol{x}^3 \ . \tag{4.12}$$

This is very useful when we choose $\psi(\boldsymbol{x},t)$ to be a quantity per unit mass. In particular, setting $\psi=1$ we recover the conservation of mass.

Linear momentum balance (Newton's second law) reads

$$\dot{\boldsymbol{P}}(t) = \frac{D}{Dt}\int_{\Omega_0} \rho_0(\boldsymbol{X})\boldsymbol{V}(\boldsymbol{X},t)d\boldsymbol{X}^3 = \frac{D}{Dt}\int_{\Omega} \rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)d\boldsymbol{x}^3 = \boldsymbol{F}(t) \ , \tag{4.13}$$

where $\boldsymbol{F}(t)$ is the total force acting on a volume element Ω (do not confuse $\boldsymbol{P}$ with the first Piola-Kirchhoff stress tensor of Eq. (3.37)). To obtain a local form of this law note that the total force is obtained by integrating local tractions (surface forces) $\boldsymbol{t}(\boldsymbol{x},t)$ and body (volume) forces $\boldsymbol{b}(\boldsymbol{x},t)$, i.e.,

$$\boldsymbol{F}(t) = \int_{\partial\Omega} \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})ds + \int_{\Omega} \boldsymbol{b}(\boldsymbol{x},t)d\boldsymbol{x}^3 \ , \tag{4.14}$$

where $\partial\Omega$ is the boundary of the volume element. Use Cauchy's stress theorem of Eq. (3.34) and the divergence (Gauss) theorem of Eq. (2.27) to obtain

$$\int_{\partial\Omega} \boldsymbol{t}(\boldsymbol{x},t,\boldsymbol{n})ds = \int_{\partial\Omega} \boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}ds = \int_{\Omega} \nabla_{\boldsymbol{x}}\cdot\boldsymbol{\sigma}(\boldsymbol{x},t)d\boldsymbol{x}^3 \ . \tag{4.15}$$

Use then Reynold's transport theorem of Eq. (4.12), with ψ replaced by the spatial velocity field $\boldsymbol{v}$, to transform the linear momentum balance of Eq. (4.13) into

$$\int_{\Omega} \left[\nabla_{\boldsymbol{x}}\cdot\boldsymbol{\sigma}(\boldsymbol{x},t) + \boldsymbol{b}(\boldsymbol{x},t) - \rho(\boldsymbol{x},t)\dot{\boldsymbol{v}}(\boldsymbol{x},t)\right] d\boldsymbol{x}^3 = 0 \ . \tag{4.16}$$

Since this result is valid for an arbitrary material volume, we obtain the following spatial (Eulerian) local form of linear momentum conservation

$$\nabla_{\boldsymbol{x}}\cdot \boldsymbol{\sigma} + \boldsymbol{b} = \rho\,\dot{\boldsymbol{v}} = \rho\,(\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\boldsymbol{v}) \ . \tag{4.17}$$

Note that this equation does not conform with the structure of a general conservation law in Eq. (4.6). This can be achieved (prove), yielding

$$\frac{\partial(\rho\,\boldsymbol{v})}{\partial t} + \nabla_{\boldsymbol{x}}\cdot(\boldsymbol{\sigma} - \rho\,\boldsymbol{v}\otimes\boldsymbol{v}) = \boldsymbol{b} \ . \tag{4.18}$$

A similar analysis can be developed for the angular momentum. However, the requirement that the angular acceleration remains finite implies that angular momentum balance, at the continuum level, is satisfied if the Cauchy stress tensor $\boldsymbol{\sigma}$ is symmetric, i.e.,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \ , \tag{4.19}$$

to be derived in the tutorial. We note that the symmetry of the Cauchy stress tensor emerges from the conservation of angular momentum if the continuum assumption is valid at all lengthscales. Real materials, however, may possess intrinsic lengthscales associated with their microstructure (e.g., grains, fibers and cellular structures). In this case, we need generalized theories which endow each material point with translational and rotational degrees of freedom, describing the displacement and rotation of the underlying microstructure. One such theory is known as Cosserat (micropolar) continuum, which is a continuous collection of particles that behave like rigid bodies. Under such circumstances one should consider a couple-stress tensor (which has the dimensions of stress $\times$ length) as well, write down an explicit angular momentum balance equation and recall that the ordinary stress tensor is no longer symmetric.

The local momentum conservation laws can be expressed in Lagrangian forms. For example, the linear momentum balance, Eq. (4.17), translates into (prove)

$$\nabla_{\boldsymbol{X}}\cdot \boldsymbol{P} + \boldsymbol{B} = \rho_0\dot{\boldsymbol{V}} \ , \tag{4.20}$$

where $\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor of Eq. (3.37) and $\boldsymbol{B}(\boldsymbol{X},t) = J(\boldsymbol{X},t)\boldsymbol{b}(\boldsymbol{x},t)$. This equation is extremely useful because it allows calculations to be done in a fixed undeformed coordinate system $\boldsymbol{X}$. It is important to note that one should also transform the boundary conditions of a given problem from the deformed configuration (where they are physically imposed) to the underformed configuration.

B. The laws of thermodynamics

Equilibrium thermodynamics is a well-established branch of physics, whose modern incarnation is deeply rooted in statistical mechanics. This framework, known as statistical thermodynamics, builds on systematic coarse-graining of statistical descriptions of microscopic dynamics. It gives rise to an effective macroscopic description of large physical systems through a small set of state variables. In the spirit of this course, we do not follow the microscopic route of statistical mechanics (a topic covered in complementary courses), but rather focus on a macroscopic perspective.

Let us consider the balance of mechanical energy (thermal energy is excluded for now and will be discussed soon). The external mechanical power $\mathcal{P}_{ext}$ is simply the rate at which mechanical work is being done by external forces, either boundary traction $\boldsymbol{t}$ or body forces $\boldsymbol{b}$. It reads

$$\mathcal{P}_{ext} = \int_{\partial\Omega} \boldsymbol{t}\cdot\boldsymbol{v} ds + \int_{\Omega} \boldsymbol{b}\cdot\boldsymbol{v} d\boldsymbol{x}^3 \ . \tag{4.21}$$

The external mechanical work is transformed into kinetic energy $\mathcal{K}$ and internal mechanical power $\mathcal{P}_{int}$. These are expressed as

$$\mathcal{K} = \int_{\Omega} \frac{1}{2}\rho\boldsymbol{v}^2\, d\boldsymbol{x}^3 \tag{4.22}$$

and

$$\mathcal{P}_{int} = \int_{\Omega} \boldsymbol{\sigma}:\boldsymbol{L}\, d\boldsymbol{x}^3 = \int_{\Omega} \boldsymbol{\sigma}:\boldsymbol{D}\, d\boldsymbol{x}^3 \ . \tag{4.23}$$

Therefore, mechanical energy balance reads

$$\mathcal{P}_{ext} = \mathcal{P}_{int} + \dot{\mathcal{K}} \ . \tag{4.24}$$

It can be easily proven using Eqs. (4.17) and (4.21).

To arrive at the first law of thermodynamics we need to consider another form of energy — thermal energy. This form of energy accounts for the random (microscopic) motion of particles, which was excluded above. To properly describe this form of energy we need two concepts, that of an internal energy $\mathcal{U}$ and that of thermal power $\mathcal{Q}$. The internal energy accounts for all microscopic forms of energy. Here, we focus on mechanical and thermal energies, but in general electric, magnetic, chemical and other forms of energy can be included. $\mathcal{U}$ can be associated with a density u (per unit mass) and hence

$$\mathcal{U} = \int_{\Omega} \rho(\boldsymbol{x},t)\, u(\boldsymbol{x},t)\, d\boldsymbol{x}^3 \ . \tag{4.25}$$

Note that the total energy is the sum of kinetic and internal energies $\mathcal{K}+\mathcal{U}$. The thermal (heat) power $\mathcal{Q}$ is expressed, as usual, in terms of fluxes and sources

$$\mathcal{Q} = -\int_{\partial\Omega} \boldsymbol{q}\cdot\boldsymbol{n}\, ds + \int_{\Omega} r\, d\boldsymbol{x}^3 \ . \tag{4.26}$$

$\boldsymbol{q}$ is the heat flux and r is a (volumetric) heat source (e.g., radiation). For simplicity we exclude r from the discussion below. Finally, we note that the rate of change of internal energy is the sum of rate of change of internal mechanical energy and thermal power

$$\dot{\mathcal{U}} = \mathcal{P}_{int} + \mathcal{Q} \ , \tag{4.27}$$

which can be regarded as a statement of *the first law of thermodynamics*. Alternatively, by eliminating $\mathcal{P}_{int}$ between Eqs. (4.24) and (4.27) we obtain

$$\dot{\mathcal{K}} + \dot{\mathcal{U}} = \mathcal{P}_{ext} + \mathcal{Q} \ , \tag{4.28}$$

which is yet another statement of the same law. It has a clear physical meaning: external mechanical work and heat supply are transformed into kinetic and internal energies. An important point to note is that this law only tells us that one form of energy can be transformed into another form, but does not tell us anything about the direction of such processes. This global law can be readily transformed into a local form (prove), which reads

$$\rho\, \dot{u} = \boldsymbol{\sigma}:\boldsymbol{D} - \nabla_{\boldsymbol{x}}\cdot\boldsymbol{q} \ . \tag{4.29}$$

It is well known that many physical processes feature a well-defined direction, e.g., heat flows from a higher temperature to a lower one. This is captured by the second law of thermodynamics. To formulate the law we need two additional concepts, entropy and temperature. The total entropy $\mathcal{S}$ is a measure of microscopic "disorder" and is well defined in the framework of statistical mechanics. The absolute temperature T (a non-negative scalar), which is also a well defined statistical mechanical concept, is introduced such that the entropy increase associated with a thermal power $\mathcal{Q}$ is $\mathcal{Q}/T$. The second law then reads

$$\dot{\mathcal{S}} \ge -\int_{\partial\Omega} \frac{\boldsymbol{q}\cdot\boldsymbol{n}}{T}\, ds + \int_{\Omega} \frac{r}{T}\, d\boldsymbol{x}^3 = \frac{\mathcal{Q}}{T} \ , \tag{4.30}$$

where the last equality is valid for space-independent T (otherwise, T is part of the integrands and one cannot globally separate the heat power $\mathcal{Q}$ and the temperature T in the second law). The inequality in (4.30) states that the increase in the entropy of a system is larger than (or equals to) the influx of entropy by heat (thermal power). When the system under consideration is closed (e.g., the universe) the second law states that the entropy is an increasing function of time (or constant), i.e., $\dot{\mathcal{S}} \geq 0$. Note that sometimes the entropy production rate Σ is defined as $\Sigma = \dot{\mathcal{S}} - \mathcal{Q}/T$, which is non-negative.

Every macroscopic physical system, and consequently every theory of such systems, cannot violate the non-negativity of the entropy production rate Σ. This is a serious and very useful constraint on developing continuum theories of non-equilibrium phenomena. Yet, applying the second law constraint to arbitrarily far-from-equilibrium phenomena is not trivial (e.g., it might raise questions about the validity of the entropy concept itself), as will be further discussed later in the course.

The most well-developed application, however, of the second law constraint emerges in the context of *Linear Response Theory*, describing systems that deviate only slightly from equilibrium (i.e., when driving forces are weak). In this case, the entropy production rate Σ can be expressed as a bilinear form in the deviation from equilibrium, defining a set of linear response coefficients. These linear response coefficients must satisfy various constraints to ensure consistency with the second law of thermodynamics. In fact, by invoking microscopic time reversibility Lars Onsager showed that these coefficients possess additional symmetries that go beyond the second law of thermodynamics, known as Onsager's reciprocal relations (derived in 1931). The Nobel Prize in Chemistry in 1968 was awarded to Onsager for this fundamental contribution (the prize citation referred to "the discovery of the reciprocal relations bearing his name, which are fundamental for the thermodynamics of irreversible processes"). It is worth mentioning in the context of linear irreversible thermodynamics that Prigogine showed in 1945 that Onsager's reciprocal relations imply that the entropy production rate Σ attains a minimum under non-equilibrium steady-state conditions (which was one of the major reasons for awarding him the Nobel prize in 1977). Prigogine's principle of minimum entropy production generated considerable excitement at the beginning, but later on it was realized that this result (like Onsager's reciprocal relations) is specific to small deviations from equilibrium (linear response).

Let us consider a simple example of the implications of the second law of thermodynamics for an isolated system that is composed of two subsystems of different temperatures $T_1 > T_2$. The subsystems are separated by a wall that allows heat transport, but not mass or mechanical work transport. Since the system as a whole is isolated and no mechanical work is involved, the first and second laws of thermodynamics read

$$\dot{\mathcal{S}} = \dot{\mathcal{S}}_1 + \dot{\mathcal{S}}_2 \geq 0 \quad \text{and} \quad \dot{\mathcal{U}} = \dot{\mathcal{U}}_1 + \dot{\mathcal{U}}_2 = 0 \ . \tag{4.31}$$

While heat cannot flow into the system from the outer world, heat can possibly flow through the wall separating the two subsystems, the rate of which is denoted as $\mathcal{Q}_{1\to 2}$. We assume that the entropy and energy changes of each subsystem is a result of the heat transfer across the wall, i.e.,

$$\dot{\mathcal{S}}_1 = -\frac{\mathcal{Q}_{1\to 2}}{T_1} = \frac{\dot{\mathcal{U}}_1}{T_1} \quad \text{and} \quad \dot{\mathcal{S}}_2 = \frac{\mathcal{Q}_{1\to 2}}{T_2} = \frac{\dot{\mathcal{U}}_2}{T_2} \ . \tag{4.32}$$

Substituting this in the second law we obtain

$$\dot{\mathcal{S}} = \mathcal{Q}_{1\to 2}\left(\frac{1}{T_2} - \frac{1}{T_1}\right) \geq 0 \ , \tag{4.33}$$

which can be satisfied by

$$\mathcal{Q}_{1\to 2} = A(T_1 - T_2) \quad \text{with} \quad A \geq 0 \tag{4.34}$$

to leading order in the temperature difference (resulting in Newton's cooling law). Therefore, heat flows from a higher temperature to a lower one.

