## Linear elasticity II

# 1 Complex representation of scalar elasticity

We study a case of scalar elasticity, where $\boldsymbol{u} = u_{x_3}(x_1, x_2)\boldsymbol{e}_{x_3}$. The strains are

$$\varepsilon_{x_2,x_3} = \frac{1}{2}(\partial_{x_2} u_{x_3} + \partial_{x_3} u_{x_2}) = \frac{1}{2}\partial_{x_2} u_{x_3} \ , \tag{1}$$

$$\varepsilon_{x_1,x_3} = \frac{1}{2}(\partial_{x_1} u_{x_3} + \partial_{x_3} u_{x_1}) = \frac{1}{2}\partial_{x_1} u_{x_3} \ , \tag{2}$$

$$\varepsilon_{x_1,x_1} = \varepsilon_{x_2,x_2} = \varepsilon_{x_3,x_3} = \varepsilon_{x_1,x_2} = 0 \ . \tag{3}$$

We have seen in class that $\nabla^2 u_{x_3} = 0$, that is, $u_{x_3}$ is a harmonic function. This means that we can write $u_{x_3}$ as

$$u_{x_3} = 2\Re(\phi) = \phi(z) + \overline{\phi(z)}, \qquad z = x_1 + i x_2 \ , \tag{4}$$

where $\phi$ is an analytic complex function. We will use the Cauchy-Riemman equations, that tell us that

$$\partial_{x_1}\phi = -i\partial_{x_2}\phi = \phi' \ , \tag{5}$$

$$\partial_{x_1}\overline{\phi} = \overline{\partial_{x_1}\phi} = i\partial_{x_2}\overline{\phi} = \overline{\phi}' \ , \tag{6}$$

and therefore the stresses are

$$\begin{aligned}
\sigma_{x_1,x_3} &= \mu\partial_{x_1} u_{x_3} = \mu\left(\partial_{x_1}\phi + \partial_{x_1}\overline{\phi}\right) = \mu\left(\phi' + \overline{\phi}'\right) = 2\mu\Re(\phi') \ , \\
\sigma_{x_2,x_3} &= \mu\partial_{x_2} u_{x_3} = \mu\left(\partial_{x_2}\phi + \partial_{x_2}\overline{\phi}\right) = i\mu\left(\phi' - \overline{\phi}'\right) = -2\mu\Im(\phi') \ , \\
&\Rightarrow \qquad 2\mu\phi' = \sigma_{x_1,x_3} - i\sigma_{x_2,x_3} \ ,
\end{aligned} \tag{7}$$

and all other components vanish.

If our domain contains a free boundary, given by a curve that is parameterized by $x_1(s), x_2(s)$ with $s$ being arc-length parametrization, then the normal to the boundary is given by $\boldsymbol{n} = (\partial_s x_2, -\partial_s x_1)$. On the boundary we thus have

$$\begin{aligned}
0 &= \sigma_{x_3,x_1} n_{x_1} + \sigma_{x_3,x_2} n_{x_2} \\
&= \mu\left[\left(\partial_{x_1}\phi + \partial_{x_1}\overline{\phi}\right)\partial_s x_2 - \left(\partial_{x_2}\phi + \partial_{x_2}\overline{\phi}\right)\partial_s x_1\right] \\
&= \mu\left[\left(-i\partial_{x_2}\phi + i\partial_{x_2}\overline{\phi}\right)\partial_s x_2 - \left(i\partial_{x_1}\phi - i\partial_{x_1}\overline{\phi}\right)\partial_s x_1\right] \\
&= -i\mu\left[\left(\frac{\partial\phi}{\partial x_2}\frac{\partial x_2}{\partial s} + \frac{\partial\phi}{\partial x_1}\frac{\partial x_1}{\partial s}\right) - \left(\frac{\partial\overline{\phi}}{\partial x_2}\frac{\partial x_2}{\partial s} + \frac{\partial\overline{\phi}}{\partial x_1}\frac{\partial x_1}{\partial s}\right)\right] \\
&= -\mu\left(\frac{\partial\phi}{\partial s} - \frac{\partial\overline{\phi}}{\partial s}\right) = 2\mu\frac{\partial\Im(\phi)}{\partial s} \ ,
\end{aligned} \tag{8}$$

so on the boundary $\Im(\phi)$ is constant. Since $\phi$ is only given up to an additive constant, we can choose $\Im(\phi) = 0$, or, in other words, $\phi = \overline{\phi}$ on the boundary. We see that solving for the displacement field is equivalent to finding an analytic function whose imaginary part is constant on the boundary.

