Linear elasticity I

In this TA session we'll start diving into linear elasticity theory. Linear elasticity by itself can be the topic of a year-long course, and in the TAs we'll try to convey a significant fraction of the richness of the theory.

1 Hooke's law, stiffness and compliance

Hooke's law is

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} = \left(\lambda + \frac{2}{3}\mu\right)\mathrm{tr}(\boldsymbol{\varepsilon})\delta_{ij} + 2\mu\left(\varepsilon_{ij} - \frac{1}{3}\mathrm{tr}(\boldsymbol{\varepsilon})\delta_{ij}\right) \ . \tag{1}$$

Let's write it explicitly, to get a better feel of what's going on

$$\begin{aligned}
\sigma_{xx} &= 2\mu\varepsilon_{xx} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = (2\mu+\lambda)\varepsilon_{xx} + \lambda\left(\varepsilon_{yy} + \varepsilon_{zz}\right) \ , \\
\sigma_{yy} &= 2\mu\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = (2\mu+\lambda)\varepsilon_{yy} + \lambda\left(\varepsilon_{xx} + \varepsilon_{zz}\right) \ , \\
\sigma_{zz} &= 2\mu\varepsilon_{zz} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}\right) = (2\mu+\lambda)\varepsilon_{zz} + \lambda\left(\varepsilon_{xx} + \varepsilon_{yy}\right) \ ,
\end{aligned} \tag{2}$$

$$\sigma_{ij} = 2\mu\varepsilon_{ij}, \qquad i \neq j \ . \tag{3}$$

The latter can be presented in matrix form (and to stress the fact that **any** two elastic constants can be used, we use here ν, instead of λ)

$$\begin{pmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{zz}\\ \sigma_{zy}\\ \sigma_{zx}\\ \sigma_{xy}\end{pmatrix} = \frac{2\mu}{1-2\nu}\begin{pmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu\end{pmatrix}\begin{pmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{zz}\\ \varepsilon_{zy}\\ \varepsilon_{zx}\\ \varepsilon_{xy}\end{pmatrix} \ . \tag{4}$$

The shear terms $i \neq j$ have a simple dependence, while the others are a bit more complicated. This equation has the general form of $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon}$, where $\boldsymbol{C}$ is called the stiffness tensor.

Let's try to invert these relations to find $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ - that is, let's find the compliance matrix for an isotropic linear elastic material. The same considerations that we used to derive Eq. (1) (i.e. that $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are related by a 4-rank isotropic tensor) allow us to write

$$\varepsilon_{ij} = a\,\sigma_{kk}\delta_{ij} + b\,\sigma_{ij} \ , \tag{5}$$

so finding the compliance reduces to finding a, b. If $\mathrm{tr}\,\boldsymbol{\varepsilon} = 0$, then Eq.(1) and (5)[1] reduce to, respectively,

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} \ , \tag{6}$$

$$\varepsilon_{ij} = b\,\sigma_{ij} \ , \tag{7}$$

[1] Recall that $\mathrm{tr}\,\boldsymbol{\sigma} \propto \mathrm{tr}\,\boldsymbol{\varepsilon}$.

so we immediately find $b = (2\mu)^{-1}$. Taking the trace of Eq. (1) and (5) gives, respectively

$$\mathrm{tr}(\boldsymbol{\sigma}) = (3\lambda + 2\mu)\,\mathrm{tr}(\boldsymbol{\varepsilon}) \ , \tag{8}$$

$$\mathrm{tr}(\boldsymbol{\varepsilon}) = (3a + b)\,\mathrm{tr}(\boldsymbol{\sigma}) \ , \tag{9}$$

which tells us that

$$\begin{aligned} &3a + b = (3\lambda + 2\mu)^{-1} \quad \Rightarrow \quad a = \frac{1}{3}\left(\frac{1}{3\lambda + 2\mu} - b\right) = \frac{-\lambda}{2\mu(3\lambda + 2\mu)} \Rightarrow \\ &\varepsilon_{ij} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\,\mathrm{tr}(\boldsymbol{\sigma})\delta_{ij} + \frac{1}{2\mu}\sigma_{ij} \ . \end{aligned} \tag{10}$$