To obtain a local version of the second law define an entropy density $s(\boldsymbol{x}, t)$ (per unit mass) such that

$$\mathcal{S} = \int_\Omega \rho(\boldsymbol{x}, t)\, s(\boldsymbol{x}, t)\, d\boldsymbol{x}^3 \ , \tag{4.35}$$

which immediately leads to (recall Eq. (4.30))

$$\rho\, \dot{s} + \nabla_{\boldsymbol{x}} \cdot (\boldsymbol{q}/T) \geq 0 \ . \tag{4.36}$$

Eliminating $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{q}$ between the first and second laws in Eqs. (4.29) and (4.36) we obtain

$$\boldsymbol{\sigma} : \boldsymbol{D} - \rho\, \dot{u} + T \rho\, \dot{s} - \frac{\boldsymbol{q} \cdot \nabla_{\boldsymbol{x}} T}{T} \geq 0 \ . \tag{4.37}$$

Usually this inequality is split into two stronger inequalities

$$\boldsymbol{\sigma} : \boldsymbol{D} - \rho\, \dot{u} + T \rho\, \dot{s} \geq 0 \quad \text{and} \quad \boldsymbol{q} \cdot \nabla_{\boldsymbol{x}} T \leq 0 \ . \tag{4.38}$$

The second inequality is satisfied by choosing

$$\boldsymbol{q} = -\kappa \nabla_{\boldsymbol{x}} T \ , \tag{4.39}$$

where $\kappa \geq 0$ is the thermal conductivity. This is Fourier's law of heat conduction. The first inequality in (4.38) is known as the dissipation inequality (or the Clausius-Planck inequality) and will play an important role later in the course. Any physical theory must satisfy this inequality. We note that the dissipation inequality can be also expressed in terms of the Helmholtz free-energy density $f = u - Ts$ (per unit mass) as

$$\boldsymbol{\sigma} : \boldsymbol{D} - \rho \dot{f} - \rho s \dot{T} \geq 0 \ . \tag{4.40}$$

Under isothermal conditions, $\dot{T} = 0$, we have

$$\boldsymbol{\sigma} : \boldsymbol{D} - \rho \dot{f} \geq 0 \ . \tag{4.41}$$

C. Heat equations

Heat equations are manifestations of the first law of thermodynamic. To see this, substitute Fourier's law of Eq. (4.39) into the first law of Eq. (4.29) to obtain

$$\rho \dot{u} = \boldsymbol{\sigma} : \boldsymbol{D} + \kappa \nabla^2_{\boldsymbol{x}} T \ . \tag{4.42}$$

This equation is transformed into a heat equation once we consider a constitutive law (see below) for the rate of deformation $\boldsymbol{D}$ and the internal energy density u. In the simplest case, which you know very well, we consider a non-deforming body $\boldsymbol{D}=0$ such that the local internal energy changes only due to heat flow. Therefore, we define the specific heat capacity through $c\dot{T} \equiv \rho \dot{u}$ and obtain

$$\dot{T} = D \nabla^2_{\boldsymbol{x}} T \ , \tag{4.43}$$

where $D \equiv \kappa/c$ is the thermal diffusion coefficient (the ratio of thermal conductivity and the specific heat capacity). This is just the ordinary heat diffusion equation, which in fact remains valid also for elastically deforming materials. Later in the course we will encounter more general heat equations that emerge in the present of more complicated constitutive laws.

A small digression

Diffusion equations possess an interesting feature, which is not always appreciated and which will teach us an important lesson about continuum physics. The modern microscopic theory of diffusion (and Brownian motion) is one of the most well-understood problems in physics and one of the greatest successes of statistical mechanics. To make things as simple and concrete as possible, we focus here on particle diffusion in 1d. Within this theory, diffusion is described by a particle which makes a random walk along the x-axis, starting at $x=0$ when the clock is set to $t=0$. At each time interval $\Delta t=1$ (in this discussion all quantities are dimensionless) the particle makes a jump of size $|\Delta x|=1$ to the right or the left, with equal probability. Statistical mechanics tells us (through the central limit theorem) that after a sufficiently long period of time (i.e., number of jumps), the probability distribution function $p(x,t)$ reads

$$p(x,t) = \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t}} \ , \tag{4.44}$$

i.e., a normal distribution with zero mean and variance which is just the time t. This continuous probability distribution function is the solution of a diffusion equation of the form

$$\partial_t p(x,t) = \frac{1}{2}\partial_{xx} p(x,t) \ , \tag{4.45}$$

which is a 1d version of Eq. (4.43) (with $D=1/2$ and recall that it is dimensionless here), with the initial condition

$$p(x,t=0) = \delta(x) \ . \tag{4.46}$$

This appears to be a strong result of continuum physics where we describe a physical system on timescales and lengthscales much larger than atomistic, forgetting about the discreteness of matter (think of this equation as describing the time evolution of an ink droplet spreading inside a water tank, where p represents for the mass density of the ink). Is there actually a problem here? Well, there is. Eq. (4.44) tells us that at time t after the initiation of the process, when the probability was localized at $x=0$ (cf. Eq. (4.46)), there is a *finite* probability to find the particle (e.g., an ink molecule) at an *arbitrarily large* x. The fact that this probability is exponentially small is beside the point. The crucial observation is that it is non-zero, implying that information propagated from $x=0$ (at $t=0$) *infinitely* fast. This violates fundamental physics (causality, relativity theory or whatever).

On the other hand, we know from the microscopic description of the problem that

$$p(x,t) = 0 \quad \text{for} \quad |x| > t \ , \tag{4.47}$$

i.e., that at most the particle could have made all of the jumps in one direction. That means that in fact the probability propagates at a finite speed (in our dimensionless units the propagation speed is 1), as we expect from general considerations.

So what went wrong in the transition from the microscopic description to macroscopic one? The answer is that Eq. (4.44) is wrong when $|x|$ becomes significantly larger than $\mathcal{O}(\sqrt{t})$. A more careful microscopic analysis (not discussed here) actually gives rise to the following continuum evolution equation for $p(x,t)$

$$\partial_t p(x,t) + \frac{1}{2}\partial_{tt} p(x,t) = \frac{1}{2}\partial_{xx} p(x,t) \ , \tag{4.48}$$

which is a combination of a wave equation and a diffusion equation. The fastest propagation of probability is limited by ordinary wave propagation of speed 1, exactly as we expect from microscopic considerations. This equation reconciles the apparent contradiction discussed above, showing that both Eqs. (4.44) and (4.47) are valid, just on different ranges of x for a given time t. This shows that the continuum limit should be taken carefully and critically. Beware. And you can also relax, the diffusion equation is in fact a very good approximation in most physical situations of interest.

We note that Eq. (4.48) is another equation of continuum physics, known as the telegrapher's equation. It was originally derived in a completely different context, that of electric transmission lines with losses, by Oliver Heaviside (1850-1925). This is the guy who invented the Heaviside step function and formulated Maxwell's equations using vector calculus in the form known to us today (the original ones were much uglier). He also independently discovered the Poynting vector (which is named after John Henry Poynting).

As noted above, the telegrapher's equation in Eq. (4.48) contains in it a 1d scalar wave equation (here for the probability distribution function $p(x,t)$). We will soon see that in the context of reversible deformation, and when higher dimensions are considered, the tensorial nature of deformation implies even more interesting and richer wave equations, and associated phenomena. Stay tuned.

D. Objectivity (frame - indifference)

To conclude this part of the course we consider the important notion of objectivity or frame-indifference. To quantify this idea we consider two observers that move (rotationally and translationally) one with respect to the other. For simplicity we assume that their watches are synchronized and that at time $t=0$ they agree on the reference configuration $\boldsymbol{X}$ of the body under consideration. The motions observed by the two observers are related by the following change-of-observer transformation

$$\boldsymbol{\varphi}^*(\boldsymbol{X},t) = \boldsymbol{Q}(t)\boldsymbol{\varphi}(\boldsymbol{X},t) + \boldsymbol{y}(t) \ , \tag{4.49}$$

where $\boldsymbol{Q}(t)$ is a proper time-dependent rotation matrix

$$\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I} \quad \text{and} \quad \det(\boldsymbol{Q}) = +1 \tag{4.50}$$

and $\boldsymbol{y}(t)$ is a time-dependent translation vector. A scalar ψ is classified as objective (frame-indifferent) if it satisfies $\psi^* = \psi$. A *spatial* vector $\boldsymbol{u}$ is objective if $\boldsymbol{u}^* = \boldsymbol{Q}\boldsymbol{u}$ and a *spatial* tensor $\boldsymbol{a}$ is objective if $\boldsymbol{a}^* = \boldsymbol{Q}\boldsymbol{a}\boldsymbol{Q}^T$ (we will try, for notational consistency, to denote spatial tensor by lowercase symbols). *Material/Lagrangian* objective vectors and tensors remain unchanged under change-of-observer transformation. Hybrid quantities, such as two-point tensors (which have a mixed spatial-material nature), have other objectivity criteria (see below).

To see how this works, consider the spatial velocity field $\boldsymbol{v} = \partial_t \boldsymbol{\varphi}$. Then we have

$$\boldsymbol{v}^* = \partial_t \boldsymbol{\varphi}^* = \boldsymbol{Q}\partial_t \boldsymbol{\varphi} + \dot{\boldsymbol{Q}}\boldsymbol{\varphi} + \dot{\boldsymbol{y}} = \boldsymbol{Q}\boldsymbol{v} + \dot{\boldsymbol{Q}}\boldsymbol{\varphi} + \dot{\boldsymbol{y}} \ . \tag{4.51}$$

Therefore, the velocity field is in general not objective. It can be made objective if we restrict ourselves to time-independent rigid transformation in which $\dot{\boldsymbol{Q}} = \dot{\boldsymbol{y}} = 0$. Likewise, the spatial acceleration field $\boldsymbol{a} = \dot{\boldsymbol{v}}$ is not an objective vector. This means that the linear momentum balance equation (i.e., Newton's second law) is not objective under the transformation in Eq. (4.49). This happens because time-dependent $\boldsymbol{Q}(t)$ and $\boldsymbol{y}(t)$ generate additional forces (centrifugal, Coriolis etc.). This is well-known to us: classical physics is invariant only under Galilean transformations, i.e., when $\dot{\boldsymbol{Q}} = 0$ and $\ddot{\boldsymbol{y}} = 0$. The classical laws of nature are the same in all inertial frames (and we know how to account for forces that emerge in non-inertial frames).

The important point to note is that for constitutive laws, i.e., physical laws that describe material behaviors, people sometimes demand something stronger: they insist that these laws remain unchanged under the change-of-observer transformation of Eq. (4.49) for general $\boldsymbol{Q}(t)$ and $\boldsymbol{y}(t)$. That is, even though Newton's second law is objective only under Galilean transformations, one usually demands constitutive laws to be objective under a more general transformation. This is called "The principle of material frame-indifference". To see how this works, consider then the deformation tensor $\boldsymbol{F}$. We have

$$\boldsymbol{F}^* = \nabla_{\boldsymbol{X}} \boldsymbol{\varphi}^*(\boldsymbol{X}, t) = \boldsymbol{Q}\nabla_{\boldsymbol{X}} \boldsymbol{\varphi}(\boldsymbol{X}, t) = \boldsymbol{Q}\boldsymbol{F} \ . \tag{4.52}$$

Therefore, $\boldsymbol{F}$ does not transform like an objective tensor, but rather like an objective vector. This is because it is not a true tensor, but rather a two-point tensor (a tensor that connects two spaces, $\boldsymbol{X}$ and $\boldsymbol{x}$ in this case). Two-point tensors (i.e., tensors of mixed spatial-material nature), that satisfy $\boldsymbol{A}^*=\boldsymbol{Q}\boldsymbol{A}$ under a change-of-observer transformation, are regarded as objective. Hence, $\boldsymbol{F}$ is regarded as an objective two-point tensor.

To see the difference between *material* and *spatial* objective tensors, consider the right Cauchy-Green deformation tensor $\boldsymbol{C}$ (see Eq. (3.26)) and the left Cauchy-Green deformation (Finger) tensor $\boldsymbol{b}$ (see Eq. (3.28)). For the former, we have

$$\boldsymbol{C}^* = \boldsymbol{F}^{*T}\boldsymbol{F}^* = \boldsymbol{F}^T\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{F} = \boldsymbol{F}^T\boldsymbol{F} = \boldsymbol{C} \ , \tag{4.53}$$

i.e., $\boldsymbol{C}$ is an objective material tensor. For the latter, we have

$$\boldsymbol{b}^* = \boldsymbol{F}^*\boldsymbol{F}^{*T} = \boldsymbol{Q}\boldsymbol{F}\boldsymbol{F}^T\boldsymbol{Q}^T = \boldsymbol{Q}\boldsymbol{b}\boldsymbol{Q}^T \ , \tag{4.54}$$

i.e., $\boldsymbol{b}$ is an objective spatial tensor.

The Cauchy stress $\boldsymbol{\sigma}$ is an objective tensor, i.e., $\boldsymbol{\sigma}^* = \boldsymbol{Q}\boldsymbol{\sigma}\boldsymbol{Q}^T$. This can be easily shown by using Eq. (3.34), $\boldsymbol{t}=\boldsymbol{\sigma}\boldsymbol{n}$, and by noting that both $\boldsymbol{t}$ and $\boldsymbol{n}$ are objective vectors. Consider then the velocity gradient, $\boldsymbol{L}=\partial\boldsymbol{v}/\partial\boldsymbol{x}=\dot{\boldsymbol{F}}\boldsymbol{F}^{-1}$, for which we have

$$\boldsymbol{L}^*\boldsymbol{F}^* = \dot{\boldsymbol{F}}^* \Longrightarrow \boldsymbol{L}^*\boldsymbol{Q}\boldsymbol{F} = \boldsymbol{Q}\dot{\boldsymbol{F}} + \dot{\boldsymbol{Q}}\boldsymbol{F} \Longrightarrow \boldsymbol{L}^* = \boldsymbol{Q}\boldsymbol{L}\boldsymbol{Q}^T + \dot{\boldsymbol{Q}}\boldsymbol{Q}^T \ . \tag{4.55}$$

Therefore, $\boldsymbol{L}$ is not an objective tensor and hence if one adopts "The principle of material frame-indifference" then $\boldsymbol{L}$ cannot be used to formulate physical laws. However, by noting that $\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}$ implies $\dot{\boldsymbol{Q}}\boldsymbol{Q}^T = -\boldsymbol{Q}\dot{\boldsymbol{Q}}^T$, we immediately conclude that the rate of deformation tensor $\boldsymbol{D}$ is objective. Many physical theories involve the time rate of change of a tensor, for example a stress rate. However, it is immediately observed that $\dot{\boldsymbol{\sigma}}$ is not an objective tensor. To see this, note that Eq. (4.55) implies $\dot{\boldsymbol{Q}}\boldsymbol{Q}^T=\boldsymbol{W}^*-\boldsymbol{Q}\boldsymbol{W}\boldsymbol{Q}^T \Longrightarrow \dot{\boldsymbol{Q}}=\boldsymbol{W}^*\boldsymbol{Q}-\boldsymbol{Q}\boldsymbol{W}$ and write

$$\begin{aligned} \dot{\boldsymbol{\sigma}}^* &= \dot{\boldsymbol{Q}}\boldsymbol{\sigma}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{\sigma}\dot{\boldsymbol{Q}}^T \\ &= (\boldsymbol{W}^*\boldsymbol{Q} - \boldsymbol{Q}\boldsymbol{W})\,\boldsymbol{\sigma}\boldsymbol{Q}^T + \boldsymbol{Q}\dot{\boldsymbol{\sigma}}\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{\sigma}\left(\boldsymbol{Q}^T\boldsymbol{W}^{*T} - \boldsymbol{W}^T\boldsymbol{Q}^T\right) \\ &= \boldsymbol{W}^*\boldsymbol{\sigma}^* + \boldsymbol{\sigma}^*\boldsymbol{W}^{*T} + \boldsymbol{Q}\left(\dot{\boldsymbol{\sigma}} - \boldsymbol{W}\boldsymbol{\sigma} - \boldsymbol{\sigma}\boldsymbol{W}^T\right)\boldsymbol{Q}^T \ . \end{aligned} \tag{4.56}$$