# 2 Conformal mapping: Inglis crack
**(Refernce: Marder & Fineberg, Physics Reports 1999)**

Complex treatment of 2D elasticity is very useful because Laplace's equation is conformally invariant, so one can use conformal mappings to deform the region over which we need to solve the equation into a more convenient geometry. Here we'll see an application of this method, which is called the Inglis (mode III) problem. In 1913 Charles Inglis solved the general problem of an elliptic hole in an infinite plate subject to distant loading. His solution turned out to be one of the cornerstones of fracture mechanics, and was later used and generalized by the works of Griffith, Irwin, and others.

So let's look at an infinite plane with an elliptic hole, subject to antiplane shear $\sigma_{x_2,x_3} = \sigma_\infty$ at $x_2 \to \pm\infty$. As working with ellipses is unpleasant, we want to find a conformal mapping that maps the region outside the ellipse to a region outside a circle. Luckily, such a mapping is well known, and is given by

$$z = f(\omega) = R\left(\omega + \frac{\rho}{\omega}\right) \;, \tag{9}$$

$$\omega = f^{-1}(z) = \frac{z}{2R} + \sqrt{\left(\frac{z}{2R}\right)^2 - \rho} \;. \tag{10}$$

$f$ maps the unit circle in the $\omega$-plane to an ellipse with axes $R(1 \pm \rho)$ in the $z$-plane. $0 \le \rho \le 1$ is a parameter that measures the ellipse's eccentricity[1] - when $\rho = 0$ the ellipse is a circle, while for $\rho = 1$ it is a 1D crack of length $4R$. The conformal mapping is shown in Fig. (1).

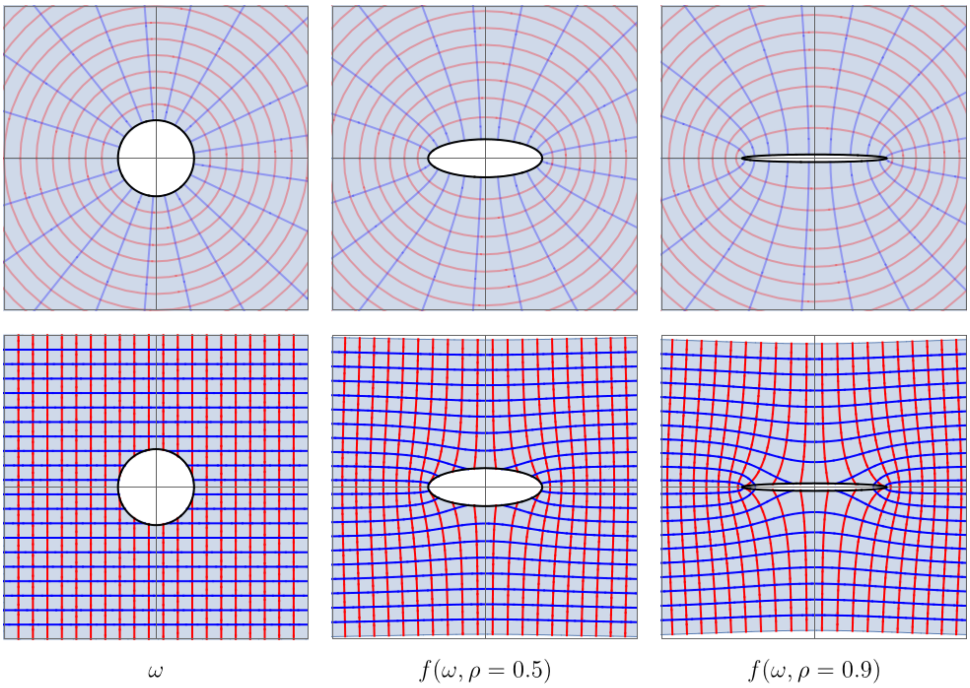


Figure 1: The conformal mapping. Polar lines (top row) and the Cartesian lines (bottom row) are shown. Note that, after the mapping, the lines remain perpendicular.

[1] Note that $\rho$ isn't the eccentricity as usually defined in geometry, which is $e = \frac{2\sqrt{\rho}}{\rho+1}$.

The crux of the conformal mapping technique is that while in the real coordinates the geometry is elliptic (and thus complicated), in the $\omega$-plane the domain is a circle (simple!), and therefore we want to reformulate the problem in terms of $\omega$. That is, we want to describe $\phi$ as a function of $\omega$, by the mapping $\phi(\omega) = \phi(\omega(z))$.

