Continuum plasticity

1 Unloading, residual stresses, shakedown (auto-frettage)

We consider a cylindrical shell under internal pressure, similar to the spherical problem fully solved by in the lectures. It's important to note that here we define $p \equiv -\sigma_{rr}(r=a)$, which is not exactly the pressure in the usual stress, but is obviously related to it, and is the main physical player in the driving. The equations of plane strain ($\epsilon_{zz} = 0$) are

$$\epsilon_{rr} = \frac{1+\nu}{E}((1-\nu)\sigma_{rr} - \nu\sigma_{\theta\theta}) \ , \tag{1}$$

$$\epsilon_{\theta\theta} = \frac{1+\nu}{E}((1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}) \ . \tag{2}$$

Plugging these into the compatibility equation, which in polar coordinates reads $\epsilon_{rr} = \frac{d}{dr}(r\epsilon_{\theta\theta})$, gives

$$\frac{d}{dr}\left[(1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right] = \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} \ . \tag{3}$$

Together with the force balance $\partial_r\sigma_{rr} + (\sigma_{rr} - \sigma_{\theta\theta})/r = 0$, we get

$$\frac{d}{dr}(\sigma_{rr} + \sigma_{\theta\theta}) = 0 \ . \tag{4}$$

1.1 Recap - elastic solution

The elastic solution may be obtained either by using the Airy stress fuction χ, or by substitution and symmetry considerations. Let us quickly recap how this is done — using χ we have $\sigma_{rr} = \frac{\partial_r\chi}{r} + \frac{\partial_{\theta\theta}\chi}{r^2}$, $\sigma_{r\theta} = -\partial_r\left(\frac{\partial_\theta\chi}{r}\right)$, and $\sigma_{\theta\theta} = \partial_{rr}\chi$. Due to the azimutal symmetry, we are looking for θ-independent solution. Substituting we have

$$\frac{\partial_r\chi}{r} + \partial_{rr}\chi = C_1 \ . \tag{5}$$

Solving the homogeneous equation we have $\chi = C_2\log(r) + C_3$. Together with the particular solution, we have $\chi = C_2\log(r) + C_3 + \frac{C_1}{4}r^2$.

To find the various constants, we need to use the boundary conditions. Take $r=a$, where we know $\sigma_{rr}(r=a) = \frac{C_1}{2} + \frac{C_2}{a^2} = -p$, which is one relation. Using the outside, traction-free, boundary we have that $\sigma_{rr}(r=b) = \frac{C_1}{2} + \frac{C_2}{b^2} = 0$. Solving for the coefficients from these two equations, we have $C_1 = -\frac{2a^2p}{a^2-b^2}$ and $C_2 = \frac{a^2b^2p}{a^2-b^2}$, and altogether

$$\sigma_{rr} = p\cdot\frac{a^2}{a^2-b^2}\left(\frac{b^2}{r^2} - 1\right) \ , \ \sigma_{\theta\theta} = -p\cdot\frac{a^2}{a^2-b^2}\left(\frac{b^2}{r^2} + 1\right) \ . \tag{6}$$

1.2 Plasticity

We now introduce the Tresca Criterion for plasticity, i.e. that $\frac{1}{2}\max\left(|\sigma_i - \sigma_j|_{i\neq j}\right) = \sigma_y$. As the loading happens from the *inside* we know the elastic region will be in the outer part, up to some internal radius c. In this elastic region $c < r < b$ we have

$$\sigma_{rr} = \sigma_y\left(\frac{c^2}{b^2} - \frac{c^2}{r^2}\right) \ , \qquad \sigma_{\theta\theta} = \sigma_y\left(\frac{c^2}{b^2} + \frac{c^2}{r^2}\right) \ , \tag{7}$$

and in the plastic region $a < r < c$ it is

$$\sigma_{rr} = \sigma_y\left[\frac{c^2}{b^2} - \log\left(\frac{c^2}{r^2}\right) - 1\right] \ , \qquad \sigma_{\theta\theta} = \sigma_y\left[\frac{c^2}{b^2} - \log\left(\frac{c^2}{r^2}\right) + 1\right] \ , \tag{8}$$

where the logarithmic contribution arises from the different constitutive law in the plastic zone, and the quasi-static momentum balance $\partial_r\sigma_{rr} + (\sigma_{rr} - \sigma_{\theta\theta})/r = 0$. Since we are in plane-strain conditions, we the zz component of the stress is given by $\sigma_{zz} = \nu(\sigma_{rr} + \sigma_{\theta\theta})$.