Writing explicitly, we have

$$\begin{aligned} \varepsilon_{xx} &= \left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\right)\sigma_{xx} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\left(\sigma_{yy} + \sigma_{zz}\right) \ , \\ \varepsilon_{yy} &= \left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\right)\sigma_{yy} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\left(\sigma_{xx} + \sigma_{zz}\right) \ , \\ \varepsilon_{zz} &= \left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\right)\sigma_{zz} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\left(\sigma_{xx} + \sigma_{yy}\right) \ , \end{aligned} \tag{11}$$

$$\varepsilon_{ij} = \frac{1}{2\mu}\sigma_{ij} \qquad \text{for } i \neq j \ . \tag{12}$$

As discussed in class, the term in parentheses in Eq.(11) is the inverse of the Young's modulus, and for uniaxial stress it reads

$$E = \sigma_{xx}/\varepsilon_{xx} = \left(\frac{1}{2\mu} - \frac{\lambda}{2\mu(3\lambda + 2\mu)}\right)^{-1} = \mu\frac{3\lambda + 2\mu}{\lambda + \mu} \ . \tag{13}$$

It is the microscopic analogue of the spring constant. If the uniaxial stress is in the x direction, then we have

$$\varepsilon_{yy} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}\sigma_{xx} = -\frac{\lambda}{2\mu(3\lambda + 2\mu)}E\varepsilon_{xx} = -\frac{\lambda}{2(\lambda + \mu)}\varepsilon_{xx} \ , \tag{14}$$

and as discussed in class, $-\varepsilon_{yy}/\varepsilon_{xx}$ is known as the Poisson ratio $\nu = \frac{\lambda}{2(\lambda+\mu)}$. Rewriting Eqs. (11) with these quantities yields a much nicer expression:

$$\begin{aligned} \varepsilon_{xx} &= \frac{1}{E}\left[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right] \ , \\ \varepsilon_{yy} &= \frac{1}{E}\left[\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right] \ , \\ \varepsilon_{zz} &= \frac{1}{E}\left[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right] \ , \\ \varepsilon_{ij} &= \frac{1 + \nu}{E}\sigma_{ij} \quad \text{for} \quad i \neq j \ , \end{aligned} \tag{15}$$

or in matrix form

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{zy} \\ \varepsilon_{zx} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E}\begin{pmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\nu & 0 & 0 \\ 0 & 0 & 0 & 0 & 1+\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 1+\nu \end{pmatrix}\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{zy} \\ \sigma_{zx} \\ \sigma_{xy} \end{pmatrix} \ . \tag{16}$$

This is just an inversion of Eq. (4) into the form $\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}$, where $\boldsymbol{S} = \boldsymbol{C}^{-1}$ is the compliance tensor (if you noticed that $\boldsymbol{C}$ is called the stiffness tensor and $\boldsymbol{S}$ is called the compliance tensor and wondered about it, this is not a mistake and there is no intention to confuse you. It is a long-time convention that cannot be reverted anymore). A useful table with all the conversions is found in Wikipedia. We are now ready to perform the reduction to 2D.

2 2D elasticity

As shown in class, the field equation of elasticity is the Navier-Lamé equation

$$\rho \partial_{tt} \boldsymbol{u} = (\lambda + \mu) \nabla (\nabla \cdot \boldsymbol{u}) + \mu \nabla^2 \boldsymbol{u} + \boldsymbol{b} \ . \tag{17}$$

It is notoriously difficult to solve. However, things become much simpler in 2D. There are several physical situations in which one can obtain an effectively 2D elastic system, by ignoring the z dimension. Here, we discuss two of these:

1. **Plane stress**: when $\sigma_{zi} = 0$. This typically holds in very thin systems (in the z direction).
2. **Plane strain**: when the system is translationally invariant in z, and therefore ∂_z of anything vanishes. This typically holds in very thick (in the z direction) systems.

We'll now develop the theory for the first two cases, and Eran demonstrated in class the formalism of scalar elasticity (the third case).

To see how one reduces elasticity to 2 dimensions, let us explicitly write Hooke's law (15) in terms of μ and ν. We note that although the stiffness matrix is a 4-rank tensor, it can be represented as a 6 by 6 matrix by rearranging the entries as in Eq. (4).