On the hole's boundary, which is the unit circle in $\omega$-plane, we have

$$\phi(\omega) = \overline{\phi(\omega)} = \overline{\phi}(\overline{\omega}) = \overline{\phi}(1/\omega) \ , \tag{11}$$

because on the unit circle $\overline{\omega} = 1/\omega$. The property (11) can be analytically extended to all the $\omega$-plane.

What are the singularities of $\phi$? Outside the hole, it must be completely regular, except at infinity where it diverges as $\phi \sim z$. This is because Eq. (7) tells us that far from the hole we have $\partial_z \phi \propto \sigma/\mu$, and therefore we conclude that

$$\phi \approx -i\frac{\sigma_\infty}{\mu} z \approx -i\frac{\sigma_\infty}{\mu} R\omega, \qquad \text{for } \omega, z \to \infty \ . \tag{12}$$

Using the analytical continuation of Eq. (11), we get that

$$\overline{\phi}(1/\omega) \approx -i\frac{\sigma_\infty}{\mu} R\omega, \qquad \text{for } \omega \to \infty \ , \tag{13}$$

or equivalently,

$$\phi(\omega) \approx i\frac{\sigma_\infty R}{\mu\omega}, \qquad \text{for } \omega \to 0 \ , \tag{14}$$

and there are no other singularities inside the unit circle. Having determined all the possible singularities of $\phi$, it is determined up to an additive constant. It must be

$$\phi(\omega) = i\frac{\sigma_\infty R}{\mu}\left(\frac{1}{\omega} - \omega\right) \ . \tag{15}$$

As discussed above, another way of finding $\phi$ is to find a function whose imaginary part vanishes on the boundary on the hole, i.e. on the unit circle. The function $i\left(1/\omega - \omega\right)$ fits this requirement, therefore, it is exactly the function we're looking for, up to a multiplicative factor which we have obtained from the external BC.

We can now calculate the displacement field in the "real" coordinate $z$ by joining Eqs. (15) and (10):

$$u_z = 2\Re\left\{-i\frac{\sigma_\infty R}{\mu}\left(\zeta + \sqrt{\zeta^2 - \rho} - \frac{1}{\zeta + \sqrt{\zeta^2 - \rho}}\right)\right\} \ , \qquad \text{where } \zeta \equiv \tfrac{z}{2R} \ . \tag{16}$$

What is the stress at the tip of the ellipse? We can differentiate $u_z(z)$ of Eq. (16) explicitly, but this gives a nasty expression that is very difficult to understand. It is simpler to use the conformal property of the mapping:

$$\begin{aligned}
\partial_z\phi(z) &= \partial_z\phi(\omega(z)) = \phi'(\omega)\frac{\partial\omega}{\partial z} \ , \\
\phi'(\omega) &= -i\frac{\sigma_\infty R}{\mu}\left(1+\frac{1}{\omega^2}\right) \ , \\
\frac{\partial\omega}{\partial z} &= \left(\frac{\partial z}{\partial w}\right)^{-1} = \frac{1}{f'(\omega)} \ , \\
f'(\omega) &= R(1-\frac{\rho}{\omega^2}) \ , \\
\phi'(z) &= \frac{-i\frac{\sigma_\infty R}{\mu}\left(1+\frac{1}{\omega^2}\right)}{R(1-\frac{\rho}{\omega^2})} = -\frac{i\sigma_\infty}{\mu}\,\frac{\omega(z)^2+1}{\omega(z)^2-\rho} \ .
\end{aligned} \tag{17}$$

Note that in the last equation $\omega$ is a function of $z$.

Now let's examine the solution. One thing we would like to know is where in space is the stress maximal. Clearly, $\phi'$ diverges for $w=\pm\sqrt{\rho}$, but remember that $\rho<1$ and our domain is outside the unit circle, so this point is inside the hole. Some trivial algebra shows that the $\phi'$ is maximal for $\omega=\pm1$, which are, not surprisingly, the closest points outside the unit circle to $\pm\rho$. When $\omega=\pm1$ we have $z=\pm R(1+\rho)$ - these are the horizontal tips of the ellipse. The stresses there are

$$\begin{aligned}
&\sigma_{x_1,x_3} - i\sigma_{x_2,x_3} = \mu\phi' = -\sigma_\infty\frac{2i}{1-\rho} \Rightarrow \\
&\sigma_{x_1,x_3} = 0 \qquad , \qquad \sigma_{x_2,x_3} = \frac{2\sigma_\infty}{1-\rho} \ .
\end{aligned} \tag{18}$$