We now evaluate p_E — the pressure above which the solution is no longer elastic (i.e. the onset of plasticity at $r = a$) —, p_U — the pressure for which the whole shell becomes plastic —, and p_c, as

$$p_c = -\sigma_{rr}(r = c) = \sigma_y\left(1 - \frac{c^2}{b^2}\right) \ , \qquad p_E = \sigma_y\left(1 - \frac{a^2}{b^2}\right) \ , \qquad p_U = \sigma_y \log\frac{b^2}{a^2} \ , \tag{9}$$

and c satisfies $\sigma_{rr}(r = a) = -p$, that is

$$\frac{p}{\sigma_y} = 1 - \frac{c^2}{b^2} + \log\frac{c^2}{a^2} \ . \tag{10}$$

Now, what happens if we remove the internal pressure? How do we deal with this kind of (un)loading? What is the constitutive law that one should use?

This is a tricky subject and there are many subtleties in the general case. In our case of perfect plasticity you can think about it in the following manner: The perfect plastic constitutive law makes sure that the stress state at any given point in the material will always be inside the yield surface (in the elastic case) or strictly on it (in the plastic case). In other words, every point which is in a plastic state is also exactly on the threshold of yielding. Thus, the unloading dynamics is governed by elasticity. Or more precisely, as we'll soon see, at least the first part of it is governed by elasticity.

So we conclude that to get the unloaded state we need to subtract the fully elastic solution from the elasto-plastic solution. That is, we need to subtract Eq. (7) with $c \to a$

and $\sigma_y \to p/(1-\frac{a^2}{b^2})$ from Eqs. (7)-(8). The result is

$$\begin{aligned} \sigma_{rr} &= -\sigma_y \left(\frac{c^2}{a^2} - \frac{p}{p_E}\right)\left(\frac{a^2}{r^2} - \frac{a^2}{b^2}\right) \\ \sigma_{\theta\theta} &= \quad \sigma_y \left(\frac{c^2}{a^2} - \frac{p}{p_E}\right)\left(\frac{a^2}{r^2} + \frac{a^2}{b^2}\right) \end{aligned} \qquad c < r < b \; , \tag{11}$$

$$\begin{aligned} \sigma_{rr} &= -\sigma_y \left[\frac{p}{p_E}\left(1 - \frac{a^2}{r^2}\right) - \log\frac{r^2}{a^2}\right] \\ \sigma_{\theta\theta} &= -\sigma_y \left[\frac{p}{p_E}\left(1 + \frac{a^2}{r^2}\right) - \log\frac{r^2}{a^2} - 2\right] \end{aligned} \qquad a < r < c \; . \tag{12}$$

Note that the system has no tractions at the boundaries but the stress field does not vanish! These stresses are called residual stresses. The largest value of $|\sigma_{\theta\theta} - \sigma_{rr}|/2$ is at $r = a$, where it is $\sigma_y(p/p_E - 1)$. Unloading is thus purely elastic if $p/p_E \leq 2$. This is surely the case if $p_U < 2p_E$. That is, if

$$\sigma_y \log\frac{b^2}{a^2} < 2\sigma_y\left(1 - \frac{a^2}{b^2}\right) \; . \tag{13}$$

The condition (13) is satisfied if $b/a \leq 2.218$. If, on the other hand, $p > 2p_E$ then the unloading itself will create a new plastic zone at $a < r < c'$.