Use now $\boldsymbol{W}^T = -\boldsymbol{W}$ to rewrite the last relation as

$$\dot{\boldsymbol{\sigma}}^* + \boldsymbol{\sigma}^*\boldsymbol{W}^* - \boldsymbol{W}^*\boldsymbol{\sigma}^* = \boldsymbol{Q}\left(\dot{\boldsymbol{\sigma}} + \boldsymbol{\sigma}\boldsymbol{W} - \boldsymbol{W}\boldsymbol{\sigma}\right)\boldsymbol{Q}^T \ , \tag{4.57}$$

or equivalently as

$$\dot{\boldsymbol{\sigma}}^* + [\boldsymbol{\sigma}^*,\boldsymbol{W}^*] = \boldsymbol{Q}\left(\dot{\boldsymbol{\sigma}} + [\boldsymbol{\sigma},\boldsymbol{W}]\right)\boldsymbol{Q}^T \ , \tag{4.58}$$

where $[\cdot]$ is the commutator of two tensors. This result shows that indeed $\dot{\boldsymbol{\sigma}}$ is not an objective tensor, but also suggests that

$$\overset{\triangle}{\boldsymbol{\sigma}} \equiv \dot{\boldsymbol{\sigma}} + [\boldsymbol{\sigma}, \boldsymbol{W}] \tag{4.59}$$

is an objective stress rate tensor. This derivative is called Jaumann derivative and is extensively used in solid mechanics. It is important to note that there is **no** unique way to define an objective tensorial time derivative, and in fact there are infinitely many others (some of which are rather common). The Jaumann derivative, as well as other objective stress rates, can be used to formulate physical theories.

What is the basic status of objectivity? Is it a fundamental principle of classical physics or is it just a very useful approximation (that, by definition, has limitations)? Since macroscopic constitutive laws should ultimately result from systematic coarse-graining of microscopic physics, and since the latter obviously satisfy Newton's second law, there must exist situations in which objectivity is violated. In particular, in situations in which centrifugal and Coriolis forces cannot be neglected at the molecular level, objectivity cannot be fully satisfied. On the other hand, in many situations this is a very useful approximation that allows us to further constrain the structure of constitutive laws. So we must conclude that objectivity cannot be a "principle" of physics, rather an approximation (possibly a very useful/fruitful one).

While objectivity is widely invoked, its basic status has been the subject of many heated debates. An example from the 1980's can be found at: Physical Review A **32**, 1239 (1985), and the subsequent comment and reply. See also some insightful comments made by de Gennes, Physica A **118**, 43 (1983).

Constitutive laws

Up to now we considered measures of deformation and the concept of stress, conservation laws, the laws of thermodynamics and symmetry principles. These are not enough to describe the behavior of materials. The missing piece is a physical theory for the response of the material to external forces, the so-called constitutive relations/laws. To see the mathematical necessity of constitutive laws, consider the momentum balance equations of (4.17) in 2D, in the quasi-static limit (no inertia) and neglecting body forces

$$\partial_x \sigma_{xx} + \partial_y \sigma_{xy} = 0 \quad \text{and} \quad \partial_x \sigma_{yx} + \partial_y \sigma_{yy} = 0 \ . \tag{4.60}$$

These are two equations for 3 fields. Additional information about how stresses are related to the state of deformation of a body is required.

In the rest of the course we will consider physical theories for the behavior of materials. These must be consistent with the laws of thermodynamics discussed above and to comply with the principle of objectivity (frame-indifference, in its either weak or strong form). While thermodynamics, objectivity and symmetry principles seriously limit and constrain physical theories and are very useful, to understand material behaviors we need additional physical input.

Reversible processes: non-dissipative constitutive behaviors

The simplest response of a solid to mechanical driving forces is elastic. By elastic we mean that the response is reversible, i.e., that when the driving forces are removed the system relaxes back to its original state. Put in other words, elasticity means that the system "remembers" its undeformed state, which can serve as a reference configuration. Later in the course we will discuss irreversible deformation processes in which the internal state of a physical system evolves, and no recovery of the original state is observed when external constraints are removed. When an elastic system is deformed, energy is being stored in it. Suppose that the deformation is described by the Green-Lagrange strain measure $\boldsymbol{E}$, cf. Eq. (3.26), then the elastic energy density is described by the functional $u(\boldsymbol{E}, s)$. All of the physics of elasticity is encapsulated in this strain-energy functional.

V. THE LINEARIZED FIELD THEORY OF ELASTICITY

A. General derivation for anisotropic and isotropic materials

In many situations solids deform only slightly within the range of relevant driving forces. Think, for example, about your teeth when you eat a nut or about all metal objects around you. This is true for "hard" solids. However, things are different when we consider "soft" solids like rubbers (e.g., your car tire), gels or various biological materials (e.g., your skin). While they also deform elastically, they require a finite deformation description, as will be explained later in the course.

If the deformation remains small, we can focus on situations in which the displacement gradient is small $|\boldsymbol{H}| \ll 1$ (in fact we also require small rotations, see below). Under these conditions, we can linearize the Green-Lagrange strain $\boldsymbol{E} \simeq \boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{H}+\boldsymbol{H}^T)$, cf. Eq. (3.27), and use only the infinitesimal strain tensor $\boldsymbol{\varepsilon}$. This will make our life much easier (but please do not relax, it will be still quite tough nonetheless). But there is a price; first, as we already discussed, $\boldsymbol{\varepsilon}$ is **not** rotationally invariant. A corollary of this lack of rotational invariance is that a constitutive law formulated in terms of $\boldsymbol{\varepsilon}$ will not be objective. To see this, note that

$$\boldsymbol{E}^* = \frac{1}{2}\left(\boldsymbol{C}^* - \boldsymbol{I}\right) = \frac{1}{2}\left(\boldsymbol{C} - \boldsymbol{I}\right) = \boldsymbol{E}\ . \tag{5.1}$$

That is, $\boldsymbol{E}$ is an objective material tensor, which immediately implies that $\boldsymbol{\varepsilon}$ is not. So already in our first discussion of a constitutive relation we violate objectivity. We can easily adhere to it formally, but the practical price will be high as it will force us to go nonlinear. In a *huge* range of problems, though, rotations remain small and $\boldsymbol{\varepsilon}$ does a remarkable job in properly describing the relevant physics.

Another great advantage of the linearity assumption is that we should no longer distinguish between the undeformed $\boldsymbol{X}$ and deformed $\boldsymbol{x}$ configurations. The reason is that while these configurations are of course distinct in the presence of deformation, and as the displacements themselves can be large the difference can be large itself, gradients remain small and all physical quantities are indistinguishable to linear order. Moreover, the convective term in the material derivative plays no role as it is intrinsically nonlinear. The linearized strain tensor takes the form

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{5.2}$$

and we can also identify $\boldsymbol{D}$ with $\dot{\boldsymbol{\varepsilon}}$. Finally, since in the context of the linearized theory of elasticity mass density variations are small (the mass density appears only as a multiplicative factor in products and hence effectively contributes only to higher orders), we can use quantities per unit volume rather than quantities per unit mass. In particular, we define $\rho\, u \equiv \bar{u}$ and $\rho\, s \equiv \bar{s}$, and for the ease of notation drop the bars.

Under these conditions, the dissipation inequality of the second law of thermodynamics of Eq. (4.38) reads $\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{u} + T\, \dot{s} \geq 0$. To proceed, we express $\dot{u}(\boldsymbol{\varepsilon}, s)$ in terms of variations in $\boldsymbol{\varepsilon}$ and s and substitute in the above inequality to get

$$\left(\boldsymbol{\sigma} - \frac{\partial u}{\partial \boldsymbol{\varepsilon}}\right) : \dot{\boldsymbol{\varepsilon}} + \left(T - \frac{\partial u}{\partial s}\right) \dot{s} \geq 0\ . \tag{5.3}$$

Since elastic response is reversible, we expect an equality to hold. Moreover, the strain and the entropy can be varied independently. Therefore, the second law analysis implies

$$\boldsymbol{\sigma} = \frac{\partial u}{\partial \boldsymbol{\varepsilon}} \quad \text{and} \quad T = \frac{\partial u}{\partial s}\ . \tag{5.4}$$

There is a related, more formal, approach to derive these relations. We focus on (5.3), without assuming an equality. However, we do note that u and $\boldsymbol{\sigma}$ are independent of $\dot{\boldsymbol{\varepsilon}}$. That is a basic property of an elastic response: the rate at which a state is reached makes no difference. Therefore, the only way to avoid violating the inequality under all circumstances is to set the brackets to zero (the same argument holds for the entropy term). This is termed the Coleman-Noll procedure.

The resulting relations are thermodynamic identities that are derived from the second law of thermodynamics. The first one says that the stress $\boldsymbol{\sigma}$ is thermodynamically conjugate to the strain $\boldsymbol{\varepsilon}$. The strain energy functional takes the form (we assume that the entropy does not change with deformation and hence is irrelevant here)

$$u = \frac{1}{2}\, \sigma_{ij}\, \varepsilon_{ij}\ , \tag{5.5}$$

where linearity implies a tensorial linear relation between the stress and the strain $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$ (also explaining the appearance of 1/2 in the expression above for the strain energy density) or in components

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}\ . \tag{5.6}$$

The response coefficients, C_{ijkl}, known as the elastic constants (the forth order tensor $\boldsymbol{C}$ is known as the stiffness tensor), are given by

$$C_{ijkl} = \frac{\partial^2 u}{\partial \varepsilon_{ij} \partial \varepsilon_{kl}}\ . \tag{5.7}$$

Equation (5.6) is a constitutive relation, i.e., a relation between the stress (driving force) and the strain (response), which is a generalization of Hooke's relation for an elastic spring ($F = -k\, x$).

How many independent numbers are needed to describe $\boldsymbol{C}$? Naively, one would think that $3^4 = 81$ independent numbers are needed. However, $\boldsymbol{C}$ possesses various general symmetries that significantly reduce the amount of independent numbers, even before considering any specific material symmetries. First, the symmetry of the Cauchy stress tensor implies, through Eq. (5.6), that

$$C_{ijkl} = C_{jikl} \ . \tag{5.8}$$

Furthermore, the symmetry of the infinitesimal strain tensor, $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^T$, implies

$$C_{ijkl} = C_{ijlk} \ . \tag{5.9}$$

These two symmetries imply that we need only 36 independent numbers (6 for the first two indices and 6 for the last two). Furthermore, since $\boldsymbol{C}$ is obtained through a second-order tensorial derivative of the energy density u, see Eq. (5.7), interchanging the order of differentiation suggests an additional symmetry of the form

$$C_{ijkl} = C_{klij} \ . \tag{5.10}$$

The latter imposes 15 additional constraints ($\frac{6\times 5}{2}$), which leaves us with 21 independent numbers. Therefore, in the most general case $\boldsymbol{C}$ contains 21 independent elastic coefficients (in fact, $\boldsymbol{C}$ can be represented as a 6 by 6 symmetric matrix — which depends on 21 independent numbers —, where the components of $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are represented as vectors with 6 components).

This is the extreme anisotropic case. However, usually materials exhibit some symmetries that further reduce the number of independent elastic constants. For example, composite materials (e.g., fiberglass, a glass-fiber reinforced plastic) may be invariant with respect to various translations and rotations. Here, we focus on isotropic materials. Since the energy functional is a scalar, it depends only on invariants of $\boldsymbol{\varepsilon}$, which can be written as $\operatorname{tr}\boldsymbol{\varepsilon}$, $\operatorname{tr}\boldsymbol{\varepsilon}^2$ and $\operatorname{tr}\boldsymbol{\varepsilon}^3$ (sometimes people use other invariants that are linearly dependent on these). Since in a linear theory the energy must be quadratic in the strain, only two combinations, $(\operatorname{tr}\boldsymbol{\varepsilon})^2$ and $\operatorname{tr}\boldsymbol{\varepsilon}^2$, can appear and hence

$$u(\boldsymbol{\varepsilon}) = \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{\varepsilon})^2 + \mu \operatorname{tr}\boldsymbol{\varepsilon}^2 \ . \tag{5.11}$$

It is important to note that we can replace $\boldsymbol{\varepsilon}$ by $\boldsymbol{E}$ in this energy functional to obtain the *simplest* possible nonlinear elastic material model

$$u(\boldsymbol{E}) = \frac{\lambda}{2}(\operatorname{tr}\boldsymbol{E})^2 + \mu \operatorname{tr}\boldsymbol{E}^2 \ . \tag{5.12}$$

This constitutive law, termed the Saint Venant-Kirchhoff material model, is both rotationally invariant and objective. Alas, it is also nonlinear due to the inherent geometric nonlinearity in

$\boldsymbol{E}$. This constitutive law is the simplest nonlinear elastic model because it is *constitutively* linear, i.e., $u(\boldsymbol{E})$ is quadratic in $\boldsymbol{E}$, but is *geometrically* nonlinear (due to the nonlinear dependence of $\boldsymbol{E}$ on $\boldsymbol{H}$). Here, we adhere to a constitutive law which is linear in $\boldsymbol{H}$, and hence use Eq. (5.11).

Equation (5.11) shows that isotropic linear elastic materials are characterized by only two elastic constants, the Lamé constants. μ is also known as the shear modulus (can we say something about the sign of λ and μ at this stage?). Using the following differential tensorial relation

$$\frac{d\,\mathrm{tr}\boldsymbol{A}^n}{d\boldsymbol{A}} = n(\boldsymbol{A}^{n-1})^T \;, \tag{5.13}$$

the constitutive law (Hooke's law) can be readily obtained

$$\boldsymbol{\sigma} = \lambda\ \mathrm{tr}\,\boldsymbol{\varepsilon}\,\boldsymbol{I} + 2\mu\,\boldsymbol{\varepsilon} \tag{5.14}$$

or in components

$$\sigma_{ij} = \lambda\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij} \;. \tag{5.15}$$

The stiffness tensor can be written as $C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$. It is important to stress that Hooke's law is simply a perturbation theory based on a gradient expansion. This is entirely analogous to the constitutive law for Newtonian fluids, which is based on a velocity gradient expansion (μ in Hooke's law plays the role of shear viscosity and λ the role of bulk viscosity).