The case $\rho=0$ gives $\sigma_{x_2,x_3}=2\sigma_\infty$, in accordance with what was done in class. In the opposite extremity, $\rho\to1$, the stress field diverges (but the displacement doesn't). We see that the stress at the tip decreases with the radius there. An interesting consequence of this is that in order to arrest a crack from propagating, one can drill a hole at its tip (!). This will reduce the radius of curvature at the tip and weaken the singularity.

The limiting case $\rho\to1$ is of particular interest, as it describes a 1-dimensional cut in the material. It is known in the literature as Mode III crack. The power with which $\sigma_{x_3,x_2}$ diverges in the case $\rho=1$ can be easily obtained. In this case we have

$$\phi = -\frac{iR\sigma_\infty}{\mu}\sqrt{\frac{z^2}{R^2}-4} \ . \tag{19}$$

Plugging in $z=2R\,(1+\delta)$ (where $\delta\in\mathbb{C}$) and keeping the leading order in $\delta$ gives

$$\begin{aligned}
&\phi = -i\frac{2\sqrt{2}R\sigma_\infty}{\mu}\sqrt{\delta} + O\left(\delta^{3/2}\right) \Rightarrow \\
&\sigma_{x_2,x_3} \sim \frac{\sigma_\infty}{\sqrt{2\delta}} \ .
\end{aligned} \tag{20}$$

The fact that near the crack tip the stress field diverges as the square root of the distance from the crack tip, and that the displacement field is regular, is of general applicability, and is true for static cracks in all loading configurations. The square-root divergence is a consequence of the branch-cut at the crack surface.

# 3 Green's function for an infinite medium

It seems that the time is ripe to fully and completely solve a problem, with all the $2\pi$'s and everything, without resorting to hand waving and scaling arguments. While the emphasis will still be on the structure of the problem, we think it will be instructive, at least once, to write down a problem and solve it exactly.

A nice problem to consider is the response of an infinite linear isotropic homogeneous elastic medium to a localized force $\vec{f} = F_i \delta(\vec{r})$, i.e. finding the Green's function of an infinite medium.

We define the Green function (matrix) $G_{ij}(\vec{r}_1, \vec{r}_2)$ as the displacement in the $i$ direction at the point $\vec{r}_1$ as a response to a localized force in the $j$ direction applied at $\vec{r}_2$. For homogeneous materials we know that $G_{ij}(\vec{r}_1, \vec{r}_2) = G_{ij}(\vec{r}_1 - \vec{r}_2)$. We therefore denote $\vec{r} = \vec{r}_1 - \vec{r}_2$. You all know well that, within the linear elastic theory, this will allow us to solve the problem of an arbitrary force distribution $f(\vec{r})$ by convolving $f(\vec{r})$ with the Green function.

Conceptually, the structure is the following. We would like to find a displacement field $u_i(\vec{r})$, from which we calculate

$$u_i \quad \Rightarrow \quad \varepsilon_{ij} \quad \Rightarrow \quad \sigma_{ij} \quad \Rightarrow \quad \partial_j \sigma_{ij} = \delta(\vec{r}) \ ,$$

but of course, we will want to do the whole thing backwards. I stress (no pun intended[2]) that we already know how to express div($\boldsymbol{\sigma}$) in terms of $u_i$ - this is what we called the Navier-Lamé equation. But for completeness, let's do it again:

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu\, \varepsilon_{ij} = \lambda \partial_k u_k \delta_{ij} + \mu \left(\partial_i u_j + \partial_j u_i\right) \ ,$$
$$\partial_j \sigma_{ij} = \partial_j \left(\lambda \partial_k u_k \delta_{ij} + \mu \left(\partial_i u_j + \partial_j u_i\right)\right) = (\lambda + \mu) \partial_i \partial_j u_j + \mu \partial_j \partial_j u_i \ ,$$

which is nothing but the $u$-dependent term of the Navier-Lamé equation. Since the equation is linear, it seems right to solve the problem by Fourier transform. We use the conventions

$$u_i(\vec{q}) = \int d^3\vec{x}\, e^{i\vec{q}\cdot\vec{r}} u_i(\vec{r}) \ , \tag{21}$$
$$u_i(\vec{r}) = \frac{1}{(2\pi)^3} \int d^3\vec{q}\, e^{-i\vec{q}\cdot\vec{r}} u_i(\vec{q}) \ . \tag{22}$$