2.1 Plane-stress

We first consider objects that are thin in one dimension, say z, and are deformed in the xy-plane. What happens in the z-direction? Since the two planes $z = 0$ and $z = h$ (where h is the thickness which is much smaller than any other lengthscale in the problem) are traction-free, we approximate $\sigma_{zz} = 0$ everywhere (an approximation that becomes better and better as $h \to 0$). Similarly, we have $\sigma_{zy} = \sigma_{zx} = 0$. We can therefore set $\sigma_{zz} = \sigma_{zy} = \sigma_{zx} = 0$ in Eq. (16) to obtain

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} \ , \tag{18}$$

and

$$\varepsilon_{zz} = -\frac{\nu}{E} (\sigma_{xx} + \sigma_{yy}) \ . \tag{19}$$

To obtain the plane-stress analog of Eq. (17), the Navier-Lamé equation, we need to invert Eq. (18), obtaining

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{E}{1-\nu^2} \begin{pmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & 1-\nu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} \ . \tag{20}$$

Note that the last equation *can not be obtained* from Eq. (4) by simply removing columns and rows. We can now substitute the last relation in the 2D momentum balance equation $\nabla \cdot \boldsymbol{\sigma} = \rho \partial_{tt} \boldsymbol{u}$ (we stress again that $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are already 2D here). The resulting 2D equation reads

$$\left[\frac{\nu E}{1-\nu^2} + \mu\right] \nabla \left(\nabla \cdot \boldsymbol{u}\right) + \mu \nabla^2 \boldsymbol{u} = \rho \partial_{tt} \boldsymbol{u} \ , \tag{21}$$

which is identical in form to Eq. (17) simply with a renormalized λ

$$\lambda \to \tilde{\lambda} = \frac{\nu E}{1-\nu^2} = \frac{2\nu\mu}{1-\nu} = \frac{2\lambda\mu}{\lambda + 2\mu} \ . \tag{22}$$

The shear modulus μ remains unchanged

$$\tilde{\mu} = \mu = \frac{E}{2(1+\nu)} \ . \tag{23}$$

Finally, we can substitute $\sigma_{xx}(x,y)$ and $\sigma_{yy}(x,y)$ inside Eq. (19) to obtain $\varepsilon_{zz}(x,y)$. Note that $u_z(x,y,z) = \varepsilon_{zz}(x,y)z$ is linear in z.

2.2 Plane-strain

We now consider objects that are very thick in one dimension, say z, and are deformed in the xy-plane with no z dependence. These physical conditions are termed plane-strain and are characterized by $\varepsilon_{zx} = \varepsilon_{zy} = \varepsilon_{zz} = 0$. Eliminating these components from Eq. (4) we obtain

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} = \frac{2\mu}{1-2\nu} \begin{pmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & 1-2\nu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} \ , \tag{24}$$

and

$$\sigma_{zz}(x,y) = \frac{2\mu\nu}{1-2\nu} \left[\varepsilon_{xx}(x,y) + \varepsilon_{yy}(x,y)\right] \ . \tag{25}$$

We can now substitute Eq. (24) in the 2D momentum balance equation $\nabla \cdot \boldsymbol{\sigma} = \rho \partial_{tt} \boldsymbol{u}$ (where again $\boldsymbol{\sigma}$ and $\boldsymbol{u}$ are 2D). The resulting 2D equation is identical to Eq. (17), both in form and in the elastic constants. With the solution at hand, we can use Eq. (25) to calculate $\sigma_{zz}(x,y)$. Finally, we note that Eq. (24) can be inverted to

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{2\mu} \begin{pmatrix} 1-\nu & -\nu & 0 \\ -\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} \ , \tag{26}$$

which can not be simply obtained from Eq. (16) by eliminating columns and rows. Using the last relation we can rewrite Eq. (25) as

$$\sigma_{zz}(x,y) = \nu \left[\sigma_{xx}(x,y) + \sigma_{yy}(x,y)\right] \ . \tag{27}$$

In summary, we see that in both plane-stress and plane-strain cases we can work with 2D objects instead of their 3D counterparts, which is a significant simplification. This allows to use the heavy mathematical tools available in 2D: complex analysis and conformal mapping. One has, though, to be careful with the elastic constants as explained above.