B. Homogeneous deformation and scaling solutions

Let us first consider a few homogeneous deformation situations. Consider a solid that is strained uniaxially (say in the x-direction) by an amount $\varepsilon_{xx} = \varepsilon$. The stress state of the solid is described by $\sigma_{xx} = \sigma$ and all of the other components vanish (because the lateral boundaries are free). Symmetry implies that the strain response is $\varepsilon_{ij} = 0$ for $i \neq j$ and $\varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_\perp$. Our goal is to calculate the response σ and $\varepsilon_\perp$ in terms of the driving ε. Using Eq. (5.15) we obtain

$$0 = \lambda(\varepsilon + 2\varepsilon_\perp) + 2\mu\varepsilon_\perp \quad \text{and} \quad \sigma = \lambda(\varepsilon + 2\varepsilon_\perp) + 2\mu\varepsilon \;. \tag{5.16}$$

Solving for the response functions $\sigma(\varepsilon)$ and $\varepsilon_\perp(\varepsilon)$ we obtain

$$\varepsilon_\perp = -\frac{\lambda}{2(\lambda+\mu)}\varepsilon \equiv -\nu\,\varepsilon \quad \text{and} \quad \sigma = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}\varepsilon \equiv E\,\varepsilon \;. \tag{5.17}$$

E is known as Young's modulus and ν as Poisson's ratio. These response coefficients are most easily measured experimentally and are therefore extensively used. In many cases, Hooke's law is expressed in terms of them (derive). The latter analysis immediately tells us something about ν in a certain limit.

Let us then consider an isotropic compression of a solid. In this case we have $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = -p$, where $p > 0$ is the hydrostatic pressure, and $\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon$ (all other components of the stress and strain tensors vanish). We then have

$$-p = \frac{3\lambda + 2\mu}{3}\,\mathrm{tr}\,\boldsymbol{\varepsilon} \equiv K\ \mathrm{tr}\,\boldsymbol{\varepsilon}\ . \tag{5.18}$$

Noting that the relative change in volume is given by $\delta V/V = \det \boldsymbol{F} - 1 = (1+\epsilon_{xx})(1+\epsilon_{yy})(1+\epsilon_{zz}) - 1 \simeq \mathrm{tr}\,\boldsymbol{\varepsilon}$, we see that K is the bulk modulus, which we are already familiar with from thermodynamics, $K = -V\partial p/\partial V$. Using the bulk modulus we can express the energy functional of Eq. (5.11) as

$$u(\boldsymbol{\varepsilon}) = \frac{1}{2}K(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 + \mu\left(\varepsilon_{ij} - \frac{1}{3}\,\mathrm{tr}\boldsymbol{\varepsilon}\,\delta_{ij}\right)^2\ . \tag{5.19}$$

The components in the second brackets are the components of the deviatoric strain tensor, $\boldsymbol{\varepsilon}^{dev} \equiv \boldsymbol{\varepsilon} - \frac{1}{3}\,\mathrm{tr}\boldsymbol{\varepsilon}\,I$, where $\mathrm{tr}\,\boldsymbol{\varepsilon}^{dev} = 0$. The representation in Eq. (5.19) shows that the total strain energy can be associated with a deviatoric (shear-like) part, weighted by the shear modulus μ, and a volumetric (dilatational) part, weighted by the bulk modulus K. Another important implication of this representation is that μ and K multiply terms which are positive definite and independently variable. The importance of this observation is that thermodynamics implies that at (stable) equilibrium the (free) energy attains a minimum and hence u must be positive under all circumstances. Since the deformation can be either volume conserving, $\mathrm{tr}\,\boldsymbol{\varepsilon} = 0$, or isotropic, $\boldsymbol{\varepsilon} \propto \boldsymbol{I}$, we must have

$$\mu, K > 0\ . \tag{5.20}$$

This has interesting implications for other elastic constants. For example, it tells us that $\lambda \geq -2\mu/3$, which shows that λ is not necessarily positive. Taking the two extreme cases, $\lambda = -2\mu/3$ and $\lambda \gg \mu$, we obtain the following constraint on Poisson's ratio

$$-1 \leq \nu \leq \frac{1}{2}\ . \tag{5.21}$$

Therefore, while a negative ν might appear counterintuitive (as it implies that a solid expands in the directions orthogonal to the uniaxial stretch direction), it does not violate any law of physics. An example for a natural material with a Poisson's ratio of nearly zero is cork, used as a stopper for wine bottles.

In the last few decades there has been an enormous interest in materials with unusual values of Poisson's ratio. In particular, materials with a negative Poisson's ratio were synthesized (they are termed auxetic materials, "auxetic"="that which tends to increase" in Greek), see the review article in Nature Materials, "Poisson's ratio and modern materials", Nature Materials 10, 823–837 (2011) (see http://www.nature.com/nmat/journal/v10/n11/pdf/nmat3134.pdf). You may also want to look at: http://silver.neep.wisc.edu/~lakes/Poisson.html. Such materials are a subset of a larger class of materials known as "Metamaterials". Metamaterials are artificial materials designed to provide properties which may not be readily available in nature, and can be very interesting and useful.

The power and limitations of the continuum assumption can be nicely illustrated through a discussion of the evolution of the concept of Poisson's ratio, from Poisson's original paper in 1827, which based on the molecular hypothesis, through the subsequent development based on the competing continuum hypothesis, to the explosion of research in this direction in recent decades based on a better microscopic/mesoscopic understanding of the structure of materials and computational capabilities.

The upper bound of ν also have a clear physical meaning. In a uniaxial test we have

$$\mathrm{tr}\,\boldsymbol{\varepsilon} = (1-2\nu)\varepsilon \ , \tag{5.22}$$

which immediately tells us that incompressible materials, i.e., materials for which $\mathrm{tr}\,\boldsymbol{\varepsilon}=0$, have $\nu = 1/2$. In fact, the incompressibility limit is a bit subtle. To see this, recall Eq. (5.18), $-p = K\ \mathrm{tr}\,\boldsymbol{\varepsilon}$. Obviously, a finite pressure can be applied to an incompressible material. This means that in the incompressibility limit we have $\mathrm{tr}\,\boldsymbol{\varepsilon} \to 0$ and $K \to \infty$, while their product is finite. It is also clear that no work is invested in applying a pressure to an incompressible material and no energy is being stored. Indeed, in the incompressibility limit we have $K(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 \to 0$. Note also that while $K \propto (1-2\nu)^{-1}$ diverges in the incompressibility limit, the shear modulus μ and Young's modulus E remain finite. Finally, we stress that the bounds on Poisson's ratio in Eq. (5.21) are valid for *isotropic* materials. *Anisotropic* materials can, and actually do, violate these bounds.

Before we move on to derive the equations of motion for a linear elastic solid, we note a few properties of the linearized (infinitesimal) strain tensor $\boldsymbol{\varepsilon}$. First, as we stressed several times above, it is not invariant under finite rotations (prove). That means that even if the relative distance change between material points remains small, $\boldsymbol{\varepsilon}$ cannot be used when rotations are not small. In that sense it is not a true strain measure. Second, the components of $\boldsymbol{\varepsilon}$ are not independent. The reason for this is that $\boldsymbol{\varepsilon}$ is derived from a continuously differentiable displacement field $\boldsymbol{u}$. The resulting relations between the different components of $\boldsymbol{\varepsilon}$, ensuring that the different parts of a material fit together after deformation, are termed "compatibility conditions". In 3D there are 6 such conditions, making a problem formulated in terms of strain components very complicated, and in 2D there is only one compatibility condition, which reads

$$\partial_{yy}\varepsilon_{xx} + \partial_{xx}\varepsilon_{yy} = 2\partial_{xy}\varepsilon_{xy} \ . \tag{5.23}$$

Finally, note that compatibility is automatically satisfied if the displacement field is used directly.

The equation of motion describing linear elastic solids is readily obtained by substituting the constitutive relation (Hooke's law) of Eq. (5.15) into the momentum balance equation of (4.17), taking the form

$$(\lambda+\mu)\nabla\left(\nabla\cdot\boldsymbol{u}\right) + \mu\nabla^2\boldsymbol{u} + \boldsymbol{b} = \rho\partial_{tt}\boldsymbol{u} \ . \tag{5.24}$$

It is called the Navier-Lamé equation. It accounts for a huge range of physical phenomena and can easily serve as the basis for a two-semester course. In spite of its linearity, its solutions in 3D and/or under dynamic situations might be very complicated and require (sometimes non-trivial) numerical methods.

There are, however, many situations in which analytical tools can be employed. One typical situation is when we are not interested in the exact solution, with all the π's 2's etc., but rather in the way the solution depends on the material's parameters, loading and geometry of a given problem. In this case we invoke everything we have at hand: physical considerations, symmetries, dimensional analysis etc. Let us demonstrate this in two examples (both are fully analytically tractable, but they will serve our purpose here).

Example: Surface Green's function

Consider a linear elastic, isotropic, half-space which is loaded at the flat surface by a concentrated force. We assume that the force is applied to an area which is small compared to the scales of interest (this will be a recurring theme later in the course), hence the pressure at the surface ($z=0$) takes the form $p_z(x,y,z=0) = F_z\delta(x)\delta(y)$, where the $x-y$ plane is parallel to the surface and z is perpendicular to it (note that $p_z = -\sigma_{zz}$). Focus then on the shape of the deformed (originally flat) surface, i.e., on $u_z(x,y,z\!=\!0)$. What form can it take? First, the azimuthal symmetry of the force with respect to the surface, implies that u_z depends on x and y

only through the radius r. Second, linearity implies that it must be proportional to F_z. Finally, since F_z has the dimension of force and u_z has the dimension of length, we need another quantity that involves force dimension. The only quantity available in the problem is the elastic modulus E (or μ). Put all these ingredients together, we must have

$$u_z(r, z\!=\!0) \sim \frac{F_z}{E\,r} \ . \tag{5.25}$$

This gives us the shape of the deformed surface. Note that since there is no lengthscale in the problem, the shape is scale-free. Moreover, the effect of the concentrated force is long-ranged, which is a generic property of elastic interactions. Of course the singularity at $r \to 0$ is not physical, it simply means that as we approach the applied force the details of how it is applied, as well as material nonlinearities, matter and actually regularize the singularity. If we compare our result to the exact one we discover that we indeed only missed a prefactor of order unity, which takes the form $(1-\nu^2)/\pi$. Since we have considered a concentrated force, our result is valid for any surface pressure distribution, i.e., we have a calculated a Green's function. Therefore, for a general surface pressure distribution $p_z(x,y,z=0)$, we have

$$u_z(x,y,z\!=\!0) = \frac{1-\nu^2}{\pi\,E} \int \frac{p_z(x',y',z=0)dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}} \ . \tag{5.26}$$

Example: Hertzian contact

Consider an isotropic linear elastic (say, of Young's modulus E and Poisson's ratio ν) sphere of radius R that is pressed against an infinitely rigid plane by a force F. This problem was considered (and solved) by Hertz in 1882 and is known as the Hertzian contact problem (in fact, Hertz considered two deforming spheres). It signaled the birth of contact mechanics and is of enormous importance and range of applications. These span the full range from friction and tribology of small structures, through earthquakes in the earth crust to rubble piles in the solar system. In order to get the essence of Hertz's solution without actually solving the partial differential equations we need a physical insight, a geometrical insight and a constitutive relation. Denote the distance by which the sphere approaches the plane by δ and the radius of the circular contact that is formed by a. The crucial physical question to ask (and to answer, of course) is what the typical lengthscale of the strain distribution is. Since strains are built in the sphere only because of the formed contact, we expect the strain distribution to be concentrated on a scale a (where the rest of the sphere responds essentially in a rigid body manner). Therefore, the displacement δ is accumulated on a scale a near the surface and the typical strain is

$$\varepsilon \sim \frac{\delta}{a} \ . \tag{5.27}$$

This is the physical insight we needed. The geometrical observation is simple and reads

$$a^2 \sim \delta R \ , \tag{5.28}$$

i.e., for a given displacement δ the contact area is linear in the sphere's radius R (to see this simply cut from the bottom of a sphere of radius R a piece of height δ and estimate the cut area). Note also that the last equation implies that $a \ll R$ since δ/a is expected to be small. Using the constitutive relation, Hooke's law, we have

$$p_0 \sim \frac{F}{a^2} \sim E\varepsilon \sim \frac{E\,\delta}{a} \quad \Rightarrow \quad F \sim E\delta a \ , \tag{5.29}$$

where p_0 is a typical pressure. Using the last two equations, we can calculate the 3 relevant "response" functions $a(F;R,E)$, $\delta(F;R,E)$ and $p_0(F;R,E)$ to be

$$a \sim \left(\frac{FR}{E}\right)^{1/3}, \quad \delta \sim \left(\frac{F^2}{R\,E^2}\right)^{1/3}, \quad p_0 \sim \left(\frac{FE^2}{R^2}\right)^{1/3} \ . \tag{5.30}$$

Therefore, we managed to express the response quantities in terms of the driving force (F), geometry (R) and constitutive parameters (E). These players (i.e., the driving forces, geometry and constitutive relations) are generic players in our game. Comparing the resulting expressions above to the exact ones indeed shows that they are correct to within numerical constants of order unity. These results are rather striking. How come a linear theory gave rise to a nonlinear response, i.e., a nonlinear dependence of the response quantities on the driving force F? The answer is that nonlinearities were hidden in the geometry of the problem. In other words, the fact that the contact area is a variable that depends self-consistently on the deformation, but is unknown a priori, makes the problem effectively nonlinear. Contact problems are highly nonlinear even within the framework of a linear elastic field theory. As a final comment, we note that p_0 represents the pressure at the center of the contact. The pressure must drop to zero at the contact line, $r\!=\!a$ (r is measured from the center). The result (not derived here) reads

$$p(r,z\!=\!0) = p_0\sqrt{1-\frac{r^2}{a^2}} \ , \tag{5.31}$$

where $z\!=\!0$ is the location of the rigid plane. This shows that while the pressure is continuous at the contact line, its derivative is not. In fact, the derivative diverges at the contact line.