We can therefore define $p_s = \min(2p_E, p_U)$ (s for shakedown). If $p_E < p < p_s$ then unloading is elastic although the loading was elastic-plastic, and *every subsequent loading/unloading with pressure up to p is also elastic!*.

Physically, the portions of the cylinder that have underwent plastic deformation are now providing additional hoop stresses to the cylinder, making it stronger than it was before the plastic flow. In the context of reinforcing metal cylinders so that they can withstand high internal pressures (you can guess what is the technological motivation for that) this is called auto-frettage ("frettage" is French for the process of putting hoops). In a more general context this is called "shakedown". A similar concept is used in prestressed concrete. To sum it up, we have four regimes

$0 < p < p_E$ System is fully elastic.

$p_E < p < p_s$ Elastic-plastic loading, elastic unloading.

$p_s < p < p_U$ Elastic-plastic loading and unloading, if exists.

$p_U < p$ No static axiosymetric solution exists.

2 Plastic cavitation

Earlier in the course, we considered the problem of elastic cavitation in soft solids. Can we analyze a similar problem for hard solids?

The answer is definitely yes, such an analogous phenomenon exists for hard solids, though the physical processes is different; while for soft solids elastic deformation can be very large and lead to cavitation, hard solids show a limited range of elastic response and the origin of cavitation is plastic deformation.

We follow the kinematic analysis of elastic cavitation (starting in Eq.(7.50) in Eran's lecture notes), which is reproduced here

$$\lambda_r = \left(1 + \frac{L^3 - \ell^3}{r^3}\right)^{2/3} \ , \tag{14}$$

where λ_r is the radial stretch, L is the radius of the undeformed cavity and ℓ is the radius of the deformed one. The logarithmic strain ϵ_r reads

$$\epsilon_r = \log \lambda_r = \frac{2}{3} \log\left(1 + \frac{L^3 - \ell^3}{r^3}\right) \ . \tag{15}$$

Note that we use the logarithmic strain because it is thermodynamically-conjugated to the Cauchy stress that we use next. The force balance equation in terms of the Cauchy stress is given by $\frac{d\sigma_r}{dr} + 2\frac{\sigma_r - \sigma_\theta}{r} = 0$, and the boundary conditions are

$$\sigma_r(r = \ell) = 0 \quad \text{and} \quad \sigma_r(r \to \infty) = \sigma^\infty \ . \tag{16}$$

Since symmetry implies $\sigma_\theta = \sigma_\phi$ and we assume incompressibility, the stress state is essentially uniaxial and we can write down a general constitutive law as

$$\sigma_r - \sigma_\theta = \sigma_y f(\epsilon_r) \ . \tag{17}$$

We then have

$$\begin{aligned} \int_\ell^\infty d\sigma_r = -2\sigma_y \int_\ell^\infty \frac{f(\epsilon_r) dr}{r} &= -2\sigma_y \int_\ell^\infty f\left[\frac{2}{3}\log\left(1 + \frac{(L/\ell)^3 - 1}{(r/\ell)^3}\right)\right] \frac{d(r/\ell)}{(r/\ell)} \\ \Longrightarrow \sigma^\infty = -2\sigma_y \int_1^\infty f\left[\frac{2}{3}\log\left(1 + \frac{(L/\ell)^3 - 1}{x^3}\right)\right] \frac{dx}{x} \ . \end{aligned} \tag{18}$$

The cavitation threshold σ_c is defined as the stress needed to grow the cavity indefinitely, i.e. $\ell \gg L$. This leads to

$$\sigma_c = \lim_{\ell \to \infty} \sigma^\infty = -2\sigma_y \int_1^\infty f\left[\frac{2}{3}\log\left(1 - x^{-3}\right)\right] x^{-1} dx \ . \tag{19}$$