The equation we want to transform is

$$(\lambda + \mu) \partial_j \partial_i u_j + \mu \partial_j \partial_j u_i = -F_i \delta(\vec{x}) \ , \tag{23}$$

which readily gives

$$-(\lambda + \mu) q_j q_i\, u_j - \mu\, q_j q_j\, u_i = -F_i \ . \tag{24}$$

This is a matrix equation:

$$\left[(\lambda + \mu) q_j q_i + \mu\, q_k q_k \delta_{ij}\right] u_j = F_i \ . \tag{25}$$

[2] Just kidding, of course it's intended.

Or even more explicitly:

$$\begin{pmatrix} (\lambda+\mu)\, q_1^2+\mu|\vec{q}|^2 & (\lambda+\mu)\, q_1 q_2 & (\lambda+\mu)\, q_1 q_3 \\ (\lambda+\mu)\, q_1 q_2 & (\lambda+\mu)\, q_2^2+\mu|\vec{q}|^2 & (\lambda+\mu)\, q_2 q_3 \\ (\lambda+\mu)\, q_1 q_3 & (\lambda+\mu)\, q_2 q_3 & (\lambda+\mu)\, q_3^2+\mu|\vec{q}|^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} . \quad (26)$$

This matrix can be inverted by using any of your favorite methods, giving

$$u_i = \frac{1}{\mu}\left[\frac{\delta_{ij}}{q_k q_k} - \frac{1}{2(1-\nu)}\frac{q_i q_j}{(q_k q_k)^2}\right] F_j \ . \quad (27)$$

Where we used $\nu = \frac{\lambda}{2(\lambda+\mu)}$. In other words, we have found the Fourier representation of the Green function:

$$G_{ij}(\vec{q}) = \frac{1}{\mu}\left[\frac{\delta_{ij}}{q_k q_k} - \frac{1}{2(1-\nu)}\frac{q_i q_j}{(q_k q_k)^2}\right] . \quad (28)$$

We now need to preform the inverse Fourier transform. We'll begin with the first term, and do it in spherical coordinates with the $q_z$ direction parallel to $\vec{r}$:

$$\frac{1}{(2\pi)^3}\int d^3\vec{q}\,\frac{e^{-i\vec{q}\cdot\vec{x}}}{q^2} = \frac{1}{(2\pi)^3}\int \frac{e^{-iqr\cos\theta}}{q^2} q^2 \sin\theta\, d\theta\, dq\, d\phi = \frac{1}{(2\pi)^2}\int e^{-iqr\cos\theta}\sin\theta\, d\theta\, dq$$
$$= \frac{1}{(2\pi)^2}\int \frac{e^{iqr}-e^{-iqr}}{iqr} dq = \frac{2}{(2\pi)^2}\int_0^\infty \frac{\sin(qr)}{qr} dq = \frac{1}{4\pi r} \ .$$

If this result surprises you, maybe you should remind yourself of the first linear field theory that you met in your life - Poisson's equation for a point charge $\nabla^2\phi = \delta(\vec{r})$. I'll let you complete the analogy by yourselves.

For the second term, we use a dirty trick:

$$\begin{aligned} \mathcal{F}^{-1}\left\{\frac{q_i q_j}{(q_k q_k)^2}\right\} &= -\frac{1}{2}\mathcal{F}^{-1}\left\{q_i \frac{\partial}{\partial q_j}\left(\frac{1}{q_k q_k}\right)\right\} = \frac{i}{2}\frac{\partial}{\partial x_i}\mathcal{F}^{-1}\left\{\frac{\partial}{\partial q_j}\left(\frac{1}{q_k q_k}\right)\right\} \\ &= \frac{i}{2}\frac{\partial}{\partial x_i}\left(-ix_j \mathcal{F}^{-1}\left\{\frac{1}{q_k q_k}\right\}\right) = \frac{1}{2}\frac{\partial}{\partial x_i}\left(\frac{x_j}{4\pi r}\right) \\ &= \frac{1}{2}\left(\frac{\delta_{ij}}{4\pi r} - \frac{x_i x_j}{4\pi r^3}\right) . \end{aligned} \quad (29)$$

Plugging in (28) we get

$$G_{ij}(\vec{r}) = \frac{1}{16(1-\nu)\pi\mu r}\left[(3-4\nu)\delta_{ij} + \frac{x_i x_j}{r^2}\right] . \quad (30)$$