Before we consider further simplifications of the Navier-Lamé equation of (5.24), let us explore some of their general properties. For that purpose, recall the identity $\nabla^2\boldsymbol{u}=\nabla(\nabla\cdot\boldsymbol{u})-\nabla\times(\nabla\times\boldsymbol{u})$. Applying it to the Navier-Lamé Eq. (5.24), in the absence of body forces ($\boldsymbol{b}=0$) and under equilibrium (static) conditions (no inertia), we obtain

$$(\lambda+2\mu)\nabla\left(\nabla\cdot\boldsymbol{u}\right)-\mu\nabla\times\left(\nabla\times\boldsymbol{u}\right)=0\ . \tag{5.32}$$

Acting with the divergence operator on this equation, we obtain

$$\nabla^2\left(\nabla\cdot\boldsymbol{u}\right)=0\ , \tag{5.33}$$

since $\nabla\cdot(\nabla\times\boldsymbol{A})=0$ for any vector field $\boldsymbol{A}$. This shows that under equilibrium (static) conditions $\nabla\cdot\boldsymbol{u}$ satisfies Laplace's equation, i.e., it is harmonic. Operating then with the Laplacian operator on the Navier-Lamé Eq. (5.24), we obtain

$$\nabla^2\nabla^2\boldsymbol{u}=0\ , \tag{5.34}$$

i.e., under equilibrium (static) conditions $\boldsymbol{u}$ satisfies bi-Laplace's equation, i.e., it is a bi-harmonic vector field. These results are useful in various contexts.

C. 2D Elasticity

We now turn to discuss further simplifications of the Navier-Lamé Eq. (5.24). In many situations the dynamics of a linear elastic body can be approximated as two-dimensional.

1. Scalar elasticity

The simplest possible such situation is when the only non-vanishing component of the displacement field is given by $u_z(x,y,t)$. This physical situation is termed anti-plane deformation, i.e., $\boldsymbol{u}(\boldsymbol{x},t)=u_z(x,y,t)\hat{z}$. In this case, we have $\nabla\cdot\boldsymbol{u}=0$ and the Navier-Lamé Eq. (5.24) reduces to

$$\mu\nabla^2 u_z=\rho\partial_{tt}u_z\ , \tag{5.35}$$

which is a scalar wave equation. $c_s=\sqrt{\mu/\rho}$ is the shear wave speed. Let us focus first on static situations in which this equation reduces to

$$\nabla^2 u_z=0\ , \tag{5.36}$$

i.e., u_z satisfies Laplace's equation (a harmonic function). Laplace's equation emerges in many branches of physics (electrostatics, fluid mechanics etc.). The theory of complex variable functions offers very powerful tools to solve 2D problems. We first discuss this approach for Laplace's equation in (5.36). Let us first briefly recall some fundamentals of complex functions theory. Let z be a complex variable, $z = x + iy$. A function $f(z)$ is called analytic if it satisfies

$$\partial_x f(z) = f'(z) \quad \text{and} \quad \partial_y f(z) = i f'(z) \ . \tag{5.37}$$

Writing $f(z)$ as $f(z) = u(x, y) + iv(x, y)$, Eq. (5.37) implies

$$f'(z) = \partial_x f(z) = \partial_x u + i\partial_x v = -i\partial_y f(z) = \partial_y v - i\partial_y u \ , \tag{5.38}$$

leading to the well-known Cauchy-Riemann conditions

$$\partial_x u = \partial_y v \quad \text{and} \quad \partial_x v = -\partial_y u \ . \tag{5.39}$$

Eq. (5.37) also implies the following operator relation

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y) \ . \tag{5.40}$$

Recalling that the complex conjugate of z is $\bar{z} = x - iy$, we also have

$$\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y) \ . \tag{5.41}$$

Any complex function $f(x, y)$ can be represented as $f(x, y) = g(z, \bar{z})$. Therefore, an analytic function is a function that is independent of $\bar{z}$. This observation immediately shows that *any* analytic function is a solution of Laplace's equation, e.g. the one in Eq. (5.36). To see this we note that

$$\nabla^2 = \partial_{xx} + \partial_{yy} = 4\partial_z\partial_{\bar{z}} \ . \tag{5.42}$$

Therefore,

$$u_z(x, y) = \Re[f(z)] \quad \text{or} \quad u_z(x, y) = \Im[f(z)] \ , \tag{5.43}$$

where $f(z)$ is sometimes called a complex potential (note the analogy with electric potential in electrostatics). Choosing the real or imaginary part is a matter of convenience. The specific solution $f(z)$ is selected so as to satisfy a specific set of boundary conditions. The stress tensor (in this case only the σ_{zx} and σ_{zy} components do not vanish) is given by (prove)

$$\sigma_{zy} + i\sigma_{zx} = \mu f'(z) \ , \tag{5.44}$$

when one chooses $u_z(x,y) = \Im[f(z)]$.

Example: Screw dislocations

Later, we will work out a more complicated example of how these powerful tools help us solving important problems. Here, we would like to consider a simple example. For that aim we introduce an object called a "dislocation". Dislocations will appear later in the course as the carriers of plastic deformation in crystalline and polycrystalline materials, and are very important and interesting objects. For our purposes here we define a dislocation as a continuum object that carries with it a topological charge and focus only on the static linear elastic consequences. By "topological charge" we mean that if we integrate the displacement field in a close loop around the dislocation core (where we define the origin of our coordinate system) we obtain a finite value, i.e.,

$$\oint du_z = b \ . \tag{5.45}$$

This implies the existence of a branch cut. The magnitude of the topological charge, b, is the size of the so-called Burgers vector $\boldsymbol{b}$, $b = |\boldsymbol{b}|$ (named after the Dutch physicist Jan Burgers, who is also known for the famous fluid mechanics equation). In this case we have $\boldsymbol{b} = b\,\hat{z}$. Note that a dislocation is a line (and not a point) defect, which extends along the z-direction. Translation symmetry along this direction allows for a 2D treatment. Obviously, the generation of a dislocation is neither a linear process, nor an elastic one. Still, once it exists, we can ask ourselves what the linear elastic fields generated by the topological charge are. The field equation is (5.36) and the boundary condition is given by Eq. (5.45). The stress must vanish far away from the topological charge that generates it. As noted above, the boundary condition implies the existence of a branch cut. Linearity implies that $u_z \sim b$. We can meet all of these constraints and solve the field equation by choosing

$$u_z = \frac{b}{2\pi}\Im[\log z] = \frac{b\,\theta}{2\pi} \ . \tag{5.46}$$

Recall that $\log z = \log(re^{i\theta}) = \log(r) + i\theta$ (note that formally, the argument of the log should be dimensionless, but here it would not make a difference). This is a solution because $\log z$ has the proper branch cut, it is analytic outside the branch cut and it satisfies

$$\oint du_z = \frac{b}{2\pi}\int_{-\pi}^{\pi} d\theta = b \ . \tag{5.47}$$

Using Eq. (5.44) we obtain

$$\sigma_{zy} + i\sigma_{zx} = \frac{\mu b}{2\pi}\frac{x - iy}{x^2 + y^2} \sim \frac{\mu b}{r} \ . \tag{5.48}$$

Such a dislocation is known as a screw dislocation. As we said above, dislocations are very interesting objects. Here, by looking at the linear elastic consequences of dislocations, we already see one aspect of it. The linear elastic stress field diverges as $1/r$ near the core of a dislocation. The size of the Burgers vector b is atomic (typically a lattice spacing). The linear elastic solution is valid at distances larger than the dislocation core, whose size c is typically of the order of a few atomic spacings. The detailed structure of the topological defect within the core regularizes the linear elastic divergence. What happens at large distances? The stress (and strain) fields of a dislocation decay very slowly in space, an observation that has profound consequences. To see this, let us calculate the energy of a single dislocation (per unit length in the z-direction)

$$dU/dz = \int u\, dxdy = \int_c^R \frac{1}{2}\sigma_{ij}\varepsilon_{ij} r dr d\theta \sim \mu b^2 \int_c^R \frac{rdr}{r^2} = \mu b^2 \log\left(\frac{R}{c}\right) \ , \tag{5.49}$$

where R is the macroscopic size of the system. We see that the energy of a single dislocation diverges logarithmically with the size of the system. That means that when many dislocations are present (a number to bear in mind for a strongly deformed metal is $10^{15}\mathrm{m}^{-2}$), they are strongly interacting. Dislocations are amongst the most strongly interacting objects in nature. In addition, they are also amongst the most dissipative objects we know of, but that has to do with their motion, which we did not consider here.

2. Conformal invariance

Many equations of mathematical physics possess an important and very useful property called conformal invariance. A conformal transformation/mapping between the complex planes ω and z is defined as

$$z = \Phi(\omega) \ , \tag{5.50}$$

where $\Phi(\omega)$ is an analytic function with a non-vanishing derivative, i.e., $\Phi'(\omega) \neq 0$. Conformal means (nearly literally) angle-preserving. To see this consider an infinitesimal line element in the ω-plane, $d\omega$, and its image in the z-plane, dz. They are related by

$$dz = \Phi'(\omega) d\omega \ . \tag{5.51}$$

However,

$$\Phi'(\omega) = |\Phi'(\omega)| e^{i \arg[\Phi'(\omega)]} \tag{5.52}$$

which means that every two infinitesimal line elements $d\omega$ going through the point ω_0 are mapped into their images dz going through $z_0 = \Phi(\omega_0)$ by a common expansion/contraction (determined by $|\Phi'(\omega_0)|$) and a common rotation (determined by the angle $\arg[\Phi'(\omega_0)]$). Therefore, the relative angle between them is preserved.

If a field equation is invariant under such a conformal transformation/mapping, then we can solve a given problem in a simple domain and immediately get the solution for a (more) complicated domain by a suitably chosen conformal transformation. This is a powerful mathematical tool. An example for a conformally invariant field equation is Laplace's equation in (5.36). To show this we need to prove that any solution $f(z)$ of Laplace's equation in the z-plane remains a solution in the ω-plane under a conformal transformation $g(\omega) = f(\Phi(\omega))$. For this particular equation this is automatically satisfied since $g(\omega)$ is also an analytic function (because it is a composition of two harmonic functions). However, it would be useful to see how it works. First, note that we have

$$\partial_\omega f(\Phi(\omega)) = \Phi'(\omega)\partial_z f(z) \ , \tag{5.53}$$

i.e., $\partial_\omega = \Phi'(\omega)\partial_z$, which immediately implies $\partial_{\bar{\omega}} = \overline{\Phi'(\omega)}\partial_{\bar{z}}$. We then have

$$\begin{aligned}&\partial_{\bar{\omega}}\partial_\omega f(\Phi(\omega)) = \partial_{\bar{\omega}}\left[\Phi'(\omega)\partial_z f(z)\right] = \\ &\partial_z f(z)\partial_{\bar{\omega}}\Phi'(\omega) + \Phi'(\omega)\partial_{\bar{\omega}}\partial_z f(z) = |\Phi'(\omega)|^2\partial_{\bar{z}}\partial_z f(z) = 0 \ .\end{aligned} \tag{5.54}$$

Therefore,

$$\partial_{\bar{\omega}}\partial_\omega f(\Phi(\omega)) = 0 \ , \tag{5.55}$$

which proves the conformal invariance of Laplace's equation. Later in the course we will use this result in the context of fracture mechanics. It is important to note that conformal invariance is a property of partial differential equations, not of differential operators. In the above example, the differential operator was not invariant, i.e., $\partial_{\bar{\omega}}\partial_\omega = |\Phi'(\omega)|^2\partial_{\bar{z}}\partial_z$, but the equation is. This, for instance, immediately implies that the Helmholtz equation, $\nabla^2 u + u = 0$ is not conformally invariant. Finally, recall that a differential equation is also defined by its boundary conditions, which should be conformally invariant as well. For Dirichlet ($u = \text{const.}$) or Neumann ($\boldsymbol{n} \cdot \nabla u = 0$) boundary conditions this is satisfied, but other boundary conditions make things more complicated.

3. In-plane elasticity, Airy stress function

The Navier-Lamé Eq. (5.24) can be also reduced to 2D under in-plane deformation conditions. There are two possibilities here, one called "plane-stress" and the other one "plane-strain". To see how it is done, let us explicitly write Hooke's law in Eq. (5.15) in terms of E and ν. As was noted above, the stiffness tensor $\boldsymbol{C}$ in the relation $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$ can be represented as a 6 by 6 matrix such that

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{zy}\\ \sigma_{zx}\\ \sigma_{xy}\end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)}\begin{pmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{zy}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{pmatrix} . \tag{5.56}$$

We can invert this relation into the form $\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}$, where $\boldsymbol{S} = \boldsymbol{C}^{-1}$ is the compliance tensor (if you noticed that $\boldsymbol{C}$ is called the stiffness tensor and $\boldsymbol{S}$ is called the compliance tensor and wondered about it, this is not a mistake and there is no intention to confuse you. It is a long-time convention that cannot be reverted anymore). We write the last relation explicitly as

$$\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{zy}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{pmatrix} = \frac{1}{E}\begin{pmatrix}1 & -\nu & -\nu & 0 & 0 & 0\\ -\nu & 1 & -\nu & 0 & 0 & 0\\ -\nu & -\nu & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1+\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1+\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1+\nu\end{pmatrix}\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{zy}\\ \sigma_{zx}\\ \sigma_{xy}\end{pmatrix} . \tag{5.57}$$

We are now ready to perform the reduction to 2D.

Plane-stress

We first consider objects that are thin in one dimension, say z, and are deformed in the xy-plane. What happens in the z-direction? Since the two planes $z = 0$ and $z = h$ (where h is the thickness which is much smaller than any other lengthscale in the problem) are traction-free, we approximate $\sigma_{zz} = 0$ everywhere (an approximation that becomes better and better as $h \to 0$). Similarly, we have $\sigma_{zy} = \sigma_{zx} = 0$. We can therefore set $\sigma_{zz} = \sigma_{zy} = \sigma_{zx} = 0$ in Eq. (5.57) to obtain

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} \tag{5.58}$$

and

$$\varepsilon_{zz}(x,y) = -\frac{\nu}{E}\left[\sigma_{xx}(x,y) + \sigma_{yy}(x,y)\right] \ . \tag{5.59}$$

To obtain the plane-stress analog of the Navier-Lamé Eq. (5.24) we need to invert Eq. (5.58), obtaining

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} \ , \tag{5.60}$$

which can not be simply obtained from Eq. (5.56) by removing columns and rows. We can now substitute the last relation in the 2D momentum balance equation $\nabla \cdot \boldsymbol{\sigma} = \rho \partial_{tt} \boldsymbol{u}$ (we stress again that $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are already 2D here). The resulting 2D equation reads

$$\left[\frac{\nu E}{1-\nu^2} + \frac{E}{2(1+\nu)}\right] \nabla\left(\nabla \cdot \boldsymbol{u}\right) + \left[\frac{E}{2(1+\nu)}\right] \nabla^2 \boldsymbol{u} = \rho \partial_{tt} \boldsymbol{u} \ , \tag{5.61}$$

which is identical in form to the Navier-Lamé Eq. (5.24) simply with a renormalized λ

$$\lambda \to \tilde{\lambda} = \frac{\nu E}{1-\nu^2} = \frac{2\nu\mu}{1-\nu} = \frac{2\lambda\mu}{\lambda + 2\mu} \ . \tag{5.62}$$

The shear modulus μ remains unchanged

$$\tilde{\mu} = \mu = \frac{E}{2(1+\nu)} \ . \tag{5.63}$$

Finally, we can substitute $\sigma_{xx}(x,y)$ and $\sigma_{yy}(x,y)$ inside Eq. (5.59) to obtain $\varepsilon_{zz}(x,y)$. Note that $u_z(x,y,z) = \varepsilon_{zz}(x,y) z$ is linear in z.