The constitutive law $(\sigma_r - \sigma_\theta)/\sigma_y = f(\epsilon_r)$ we adopt is that of perfect plastic material, which we interpret here as pertaining to the logarithmic strain and also allow all quantities to be signed,

$$\begin{aligned} f(\epsilon_r) &= \frac{\epsilon_r}{\epsilon_y} \quad \text{for} \quad |\epsilon_r| < \epsilon_y \\ f(\epsilon_r) &= \text{sign}(\epsilon_r) \quad \text{for} \quad |\epsilon_r| \geq \epsilon_y \ , \end{aligned} \tag{20}$$

where $\epsilon_y \equiv \sigma_y / E$. With this law at hand, after a few rather simple mathematical manipulations, we obtain a nice analytic result. First, we use the yield strain ϵ_y inside the argument of $f(\cdot)$ in the above integral

$$-\epsilon_y = \frac{2}{3}\log\left(1 - x_y^{-3}\right) \qquad \Longrightarrow \qquad x_y = \left[1 - \exp(-3\epsilon_y/2)\right]^{-1/3} \ . \tag{21}$$

This allows us to use the constitutive law in order to split the integral into its elastic and plastic contributions as

$$\frac{\sigma_c}{\sigma_y} = \underbrace{2\int_1^{x_y} x^{-1}dx}_{\text{Plastic domain}} - \underbrace{\frac{4}{3\epsilon_y}\int_{x_y}^{\infty} \log\left(1-x^{-3}\right)x^{-1}dx}_{\text{Elastic domain}} \ . \tag{22}$$

We now recall that there exists a small parameter in the problem, $\epsilon_y \ll 1$ (since for ordinary hard solids the yield stress is much smaller than the elastic modulus). Therefore, we have

$$x_y \simeq \left(\frac{2}{3\epsilon_y}\right)^{1/3} \gg 1 \quad \text{and} \quad \log\left(1-x^{-3}\right) \simeq -x^{-3} \ . \tag{23}$$

This immediately yields

$$\frac{\sigma_c}{\sigma_y} \simeq 2\log(x)\Big|_1^{\left(\frac{2}{3\epsilon_y}\right)^{1/3}} - \frac{4}{3\epsilon_y}\times\frac{x^{-3}}{3}\Big|_{\left(\frac{2}{3\epsilon_y}\right)^{1/3}}^{\infty} = \frac{2}{3}\log\left(\frac{2}{3\epsilon_y}\right) + \frac{2}{3} \ . \tag{24}$$

Therefore,

$$\frac{\sigma_c}{E} \simeq \frac{2\epsilon_y}{3}\left[1+\log\left(\frac{2}{3\epsilon_y}\right)\right] \ . \tag{25}$$

As expected, σ_c is an increasing function of σ_y (for a fixed E), but the dependence is not trivial and could not have been guessed to begin with (note, though, that the elastic term is related to the elasticity limit presented in the spherical shell example when in the $b/a\to\infty$ limit). This is an example of unlimited plastic flow under a fixed applied stress ("plastic collapse").

We're going now to solve in detail two problems in plasticity that nonetheless make use of all the tools we've developed in the course, so they are optimal preparation for the exam.

3 Spherical shell

In class, we've found the elasto-plastic solution for a spherical shell. We now look at some interesting aspects of the results.

1. Examine numerically Eq. (12.38) from the lecture notes. For the case that $b = 10a$, plot c as a function of p. Can you analytically explain what happens when $p \to p_U$?