A more elegant way to go, is to write $G_{ij}$ as gradients of $r$ (I mean $|r|$, not $\vec{r}$):

$$G_{ij}(\vec{r}) = \frac{1}{8\pi\mu}\left[\partial_k\partial_k r\,\delta_{ij} - \frac{1}{2(1-\nu)}\partial_i\partial_j r\right] = \frac{1}{8\pi\mu}\left[\boldsymbol{I}\,\nabla^2 r - \frac{\nabla\nabla r}{2(1-\nu)}\right] . \quad (31)$$

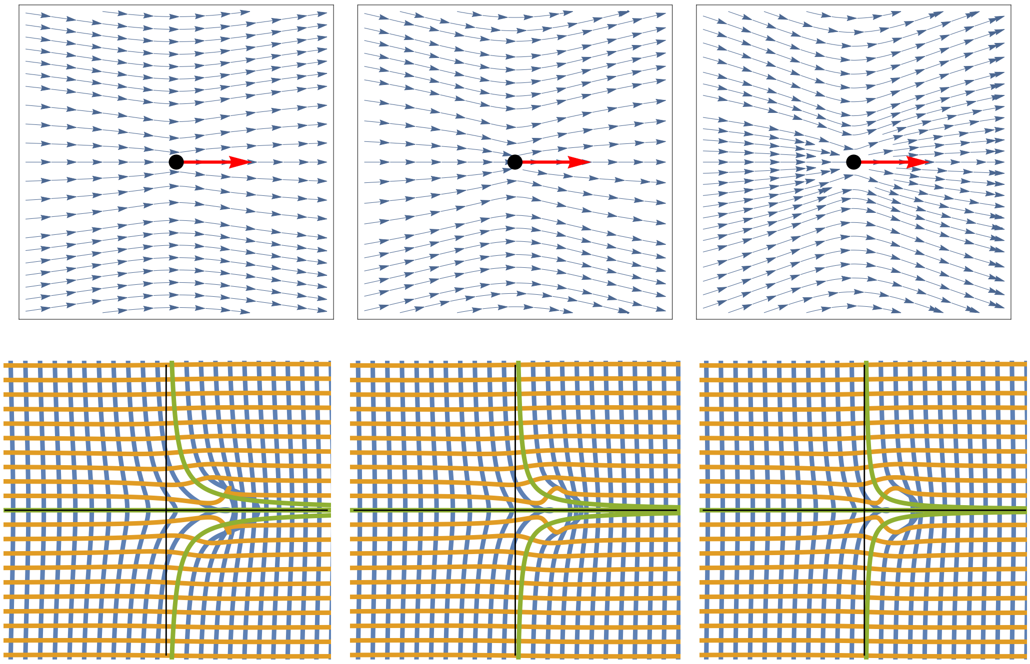

Figure 2: Top: Displacement field lines in the $x-y$ plane for point $\vec{F} = F\hat{x}$ in the horizontal direction. From left to right, with $\nu = 0$, 0.33, 0.5. Bottom: deformation of a regular mesh under this motion. Note that crossing of two lines of the same color is physically forbidden (why?).

## 3.1 Notes about the solution

1. The displacements go as $1/r$, which means that the strain/stress go as $1/r^2$. Therefore the elastic energy density, which goes like $\varepsilon^2$, goes like $1/r^4$ and *its integral diverges*. This is much like the case of electrostatics, where the total energy of the electrical field of a point charge diverges.

2. The scaling $u \sim 1/r$ could also have been obtained from simple dimensional analysis. It is quite common that dimensional considerations in elasticity take the "dimension" from the shear modulus $\mu$, and then there's an unknown (and usually uninteresting) function of $\nu$.

3. You might have noticed that $G_{ij}$ is a symmetric matrix. This might look at first glance as a trivial property that stems from the translational symmetry or rotational symmetry (=isotropy), but this is not the case. This symmetry property does not stem from any simple argument (that I can think of). Instead, this symmetry is a special case of a more general property that is called *reciprocity*. For a general linear elastic solid, and by general I mean that $C_{ijkl}(\boldsymbol{r})$ can have any symmetry and can even depend on space, the static Green function satisfies

$$G_{ij}(\boldsymbol{r}, \boldsymbol{r}') = G_{ji}(\boldsymbol{r}', \boldsymbol{r}) \ . \tag{32}$$