Plane-strain

We now consider objects that are very thick in one dimension, say z, and are deformed in the xy-plane with no z dependence. These physical conditions are termed plane-strain and are characterized by $\varepsilon_{zx} = \varepsilon_{zy} = \varepsilon_{zz} = 0$. Eliminating these components from Eq. (5.56) we obtain

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} \tag{5.64}$$

and

$$\sigma_{zz}(x,y) = \frac{\nu E}{(1+\nu)(1-2\nu)}\left[\varepsilon_{xx}(x,y) + \varepsilon_{yy}(x,y)\right] \ . \tag{5.65}$$

We can now substitute Eq. (5.64) in the 2D momentum balance equation $\nabla \cdot \boldsymbol{\sigma} = \rho \partial_{tt}\boldsymbol{u}$ (where again $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are 2D). The resulting 2D equation is identical to the Navier-Lamé Eq. (5.24), both in form and in the elastic constants. With the solution at hand, we can use Eq. (5.65) to calculate $\sigma_{zz}(x,y)$. Finally, we note that Eq. (5.64) can be inverted to

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1+\nu}{E} \begin{pmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} \ , \tag{5.66}$$

which can not be simply obtained from Eq. (5.57) by eliminating columns and rows. Using the last relation we can rewrite Eq. (5.65) as

$$\sigma_{zz}(x,y) = \nu\left[\sigma_{xx}(x,y) + \sigma_{yy}(x,y)\right] \ . \tag{5.67}$$

In summary, we see that in both plane-stress and plane-strain cases we can work with 2D objects instead of their 3D counterparts, which is a significant simplification. One has, though, to be careful with the elastic constants as explained above.

Airy stress function (potential)

Focus now on 2D static deformation conditions (either plane-stress or plane-strain) and write down the momentum balance equations under static conditions

$$\partial_x \sigma_{xx} + \partial_y \sigma_{xy} = 0 \quad \text{and} \quad \partial_x \sigma_{yx} + \partial_y \sigma_{yy} = 0 \ . \tag{5.68}$$

These equations are automatically satisfied if $\boldsymbol{\sigma}$ is derived from a scalar potential χ following

$$\sigma_{xx} = \partial_{yy}\chi, \quad \sigma_{xy} = -\partial_{xy}\chi, \quad \sigma_{yy} = \partial_{xx}\chi \ . \tag{5.69}$$

χ is called Airy stress potential. This implies

$$\sigma_{xx} + \sigma_{yy} = \nabla^2 \chi \ . \tag{5.70}$$

What a differential equation does χ satisfy? Obviously up to now we did not use the fact that we are talking about a linear elastic solid (we just used linear momentum balance). To incorporate the linear elastic nature of the problem we use Hooke's law, which implies

$$\sigma_{xx} + \sigma_{yy} = \operatorname{tr}\boldsymbol{\sigma} \propto \operatorname{tr}\boldsymbol{\varepsilon} = \nabla \cdot \boldsymbol{u} \ . \tag{5.71}$$

However, we already proved that $\nabla\cdot\boldsymbol{u}$ is harmonic under static conditions (cf. Eq. (5.33)), leading to $\nabla^2(\sigma_{xx}+\sigma_{yy})=0$, which in turn implies

$$\nabla^2\nabla^2\chi = 0\ . \tag{5.72}$$

Therefore, χ satisfies the bi-Laplace equation, i.e., it is a bi-harmonic function.

Example: Cylindrical cavity

Consider a large linear elastic solid containing a cylindrical hole of radius R under uniform radial tensile loading σ^∞ far away (plane-strain conditions). The hole can be regarded as a defect inside a perfect solid. What is the emerging stress field? What can we learn from it? First, the geometry of the problem suggests we should work in polar coordinates (derive)

$$\left(\partial_{rr}+\frac{\partial_r}{r}+\frac{\partial_{\theta\theta}}{r^2}\right)\left(\partial_{rr}+\frac{\partial_r}{r}+\frac{\partial_{\theta\theta}}{r^2}\right)\chi(r,\theta)=0 \tag{5.73}$$

and

$$\sigma_{rr}=\frac{\partial_r\chi}{r}+\frac{\partial_{\theta\theta}\chi}{r^2},\quad \sigma_{r\theta}=-\partial_r\left(\frac{\partial_\theta\chi}{r}\right),\quad \sigma_{\theta\theta}=\partial_{rr}\chi\ . \tag{5.74}$$

Furthermore, azimuthal symmetry implies $\sigma_{r\theta}=0$ and no θ-dependence. Moreover, since the only lengthscale in the problem is R, we expect the result to be a function of r/R alone. Finally, linearity implies that σ_{rr} and $\sigma_{\theta\theta}$ are proportional to σ^∞. We should now look for θ-independent solutions of the bi-harmonic equation of (5.72) with the following boundary conditions

$$\sigma_{rr}(r\!=\!R)=0 \quad \text{and} \quad \sigma_{rr}(r/R\!\to\!\infty)=\sigma^\infty\ . \tag{5.75}$$

The θ-independent solutions of Eq. (5.73) are r^2, $\log(r)$ and $r^2\log(r)$ (show that $\chi(r,\theta)\!=\!\phi_0(r,\theta)+\phi_1(r,\theta)r\cos\theta+\phi_2(r,\theta)r\sin\theta+\phi_3(r,\theta)r^2$ is the general solution of the bi-harmonic equation, where $\{\phi_i(r,\theta)\}$ are harmonic). The $r^2\log r$ solution gives rise to a logarithmically diverging stress as $r\!\to\!\infty$, and hence should be excluded here. We therefore have

$$\chi(r)=a\log(r)+b\,r^2\ . \tag{5.76}$$

Satisfying the boundary conditions implies that $a\!=\!-\sigma^\infty R^2$ and $b\!=\!\sigma^\infty/2$, leading to

$$\sigma_{rr}=\sigma^\infty\left(1-\frac{R^2}{r^2}\right),\quad \sigma_{\theta\theta}=\sigma^\infty\left(1+\frac{R^2}{r^2}\right)\ . \tag{5.77}$$

Note that $\sigma_{\theta\theta}$ at the surface of the cylinder, which tends to break the material apart, is two times larger than σ^∞. This amplification factor, which is mild in this case, is a generic property of defects which plays a crucial role in determining the strength of solids. We will discuss this later in the course when dealing with failure. Another interesting feature of the solution is that $\sigma_{\theta\theta} + \sigma_{rr}$ is a constant.

Complex variable methods are applicable to Eq. (5.72) as well. We first rewrite it in terms of complex differential operators as

$$\partial_{zz}\partial_{\bar{z}\bar{z}}\chi = 0 \ . \tag{5.78}$$

It is obvious that analytic functions are solutions of this equation. However, there are more solutions because of the appearance of another derivative with respect to $\bar{z}$. In fact it is clear that $\bar{z}f(z)$, where $f(z)$ is an analytic function, is also a solution. As no other solutions can be found, the most general solution of the bi-Laplace equation is given in terms of *two* analytic functions $f(z)$ and $g(z)$ as

$$\chi = \Re\left[\bar{z}f(z) + g(z)\right] \ . \tag{5.79}$$

Of course the imaginary part can be used as well. It is important to understand that while this solution is given in terms of analytic function it is by itself not an analytic function. The reason is obvious: it depends on $\bar{z}$. The stress tensor can be easily derived using complex derivatives, yielding (derive)

$$\sigma_{xx} + \sigma_{yy} = 4\Re\left[f'(z)\right] \ , \tag{5.80}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\bar{z}f''(z) + g''(z)\right] \ . \tag{5.81}$$

Finally, we note that while the bi-Laplace equation is *not* conformally invariant (prove), conformal methods are useful for its solution.

D. Elastic waves

Up to now we did not discuss dynamic phenomena. However, the most basic solutions of the Navier-Lamé Eq. (5.24) are dynamic and well-known to you from everyday life: elastic waves. This might appear strange at first sight because the Navier-Lamé Eq. (5.24) does not take the form of an ordinary wave equation. The reason will become clear soon. The first step to address this question would be to decompose the general displacement field $\boldsymbol{u}$ into a curl-free component and a divergence-free component (Helmholtz decomposition)

$$\boldsymbol{u} = \nabla\phi + \nabla \times \boldsymbol{\psi} \ , \tag{5.82}$$

where ϕ and $\boldsymbol{\psi}$ are scalar and vector displacement potentials, respectively. Recall that $\nabla\cdot(\nabla\times\boldsymbol{\psi}) = 0$ and $\nabla\times(\nabla\phi) = 0$. Note that the vector potential features a gauge freedom, i.e., $\boldsymbol{\psi}\to\boldsymbol{\psi}+\nabla\varphi$ with a scalar field φ leaves $\boldsymbol{u}$ unchanged. A common gauge choice is $\nabla\cdot\boldsymbol{\psi} = 0$ (e.g., as adopted in seismology). In 2D, it can be satisfied by choosing $\boldsymbol{\psi}=\psi_z(x,y,t)\,\hat{z}$.

Substituting Eq. (5.82) into the Navier-Lamé Eq. (5.24), we obtain

$$\nabla\left[(\lambda+2\mu)\nabla^2\phi-\rho\partial_{tt}\phi\right]+\nabla\times\left[\mu\nabla^2\boldsymbol{\psi}-\rho\partial_{tt}\boldsymbol{\psi}\right]=0\ . \tag{5.83}$$

Using the analogy of this equation with Eq. (5.82) we see that each term in the square bracket should vanish independently, yielding

$$c_d^2\nabla^2\phi=\partial_{tt}\phi \quad \text{and} \quad c_s^2\nabla^2\boldsymbol{\psi}=\partial_{tt}\boldsymbol{\psi}\ , \tag{5.84}$$

where

$$c_d=\sqrt{\frac{\lambda+2\mu}{\rho}},\quad c_s=\sqrt{\frac{\mu}{\rho}}\ , \tag{5.85}$$

are the dilatational (longitudinal, sound) and shear wave speeds, respectively. Linear elastodynamics is therefore characterized by two different wave equations with two different wave speeds, $c_d > c_s$ (recall that $\lambda > -2\mu/3$, which implies $\lambda+2\mu > 4\mu/3 > \mu$). In that sense, while this theory shares various features with electrodynamics (electromagnetism), it is more complicated because of the presence of two wave speeds instead of one (the speed of light). It is also important to note that while the two wave equations in (5.84) are independent inside the bulk of the solid, they are coupled on the boundaries, which of course makes things more complicated (we will see this explicitly when discussing fracture later in the course). Finally, note that there exist also surface (Rayleigh) waves whose propagation velocity c_R is different from both c_s and c_d. In general, we have $c_R < c_s < c_d$.

How do we actually know that c_s corresponds to shear waves and c_d to dilatational waves? This is implicit in the fact that the latter are curl-free and the former are divergence-free, but can we find more explicit distinguishing features? To that aim, consider plane-wave solutions of the form

$$\boldsymbol{u}=g(\boldsymbol{x}\cdot\boldsymbol{n}-c\,t)\boldsymbol{a}\ , \tag{5.86}$$

where $\boldsymbol{n}$ is the propagation direction, $\boldsymbol{a}$ is the direction of the displacement and $|\boldsymbol{n}|=|\boldsymbol{a}|=1$. Substituting this expression into the Navier-Lamé Eq. (5.24), we obtain (see tutorial)

$$(c_d^2-c_s^2)(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}+(c_s^2-c^2)\boldsymbol{a}=0\ . \tag{5.87}$$

There are two independent solutions to this equation; either $c = c_s$ and $\boldsymbol{a} \cdot \boldsymbol{n} = 0$ or $c = c_d$ and $\boldsymbol{a} \cdot \boldsymbol{n} = \pm 1$ (recall that both $\boldsymbol{n}$ and $\boldsymbol{a}$ are unit vectors). Therefore, shear waves are polarized such that the displacement is always orthogonal to the propagation direction and dilatational waves are polarized such that the displacement is parallel to the propagation direction.