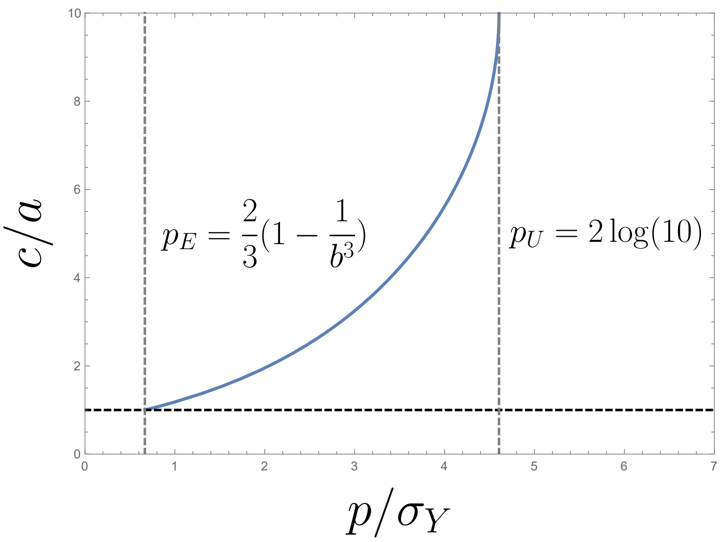
   The equation is

   $$p = \frac{2\sigma_y}{3}\left[1 - \frac{c^3}{b^3} + 3\log\left(\frac{c}{a}\right)\right] .$$

   As always, we should non-dimensionalize the equation. Measuring stresses in terms of σ_Y and lengths in terms of a, the equation becomes

   $$p = \frac{2}{3}\left[1 - \frac{c^3}{b^3} + 3\log(c)\right] , \tag{26}$$

   where all quantities should have tildes. In these units, $b = 10a$ actually means $b = 10$. Inverting this relation numerically gives the following dependence:



   The slope of the curve is

   $$\frac{\partial c}{\partial p} = \left(\frac{\partial p}{\partial c}\right)^{-1} = \frac{1}{2c^2}\left(\frac{1}{c^3} - \frac{1}{b^3}\right)^{-1} \tag{27}$$

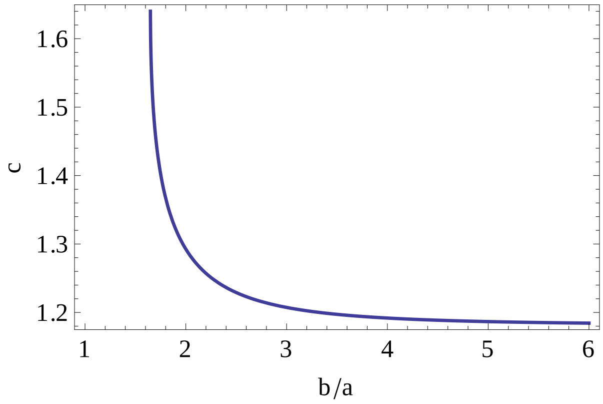
   As $p \to p_U$, we have $c \to b$ so the term in parentheses vanishes and the slope diverges (but the curve reaches the finite value b/a). This happens when $\frac{p}{\sigma_Y} = 2\log\left(\frac{b}{a}\right)$.

2. For the case that $p = \sigma_Y$, plot c/a as a function of b/a. What is the asymptotic value of c when $b/a \to \infty$?

   The dimensionless pressure is 1, so our equation takes the form

$$1 = \frac{2}{3}\left[1 - \frac{c^3}{b^3} + 3\log(c)\right] , \tag{28}$$

   and the solution is shown here:



   When $b/a = \tilde{b} \to \infty$ Eq. (28) reduces to

$$1 = \frac{2}{3}(1 + 3\log(c)) , \tag{29}$$

   which is solved by $c = e^{1/6} \approx 1.18$.

3. Find the displacement field $u_r(r)$ (from symmetry, $\vec{u}$ is a function of r only and other components vanish). Is the stress/strain/displacement field continuous/differentiable across the elasto-plastic boundary? In the elastic region, there's a particularly simple relation between u_r and some of the strain components. In the plastic region, the volumetric part of the deformation is still elastic - we still have $\mathrm{tr}\,\boldsymbol{\sigma} = K\,\mathrm{tr}\,\boldsymbol{\epsilon}$, where K is the bulk modulus.

   In the elastic domain we have $\epsilon_{\theta\theta} = \epsilon_{\phi\phi} = u_r/r$ (that's a general kinematic formula for radial motion). Since $\epsilon_{\theta\theta} = E^{-1}[\sigma_{\theta\theta} - \nu(\sigma_{rr} + \sigma_{\phi\phi})]$, we obtain

$$u_r = \frac{r}{E}\left((1-\nu)\sigma_{\theta\theta} - \nu\sigma_{rr}\right) . \tag{30}$$

   Plugging in Eqs. (11.28)-(11.29) we get

$$u_r = \frac{r}{E}\,\frac{p_c}{b^3/c^3 - 1}\left(1 - 2\nu + (1+\nu)\frac{b^3}{2r^3}\right) \tag{31}$$

   In the plastic regime, the volumetric response is elastic, that is $\mathrm{tr}\,\boldsymbol{\sigma} = 3K\,\mathrm{tr}\,\boldsymbol{\epsilon}$, with $K = \frac{E}{3(1-2\nu)}$:

$$\mathrm{tr}\,\boldsymbol{\epsilon} = \epsilon_{\theta\theta} + \epsilon_{\phi\phi} + \epsilon_{rr} = \frac{\partial u_r}{\partial r} + 2\frac{u}{r} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) \tag{32}$$

$$\mathrm{tr}\,\boldsymbol{\sigma} = \sigma_{rr} + 2\sigma_{\theta\theta} = (3\sigma_{rr} + 2\sigma_Y) \tag{33}$$

Where we used the fact that in the plastic zone we have $\sigma_{\theta\theta} = \sigma_{rr} + \sigma_Y$ (Eq. (11.37)). Plugging in the expression for σ_{rr} (Eq. (11.36)) we arrive at

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 u_r) = \frac{2(1-2\nu)\sigma_Y}{E}\left[\frac{c^3}{b^3} - 3\log\left(\frac{c}{r}\right)\right] \tag{34}$$

Which is solved by

$$u_r = \frac{A}{r^2} + \frac{2(1-2\nu)\sigma_Y r}{E}\left[\frac{1}{3}\left(\frac{c^3}{b^3} - 1\right) - \log\left(\frac{c}{r}\right)\right] \tag{35}$$

The integration constant A is determined form continuity at $r = c$. The stress field is continuous across the boundary. This is because σ_{rr} must be continuous for static equilibrium to exist, and the other components of the stress depend continuously on σ_r (remember that $\sigma_{\theta\theta} = \sigma_{rr} + \sigma_Y$). The strain is not continuous, and neither the stress nor the strain are differentiable.

4 Cylindrical shell

Continuing our previous hour in the TA session, consider an elastic-perfect-plastic 2D annulus with internal and external radii a, b, subject to internal pressure p and zero outer pressure, under *plane-stress* conditions. Use the Tresca yield criterion, and preform the analysis that was done in class:

1. Find the stress field $\sigma_{ij}(r)$, the minimal internal pressure that induces plastic flow (p_E), the ultimate pressure for which the entire annulus is plastic p_U, and give an equation that determines the radius of the elasto-plastic boundary c. Try and solve this in a different method than the one shown before

   The purpose of this exercise is that you redo the algebra in a slightly different setting. The calculations are practically the same, so I will only give hints here. The full thing is derived in Lubliner's book (section 4.3.5).

   Elastic solution
   The elastic solution is obtained in the following way. In 2D the force balance equation (11.14) takes the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0\ . \tag{36}$$

   As in 3D, we use Hooke's law, combined with the compatibility equation, to obtain the equivalent of Eq. (11.18):

$$\frac{\partial}{\partial r}(\sigma_r + \sigma_\theta) = 0 \tag{37}$$

This is solved under the proper boundary conditions to yield

$$\sigma_r = -\frac{p}{b^2/a^2 - 1}\left[\frac{b^2}{r^2} - 1\right] \tag{38}$$

$$\sigma_\theta = \frac{p}{b^2/a^2 - 1}\left[\frac{b^2}{r^2} + 1\right] \tag{39}$$