Dilatational (pressure/density) waves in fluids and solids

Let us briefly discuss the difference between dilatational (pressure/density) waves in fluids and solids. Since we consider linear waves we can neglect convective nonlinearities and hence the momentum balance equation for both fluids and solids reads

$$\rho \partial_t \boldsymbol{v} = \nabla \cdot \boldsymbol{\sigma} \ . \tag{5.88}$$

The difference stems from the different constitutive laws that relate the stress tensor $\boldsymbol{\sigma}$ to particles motions. To see this, we use Eq. (5.19) to write Hooke's law as

$$\boldsymbol{\sigma} = K \, \mathrm{tr} \, \boldsymbol{\varepsilon} \boldsymbol{I} + 2\mu \left(\boldsymbol{\varepsilon} - \frac{1}{3} \, \mathrm{tr} \, \boldsymbol{\varepsilon} \boldsymbol{I} \right) \ . \tag{5.89}$$

For a fluid we have $\mu = 0$, i.e., fluids cannot sustain shear stresses at all (note that since we focus on non-dissipative waves we exclude viscous stresses here). Consider then small density perturbations, $\rho = \rho_0 + \delta\rho$, such that

$$\mathrm{tr} \, \boldsymbol{\varepsilon} = \frac{\delta V}{V} = -\frac{\delta \rho}{\rho_0} \ . \tag{5.90}$$

Therefore, to linear order in density perturbations the momentum balance equation for fluids reads

$$\rho_0 \partial_t \boldsymbol{v} = -\frac{K}{\rho_0} \nabla \rho \ . \tag{5.91}$$

Operating with the divergence operator on both sides of this equation we obtain

$$\rho_0 \partial_t \nabla \cdot \boldsymbol{v} = -\frac{K}{\rho_0} \nabla^2 \rho \ . \tag{5.92}$$

Finally, linearizing the mass conservation equation of (4.5)

$$\partial_t \rho + \rho_0 \nabla \cdot \boldsymbol{v} = 0 \ , \tag{5.93}$$

we obtain

$$\partial_{tt} \rho = \frac{K}{\rho_0} \nabla^2 \rho \ . \tag{5.94}$$

Therefore, the speed of sound (dilatational/desnity/pressure waves) in fluids is $\sqrt{K/\rho_0}$. What happens in solids? One may naively think that even though solids feature a finite shear modulus μ, the combination $\boldsymbol{\varepsilon}-\frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\varepsilon}\boldsymbol{I}$ — which describes shear/deviatoric deformation — does not contribute to dilatational waves. This is wrong. In fact, we have

$$\nabla\cdot\left(\boldsymbol{\varepsilon}-\frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\varepsilon}\,\boldsymbol{I}\right)=\frac{2}{3}\nabla\,\mathrm{tr}\,\boldsymbol{\varepsilon}-\frac{1}{2}\nabla\times(\nabla\times\boldsymbol{u})\ . \tag{5.95}$$

That is, $\boldsymbol{\varepsilon}^{dev}=\boldsymbol{\varepsilon}-\frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\varepsilon}\,\boldsymbol{I}$ is trace-less, but not divergence-free. Using this result in the momentum balance equation (through Hooke's law) and operating with the divergence operator on both sides we obtain

$$\rho_0\partial_t\nabla\cdot\boldsymbol{v}=\left(K+\frac{4\mu}{3}\right)\nabla^2\,\mathrm{tr}\,\boldsymbol{\varepsilon}\ . \tag{5.96}$$

Following the steps as in the fluid case, we immediately see that the speed of sound in solids is $\sqrt{\frac{K+\frac{4\mu}{3}}{\rho_0}}$, which is of course identical to the result obtained in Eq. (5.85) since $K+\frac{4\mu}{3}=\lambda+2\mu$. We thus conclude that the shear modulus contributes to the speed of sound in solids, which is different from the speed of sound in fluids.

VI. THE LINEARIZED FIELD THEORY OF THERMO-ELASTICITY

What happens when additional fields play a role? Up to now we did not consider explicitly the role of temperature. We know that ordinary solids expand when heated. Therefore, we expect that differential heating, i.e., temperature gradients, would give rise to nontrivial thermal stresses. Such processes are important in a wide range of physical systems, from heat engines, through blood vessels to the deformation of the earth. In situations in which the temperature T plays a role, the relevant thermodynamic potential is the Helmholtz free energy, which is obtained by a Legendre transformation of the internal energy

$$f(\boldsymbol{\varepsilon}, T) = u(\boldsymbol{\varepsilon}, T) - T\, s(\boldsymbol{\varepsilon}, T) \ . \tag{6.1}$$

Therefore, the second law of thermodynamics (dissipation inequality) in Eq. (4.38) reads

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{f} - s\,\dot{T} \geq 0 \ , \tag{6.2}$$

where we identified $\boldsymbol{D} = \dot{\boldsymbol{\varepsilon}}$. Using the chain rule to express $\dot{f}$, we obtain

$$\left(\boldsymbol{\sigma} - \frac{\partial f}{\partial \boldsymbol{\varepsilon}}\right) : \dot{\boldsymbol{\varepsilon}} - \left(s + \frac{\partial f}{\partial T}\right)\dot{T} \geq 0 \ . \tag{6.3}$$

Since elastic response in reversible, we expect an equality to hold. Moreover, the strain and the temperature can be varied independently. Therefore, the second law analysis implies

$$\boldsymbol{\sigma} = \frac{\partial f}{\partial \boldsymbol{\varepsilon}} \quad \text{and} \quad s = -\frac{\partial f}{\partial T} \ . \tag{6.4}$$

These relations are the macro-canonical counterparts of Eqs. (5.4). What form then $f(\boldsymbol{\varepsilon}, T)$ can take within a linear theory? Obviously the temperature independent terms in Eq. (5.11) still appear. To couple temperature variations to deformation we need to construct a scalar, which within a linear theory must take the form $(T - T_0)\ \mathrm{tr}\,\boldsymbol{\varepsilon}$ (where T_0 is some reference temperature). Therefore, $f(\boldsymbol{\varepsilon}, T)$ takes the form

$$f(\boldsymbol{\varepsilon}, T) = \frac{1}{2}K(\mathrm{tr}\,\boldsymbol{\varepsilon})^2 + \mu\left(\varepsilon_{ij} - \frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\varepsilon}\delta_{ij}\right)^2 - K\alpha_T\,(T - T_0)\ \mathrm{tr}\,\boldsymbol{\varepsilon} + f_0(T) \ , \tag{6.5}$$

where the physical meaning of α_T will become clear soon and $f_0(T)$ is a temperature dependent function that plays no role here. The constitutive relation reads

$$\sigma_{ij} = -K\alpha_T\,(T - T_0)\,\delta_{ij} + K\,\mathrm{tr}\,\boldsymbol{\varepsilon}\delta_{ij} + 2\mu(\varepsilon_{ij} - \frac{1}{3}\,\mathrm{tr}\,\boldsymbol{\varepsilon}\delta_{ij}) \ . \tag{6.6}$$

First consider free thermal expansion of a body (i.e., the temperature is increased from T_0 to T). In this case no stresses emerge, $\boldsymbol{\sigma} = 0$ (why is this the case?), and the deformation is isotropic, hence Eq. (6.6) implies

$$\operatorname{tr}\boldsymbol{\varepsilon} = \alpha_T\,(T - T_0)\ . \tag{6.7}$$

Since $\operatorname{tr}\boldsymbol{\varepsilon}$ is the relative volume change, $\delta V/V$, α_T is simply the thermal expansion coefficient $\alpha_T = \frac{1}{V}\frac{\partial V}{\partial T}$.

A side comment: While the thermal expansion coefficient appears as a linear response coefficient, it is not a harmonic (linear) material property (i.e., it cannot be obtained from a quadratic approximation to the energy). To see this, convince yourself that the thermal average $\delta V = V\langle \operatorname{tr}\boldsymbol{\varepsilon}\rangle_T$ vanishes when a quadratic approximation to the energy, $u \sim (\operatorname{tr}\boldsymbol{\varepsilon})^2$, is used. You need to go nonlinear, i.e., invoke anharmonic contributions to the energy.

The equations of motion for a linear thermo-elastic solid take the form (neglecting inertia and body forces, and using λ and μ again)

$$(\lambda + \mu)\nabla\left(\nabla\cdot\boldsymbol{u}\right) + \mu\nabla^2\boldsymbol{u} = \alpha_T\,K\nabla T\ . \tag{6.8}$$

This shows that thermal gradients appear as a source term (inhomogeneous term) in the standard linear elastic equations of motion.

Example: Heated annulus

Consider a thin annulus of internal radius R_1 and external radius R_2. Consider then a nonuniform, purely radial, temperature field $T(r)$ and determine the resulting displacement field. The geometry of the problem implies that the only non-vanishing displacement component is $u_r(r,\theta) = u(r)$. Writing then Eqs. (6.8) in polar coordinates we obtain

$$\partial_{rr}u + \frac{\partial_r u}{r} - \frac{u}{r^2} = \frac{\alpha_T\,K}{\lambda + 2\mu}\partial_r T\ . \tag{6.9}$$

What are the boundary conditions? As the inner and outer surfaces of the annulus are traction-free, we have the following boundary conditions

$$\sigma_{rr}(r = R_1) = \sigma_{rr}(r = R_2) = 0\ . \tag{6.10}$$

The key point for solving Eq. (6.9) is to note that the operator on the left-hand-side can be rewritten in compact form as

$$\partial_{rr}u + \frac{\partial_r u}{r} - \frac{u}{r^2} = \partial_r\left[\frac{\partial_r(r\,u)}{r}\right]\ . \tag{6.11}$$

Therefore, integrating twice Eq. (6.9) readily yields

$$u(r) = \frac{\alpha_T K}{\lambda + 2\mu}\frac{1}{r}\int_{R_1}^{r} T(r')r'dr' + \frac{c_1 r}{2} + \frac{c_2}{r} \ , \tag{6.12}$$

where c_1 and c_2 are two integration constants. These are being determined (derive) by the traction-free boundary conditions of Eq. (6.10) and turn out to be proportional to $\int_{R_1}^{R_2} T(r')r'dr'$. Are these results valid for $T(r)\!=\!$const., i.e., for a spatially uniform temperature field?

VII. THE NON-LINEAR FIELD THEORY OF ELASTICITY

Our previous discussion focussed on linear elastic deformation. Why is it such a useful theory? After all it is a linear perturbation theory, so what makes it so relevant in a wide range of situations? In other words, why ordinary solids do not typically experience large elastic deformations? The answer is hidden in a small parameter that we have not yet discussed. Until now the only material parameter of stress dimensions was the elastic modulus, say μ. In ordinary solids the elastic modulus is "large". Compared to what? What other typical, intrinsic, stress scales exist? The answer is that ordinary solids start to respond irreversibly (flow plastically, break, etc.) at a typical stress level that is usually much smaller than the elastic modulus. In other words, reversibility breaks down at a typically small displacement gradient. As reaching the onset of irreversibility limit still requires relatively large stresses, this explains why a small elastic deformation perturbation theory is useful. We will focuss on irreversible processes later in the course.

Everyday life experience, however, tells us that there are many materials that respond reversibly at large deformation. Think, for example, of a rubber band, of your skin or of jelly. Such materials can deform to very large strains (of order unity or more) under mild stresses and recover their original shape when the stress is removed. They are "soft". Such soft materials are of enormous importance and range of applicability, and have attracted lots of attention in recent years. What makes them significantly softer than ordinary solids? The answer is that their elasticity has a difference origin.

A. Entropic elasticity ("Rubber elasticity")

The paradigmatic example of an elastic behavior is a Hookean spring in which a restoring force is exerted in response to length/shape variations. In this case, the restoring force has an energetic origin: the interatomic interaction energy changes with the length/shape variations. However, this is not the only form of an elastic behavior. Consider the Helmholtz free energy density

$$f(\boldsymbol{E},T) = u(\boldsymbol{E},T) - T\,s(\boldsymbol{E},T)\ , \tag{7.1}$$

where $\boldsymbol{E}$ now is the Green-Lagrange (metric) strain tensor. A stress measure is obtained by the variation of f with respect to $\boldsymbol{E}$. Ordinary elasticity of "hard" materials, e.g., of metals, has an energetic origin. In this case, the entropy s does not depend on the deformation, while the internal energy u does. On the other hand, "rubber" elasticity of "soft" materials, e.g., of gels, rubber and various polymeric materials, has an entropic origin. In this case, the internal energy does not depend on the deformation, but the entropy does.

To understand the physics underlying entropic elasticity we consider a network of long-chain polymers within a fixed unit volume. We assume that the network is incompressible. Consider first a single polymer chain of length $L = m\ell$, where m is the number of monomers and ℓ is the length of a single monomer. Suppose now that one end of the chain is fixed (say at the origin) and the other end is free to wander in space. Denote the end-to-end vector by $\boldsymbol{r}$. In situations in which $r = |\boldsymbol{r}| \ll L$ and under the assumption that there is no correlation between the orientation of successive monomers, the probability distribution function of the end-to-end distance $p(r)$ can be easily determined, in analogy to a random walk in time, to be

$$p(r) \propto e^{-\frac{3r^2}{2\langle r^2 \rangle}} \ , \tag{7.2}$$

where $\langle r^2 \rangle = m\,\ell^2$ is the mean-square value of r (in the analogy to random walk in time, m plays the role of time t). $p(r)$ measures the number of *configurations* the chain can be in for a given end-to-end distance r. Also note that no elastic energy is involved here, i.e., the "joints" of size ℓ can move freely (in principle, ℓ can be larger than the monomer size, i.e., the so-called persistence length above which correlations fade away. In this case, the polymer is termed "semi-flexible", but we do not discuss this here). A chain with these properties is termed a Gaussian chain (note that despite the name, $p(r)$ of Eq. (7.2) is not strictly Gaussian, but rather takes the form $p(r) = 4\pi r^2 \left(\frac{3}{2\,\pi\langle r^2\rangle}\right)^{3/2} e^{-\frac{3r^2}{2\langle r^2\rangle}}$, featuring $p(0) = 0$ and $r > 0$. Consequently, one can say that $p(r)$ is predominantly Gaussian). The configurational entropy of a single polymer chain with an end-to-end distance r is given by

$$\bar{s} = s_0 + k_B \ln\left[p(r)\right] = \bar{s}_0 - k_B \frac{3r^2}{2\langle r^2 \rangle} \ , \tag{7.3}$$

where s_0 and $\bar{s}_0$ are unimportant constants. In order to understand the effect of deformation on the entropy of the i$^{\text{th}}$ polymer chain we denote the undeformed end-to-end distance by $\boldsymbol{r}^{(i)} = (X_1^{(i)}, X_2^{(i)}, X_3^{(i)})$ and the deformed one by $\tilde{\boldsymbol{r}}^{(i)} = (x_1^{(i)}, x_2^{(i)}, x_3^{(i)}) = (\lambda_1^{(i)} X_1^{(i)}, \lambda_2^{(i)} X_2^{(i)}, \lambda_3^{(i)} X_3^{(i)})$, where $\{\lambda_{k=1-3}^{(i)}\}$ are the (principal) stretches. Therefore, the entropy change of a single chain due to deformation reads

$$\Delta\bar{s}^{(i)} = -\frac{3k_B}{2\langle r^2 \rangle}\left(([\lambda_1^{(i)}]^2 - 1)[X_1^{(i)}]^2 + ([\lambda_2^{(i)}]^2 - 1)[X_2^{(i)}]^2 + ([\lambda_3^{(i)}]^2 - 1)[X_3^{(i)}]^2\right) \ . \tag{7.4}$$

We now assume that the deformation is *affine*, i.e., that the macroscopic and microscopic strains are the same, $\lambda_k^{(i)} = \lambda_k$. Hence, the entropy change per unit volume of the part of the polymeric network that contains N chains and occupies a volume V reads

$$s = \sum_{i=1}^{N} \frac{\Delta \bar{s}^{(i)}}{V} = \tag{7.5}$$
$$-\frac{3k_B}{2V\langle r^2\rangle}\left((\lambda_1^2 - 1)\sum_{i=1}^{N}[X_1^{(i)}]^2 + (\lambda_2^2 - 1)\sum_{i=1}^{N}[X_2^{(i)}]^2 + (\lambda_3^2 - 1)\sum_{i=1}^{N}[X_3^{(i)}]^2 \right) .$$

We now invoke isotropy and assume that we can treat $\{[X_k^{(i)}]^2\}$ as independent variables, to obtain

$$\sum_{i=1}^{N}[X_1^{(i)}]^2 = \sum_{i=1}^{N}[X_2^{(i)}]^2 = \sum_{i=1}^{N}[X_3^{(i)}]^2 = \frac{1}{3}N\langle r^2\rangle \ . \tag{7.6}$$

Therefore, the free energy density of the polymeric network (due to deformation) is given by

$$f = -Ts = \frac{1}{2}nk_BT\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) \ , \tag{7.7}$$

where $n \equiv N/V$ is the number of chains per unit volume (density of chains). Recall that we also assume incompressibility (consequently we did not consider the variation of the entropy with volume changes), i.e., that the constitutive law also includes the incompressibility condition

$$J = \det \boldsymbol{F} = \lambda_1\lambda_2\lambda_3 = 1 \ . \tag{7.8}$$

We can immediately identify nk_BT in Eq. (7.7) as an elastic modulus (it has the dimensions of energy density, i.e., of stress), which actually corresponds to the shear modulus

$$\mu = nk_BT \ . \tag{7.9}$$

This dependence on T, i.e., $d\mu/dT > 0$, has remarkable consequences that distinguish entropic elasticity from energetic one. For example, a piece of rubber under a fixed force will shrink/expand in response to heating/cooling, just the opposite of the behavior of a metallic spring! Another related effect, that we do not discuss in detail here, is that of adiabatic stretching. When we rapidly (and elastically) stretch a piece of metal it cools down. However, a rubber band under the same conditions warms up. You can easily experience it yourself by rapidly stretching a piece of rubber and using your lips as a thermo-sensitive device. We note that f, which was calculated above, is the free-energy in the deformed configuration per unit volume in the undeformed configuration.