The maximal value of $\sigma_\theta - \sigma_r$ is obtained at $r = a$ where it equals $\frac{2p}{1-a^2/b^2}$ and therefore the system will begin to yield when

$$p = p_E \equiv \sigma_Y \left(1 - \frac{a^2}{b^2}\right) \ . \tag{40}$$

Elasto-Plastic solution
The elastic part of the elasto-plastic solution is obtained by substituting b with c and p with p_c in the above equations. The plastic part is obtained by assuming that

$$\sigma_r < \sigma_z = 0 < \sigma_\theta \tag{41}$$

(this will be checked later for consistency) and therefore the Tresca criterion reads

$$|\sigma_\theta - \sigma_r| = \frac{2p}{1 - c^2/b^2} = 2\sigma_Y \quad \Rightarrow \quad p_c = \sigma_Y \left(1 - \frac{c^2}{b^2}\right) \tag{42}$$

p_E is obtained by plugging $c \to a$ in the above. Eq. (36) can then be integrated to give

$$\sigma_r = -p + \sigma_Y \log \frac{r^2}{a^2} \ . \tag{43}$$

Continuity of stresses then yields the transcendental equation for c:

$$p = \sigma_Y \left(1 - \frac{c^2}{b^2} + \log \frac{c^2}{a^2}\right) \tag{44}$$

$p_U = 2\sigma_Y \log \frac{b}{a}$ is the solution of this equation for $c = b$. Plugging (44) into (43), and using $\sigma_\theta = \sigma_r + 2\sigma_Y$ in the plastic zone, we get the stress field:

$$\sigma_r = \sigma_Y \left(\frac{c^2}{b^2} - \log \frac{c^2}{r^2} - 1\right) \tag{45}$$

$$\sigma_\theta = \sigma_Y \left(\frac{c^2}{b^2} - \log \frac{c^2}{r^2} + 1\right) \tag{46}$$

2. Show that your solution is valid only if

$$1 + \frac{c^2}{b^2} - \log \frac{c^2}{a^2} \geq 0 \ . \tag{47}$$

What happens if this criterion is not satisfied? Why is this problem not present in plane strain conditions?

Take a look at Eq. (46) and remind yourself that we assumed $\sigma_\theta > 0$. If this is not the case, then $\sigma_z = 0 > \sigma_\theta$ and then the form of the Tresca criterion changes and everything we did is invalid. The smallest value of σ_θ occurs on $r = a$ so in order for our solution to be valid we need to demand $\sigma_\theta(r=a) > 0$, and this is exactly the condition (47).

In plane-strain conditions, we have $\sigma_z = \nu(\sigma_r + \sigma_\theta)$. At $r = c$ we have

$$\begin{aligned} \sigma_r &= \frac{p_c}{(b/c)^2 - 1}\left(-\frac{b^2}{c^2} + 1\right) \\ \sigma_\theta &= \frac{p_c}{(b/c)^2 - 1}\left(\frac{b^2}{c^2} + 1\right) \\ \sigma_z &= 2\nu\frac{p_c}{(b/c)^2 - 1} \end{aligned} \tag{48}$$

and since $1 - \frac{b^2}{c^2} < 2\nu < 1 + \frac{b^2}{c^2}$ our assumption is always valid (remember that $0 < \nu < \frac{1}{2}$).

3. Considering this, what is the condition on a/b that ensures that p_U exists? Give an equation that describes, for a given value of a/b, the maximal possible value of c/a. What is this value when $b/a \to \infty$?

p_U describes the situation that the entire disk can become plastic, that is, $c = b$. plugging that in the condition, we get $1 - \log(b/a) \geq 0$, or more nicely $b/a \leq e$. For larger values of b/a our solution breaks down before the entire disk have flowed.

The maximal possible value of c is obtained by turning the condition (47) into an equality. In the limit $b \gg a$ (a hole in an infinite plane), this turns to be $1 - 2\log(c/a)$, and the limiting value is therefore $c = a\sqrt{e}$.