Equations (7.7)-(7.8) constitute the incompressible neo-Hookean model, which is one of the first and most useful nonlinear elastic models. The statistical mechanical model that was used to derived it, originally due to Flory in the early 1940's, is called the Gaussian-chain model. The name "neo-Hookean" has to do with the intimate relation of this model to the small strains linear elastic Hookean model. Noting that

$$\mathrm{tr}\,\boldsymbol{E} = \frac{1}{2}\,\mathrm{tr}(\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I}) = \frac{1}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) \ , \tag{7.10}$$

we can rewrite Eq. (7.7) as

$$f = \mu\ \mathrm{tr}\,\boldsymbol{E} = \frac{1}{2}\mu\left[\mathrm{tr}(\boldsymbol{F}^T\boldsymbol{F}) - 3\right] \ . \tag{7.11}$$

We can incorporate Eq. (7.8) into this free-energy function by writing

$$f = \frac{1}{2}\mu\left[\mathrm{tr}(\boldsymbol{F}^T\boldsymbol{F}) - 3\right] - \alpha\,(J-1) \ , \tag{7.12}$$

where α is a Lagrange multiplier introduced to enforce incompressibility. This is the simplest possible model that is quadratic in $\boldsymbol{F}$ and reduces to Hookean elasticity at small stretches. This phenomenological approach cannot, of course, predict the exact expression of μ in Eq. (7.9), which requires a statistical mechanical derivation, though the T-dependence is expected on general grounds. Note also that unlike Hookean linear elasticity the neo-Hookean model is rotationally invariant under **finite** rotations and is also objective.

To appreciate the "softness" of materials that are governed by entropic (rubber-like) elasticity, let us make some rough estimates. First, consider ordinary (say, metallic) solids. The elastic modulus has the dimensions of stress, which is equivalent to energy density. The typical energy scale for metals is roughly 1eV. Divide this by an atomic volume, $\Omega \simeq 10^{-29}\mathrm{m}^3$, and you get 10GPa which is a reasonable rough estimate (the Young's modulus of metals can reach 100GPa). Consider now Eq. (7.9), $\mu = nk_BT$. At room temperature we have $k_BT_{\mathrm{R}}\simeq 1/40eV$, which sets the energy scale for rubber elasticity. If we assume a chain density n of 10^{-2} per atomic volume, we get an elastic modulus which is about 3 orders of magnitude smaller for rubber. Indeed, 10MPa is a reasonable rough estimate for the modulus of rubber. When we consider hydrogels, which are filled with water (or other solvents), the effective chain density can be significantly smaller and the modulus drops down to the 10KPa range, which is 6 orders of magnitude smaller than ordinary solids. These rigidity levels are also characteristic of biological substance such as tissues and cells.

B. Geometric nonlinearities and stress measures

Many other useful nonlinear elastic models were developed based on either statistical mechanical or phenomenological approaches that employ symmetry principles and experimental observations. It is usually very difficult to solve nonlinear elastic problems analytically. The inherent difficulty goes beyond the usual statement that nonlinear differential equations are not analytically tractable in general. The reason for that is geometrical in nature and has to do with the fact that in nonlinear elastic problems the domain in which we solve the differential equations depends itself on the solution that is sought for (and of course unknown to begin with). Think, for example, of the Cauchy stress $\boldsymbol{\sigma}$, defined as the force per unit area in the deformed configuration, and consider a free boundary. Since the boundary is traction-free, $\sigma_{nn} = \sigma_{tn} = 0$ on it, where n and t denote the normal and tangent to the free boundary, respectively. In order to satisfy this boundary condition throughout the deformation process, the location of the boundary should be known, but this usually requires to know the solution. We did not encounter this problem in the linearized theory since the deformed and undeformed configurations are distinguishable only to second order in the displacement gradient.

One way to deal with this situation is to formulate problems in the undeformed configuration. This was briefly discussed in Eqs. (3.37), (3.38) and (4.20), and will be repeated here within a thermodynamic context. Consider a small incremental deformation of a body (that might be already deformed) and ask how much stress work was done within a volume element $\delta \boldsymbol{x}^3$. For that aim, define an incremental strain measure $d\epsilon$ as the change in length of a material element relative to the current (deformed) state of the material. To (re)stress the difference between ϵ and ε, we resort to 1D and we discuss again Eqs. (3.19)-(3.20). ε is defined as the change in length with respect to the undeformed state ℓ_0, $\ell-\ell_0$, relative to the undeformed state

$$d\varepsilon = \frac{d\ell}{\ell_0} \quad \Longrightarrow \quad \varepsilon = \int_{\ell_0}^{\ell} \frac{d\ell}{\ell_0} = \frac{\ell-\ell_0}{\ell_0} = \lambda - 1 \quad \Longrightarrow \quad \lambda = \varepsilon + 1 \ , \tag{7.13}$$

where $\lambda = \ell/\ell_0$ is the stretch. $d\epsilon$ is defined similarly to $d\varepsilon$, but with respect to the deformed (current) state, implying

$$d\epsilon = \frac{d\ell}{\ell} \quad \Longrightarrow \quad \epsilon = \int_{\ell_0}^{\ell} \frac{d\ell}{\ell} = \ln\left(\frac{\ell}{\ell_0}\right) = \ln\lambda \quad \Longrightarrow \quad \lambda = e^{\epsilon} \ . \tag{7.14}$$

While these two strain measures (as every other two strain measures) agree to linear order, they differ dramatically in general; $\lambda(\varepsilon)$ is a linear function, while $\lambda(\epsilon)$ is exponential. Going back to our original question, the stress work done by the Cauchy stress $\boldsymbol{\sigma}$ in the (current) volume element $\delta\boldsymbol{x}^3$ is

$$\boldsymbol{\sigma} : d\boldsymbol{\epsilon}\,\delta\boldsymbol{x}^3 \ , \tag{7.15}$$

where $d\boldsymbol{\epsilon}$ is a tensorial generalization of $d\epsilon$ (cf. Eq. (3.23), where a slightly different notation was used). We can now associate a new stress measure that is thermodynamically conjugate to a given strain measure by demanding that the stress work produced would equal the above expression. To see how this works let us focus on the case in which the deformation measure we use is $\boldsymbol{F}$, which connects the deformed (reference) and undeformed (current) configurations. Since $\boldsymbol{F}$ is defined in terms of the reference coordinates $\boldsymbol{X}$, the relevant volume element is $\delta\boldsymbol{X}^3$. We then define a stress tensor $\boldsymbol{P}$ such that

$$\boldsymbol{P} : d\boldsymbol{F}\,\delta\boldsymbol{X}^3 = \boldsymbol{\sigma} : d\boldsymbol{\epsilon}\,\delta\boldsymbol{x}^3 \ . \tag{7.16}$$

$\boldsymbol{P}$ is the first Piola-Kirchhoff stress tensor, that was already defined in Eq. (3.37) using other considerations, which is related to $\boldsymbol{\sigma}$ through Eq. (3.38), $\boldsymbol{P} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$. This is perfectly consistent (prove) with the thermodynamic definition of Eq. (7.16). It is very important to note that $\boldsymbol{P} : d\boldsymbol{F}$ is a work increment in the deformed (current) configuration per unit volume in the reference configuration. Since df has the very same meaning, we can identify $df = \boldsymbol{P} : d\boldsymbol{F}$, which leads to

$$\boldsymbol{P} = \frac{\partial f}{\partial \boldsymbol{F}} \ . \tag{7.17}$$

Therefore, $\boldsymbol{P}$ is the force per unit area in the reference configuration acting on its image in the deformed (current) configuration. These quantities might appear (very?) strange at first sight (even at the second and third ones), but they are enormously useful in real calculations since these can be done in the reference configuration. For that aim, we need to express the momentum balance equation in terms of $\boldsymbol{P}$ in the reference configuration, which was already done in Eq. (4.20).

As we said above, this procedure can be followed for *any* strain measure. As another example, consider the Green-Lagrange strain tensor $\boldsymbol{E}$. In that case, we define a stress measure $\boldsymbol{S}$, termed the second Piola-Kirchhoff stress tensor, such that $df = \boldsymbol{S} : d\boldsymbol{E}$. Therefore,

$$\boldsymbol{S} = \frac{\partial f}{\partial \boldsymbol{E}} \ . \tag{7.18}$$

This stress measure is rather commonly used.

To demonstrate how these stress measures (and the associated geometric nonlinearities) appear in physical situations, let us consider the incompressible neo-Hookean material characterized by

$$f = \frac{\mu}{2}\left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right) \tag{7.19}$$

and

$$\det \boldsymbol{F} = \lambda_1 \lambda_2 \lambda_3 = 1 \ , \tag{7.20}$$

where λ_i are the principal stretches. Consider a cylinder under a uniaxial stress state with $P_1 = P > 0$ (along the main axis of the cylinder) and $P_2 = P_3 = 0$ (traction-free lateral boundaries). The stretches take the form $\lambda_1 = \lambda$ and $\lambda_2 = \lambda_3 = \lambda^{-1/2}$, where we used isotropy and incompressibility. The relation between the Cauchy stress and the first Piola-Kirchhoff stress reads (recall that $J = \det \boldsymbol{F} = 1$)

$$\sigma = \lambda P \ . \tag{7.21}$$

Eq. (7.17) implies that in our uniaxial example we simply have $P = \partial f / \partial \lambda$ (this is proved below). Therefore, we have

$$f = \frac{\mu}{2}\left(\lambda^2 + 2\lambda^{-1} - 3\right) \quad \Longrightarrow \quad P = \frac{\partial f}{\partial \lambda} = \mu\left(\lambda - \lambda^{-2}\right) \ . \tag{7.22}$$

The constitutive relation $P = \mu\left(\lambda - \lambda^{-2}\right)$ is different from the constitutive relation

$$\sigma = \lambda P = \mu\left(\lambda^2 - \lambda^{-1}\right) \tag{7.23}$$

due to deformation-induced nonlinearities. Note that all of these effects disappear when we linearize with respect to ε ($\lambda = 1 + \varepsilon$)

$$\sigma \simeq P \simeq 3\,\mu\,\varepsilon = E\,\varepsilon \ . \tag{7.24}$$

Therefore, $E = 3\mu$ is the Young's modulus (did you expect this? what is Poisson's ratio of this material?). One immediate consequence of nonlinearities in the constitutive law is that the symmetry between tension and compression observed in the linear theory is typically broken. We will now work out a few examples to demonstrate the rather dramatic physical effects that emerge in nonlinear elasticity.

In the example above, it was stated/argued that $P = \partial f / \partial \lambda$ is satisfied along the uniaxial tension axis. Let us prove it. Our starting point is the free-energy functional in Eq. (7.12), where α is a Lagrange multiplier introduced to enforce incompressibility (i.e., $\partial f / \partial \alpha = 0$ implies $J = 1$). Recalling that $\partial\,\mathrm{tr}(\boldsymbol{F}^T \boldsymbol{F}) / \partial \boldsymbol{F} = 2\boldsymbol{F}$ and $\partial \det \boldsymbol{F} / \partial \boldsymbol{F} = \det \boldsymbol{F}\ \boldsymbol{F}^{-T}$, we obtain

$$\boldsymbol{P} = \frac{\partial f}{\partial \boldsymbol{F}} = \mu \boldsymbol{F} - \alpha \ \det \boldsymbol{F} \boldsymbol{F}^{-T} \ . \tag{7.25}$$

Consider then, as above, a deformation state of the form

$$\boldsymbol{F} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \ , \tag{7.26}$$

which implies

$$\boldsymbol{P} = \mu \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} - \alpha\, \lambda_1 \lambda_2 \lambda_3 \begin{pmatrix} \lambda_1^{-1} & 0 & 0 \\ 0 & \lambda_2^{-1} & 0 \\ 0 & 0 & \lambda_3^{-1} \end{pmatrix} \tag{7.27}$$

$$= \begin{pmatrix} \mu\lambda_1 - \alpha\lambda_2\lambda_3 & 0 & 0 \\ 0 & \mu\lambda_2 - \alpha\lambda_1\lambda_3 & 0 \\ 0 & 0 & \mu\lambda_3 - \alpha\lambda_1\lambda_2 \end{pmatrix} . \tag{7.28}$$

Applying this to the uniaxial tension state considered above together with incompressibility yields

$$\boldsymbol{P} = \begin{pmatrix} \mu\,\lambda - \alpha\,\lambda^{-1} & 0 & 0 \\ 0 & \mu\,\lambda^{-1/2} - \alpha\,\lambda^{1/2} & 0 \\ 0 & 0 & \mu\,\lambda^{-1/2} - \alpha\,\lambda^{1/2} \end{pmatrix} . \tag{7.29}$$

The traction-free boundary conditions, $P_2 = P_3 = 0$, allow to determine the Lagrange multiplier, leading to $\alpha = \mu/\lambda$. Using the latter, one obtains $P = P_1 = \mu\,(\lambda - \lambda^{-2}) = \partial f/\partial\lambda$, as stated.