Waves and Thermo-Elasticity

1 Elastic waves

I remind you that you have shown in class that the Navier-Lamé Equation,

$$(\lambda+\mu)\nabla\left(\nabla\cdot\boldsymbol{u}\right)+\mu\nabla^2\boldsymbol{u}+\boldsymbol{b}=\rho\partial_{tt}\boldsymbol{u}\ , \tag{1}$$

is basically two uncoupled wave equations for dilatational and shear waves. They propagate at speeds

$$c_s=\sqrt{\frac{\mu}{\rho}}\ , \qquad c_d=\sqrt{\frac{\lambda+2\mu}{\rho}}\ . \tag{2}$$

Thus, Eq. (1) can also be written as

$$(c_d^2-c_s^2)\nabla\left(\nabla\cdot\boldsymbol{u}\right)+c_s^2\nabla^2\boldsymbol{u}+\boldsymbol{b}/\rho=\partial_{tt}\boldsymbol{u}\ . \tag{3}$$

The two wave speeds differ by a significant factor. c_s is always smaller than c_d and their ratio is

$$\beta\equiv\frac{c_s}{c_d}=\sqrt{\frac{\mu}{\lambda+2\mu}}=\sqrt{\frac{1-2\nu}{2(1-\nu)}}\ . \tag{4}$$

For a typical value of $\nu=1/3$, this gives a ratio of $\frac{1}{2}$. This function is plotted in Fig. 1. Note that the ratio goes to 0 for $\nu\to\frac{1}{2}$. This is because in incompressible materials ($\nu=1/2$) the dilatational velocity c_d diverges (as the bulk modulus K diverges). Seismographers use the difference in propagation velocity to determine the distance to an earthquake source, as is seen in Fig. 1.

1.1 Propagation-displacement relation

In class, you have discussed the polarization of these two waves by writing

$$\boldsymbol{u}=g(\boldsymbol{x}\cdot\boldsymbol{n}-c\,t)\boldsymbol{a}\ , \tag{5}$$

where $\boldsymbol{n}$ is the propagation direction, $\boldsymbol{a}$ is the direction of the displacement and $|\boldsymbol{n}|=|\boldsymbol{a}|=1$. The identity

$$(c_d^2-c_s^2)(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}+(c_s^2-c^2)\boldsymbol{a}=0 \tag{6}$$

is obtained by direct application of the corresponding operators on $\boldsymbol{u}$:

$$\nabla g=\partial_i g=g'n_i=g'\boldsymbol{n}\ , \tag{7a}$$
$$\nabla\boldsymbol{u}=\partial_j u_i=\partial_j(ga_i)=g'n_ja_i=g'\boldsymbol{a}\otimes\boldsymbol{n}\ , \tag{7b}$$
$$\nabla^2\boldsymbol{u}=\nabla\cdot\nabla\boldsymbol{u}=\partial_j(g'n_ja_i)=g''n_jn_ja_i=g''\boldsymbol{a}\ , \tag{7c}$$
$$\nabla\cdot\boldsymbol{u}=\mathrm{tr}\left(\nabla\boldsymbol{u}\right)=g'\boldsymbol{a}\cdot\boldsymbol{n}\ , \tag{7d}$$
$$\nabla\left(\nabla\cdot\boldsymbol{u}\right)=\partial_i(g'\boldsymbol{a}\cdot\boldsymbol{n})=g''(\boldsymbol{a}\cdot\boldsymbol{n})n_i=g''(\boldsymbol{a}\cdot\boldsymbol{n})\boldsymbol{n}\ , \tag{7e}$$
$$\partial_{tt}\boldsymbol{u}=c^2g''\boldsymbol{a}\ . \tag{7f}$$

Plugging Eqs. (7e)-(7f) into (3) gives immediately Eq. (6).

The two waves are independent in the bulk. However, on the boundary of a body the traction-free condition $\sigma_{ij}n_j = 0$ couples between the two modes, and some others arise with a distinct propagation velocity. These are called Rayleigh waves.

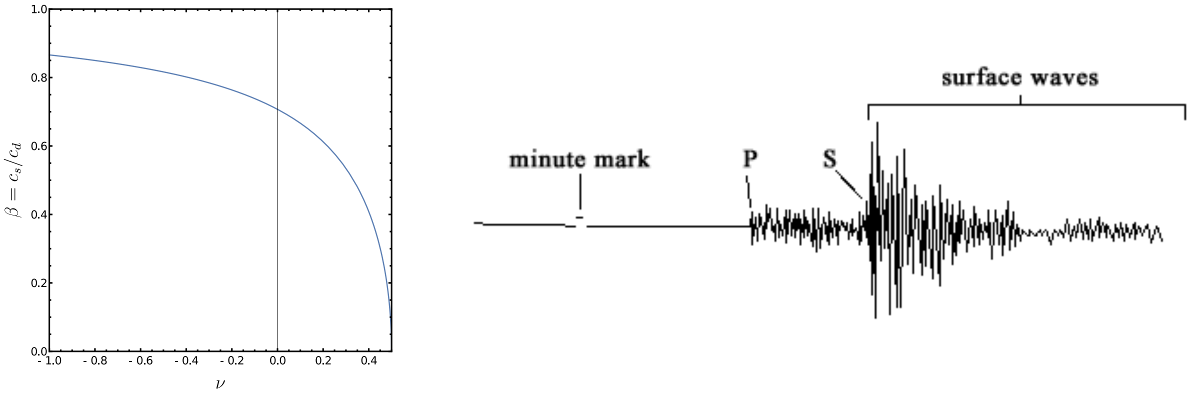


Figure 1: Left: c_s/c_d as a function of Poisson's ratio (Eq. (4)). Right: Seismograph reading of an earthquake. One can clearly see a P-wave (longitudinal) and an S-wave (transverse) arriving at different times. Later, surface waves are visible. The time difference can be used to obtain the distance from the earthquake source.

1.2 Rayleigh waves

So let's see exactly how this works. We want to look at surface waves which propagate, say, in the x-direction. To this end, consider a material that fills the lower half-space $z < 0$, and assume that

$$\boldsymbol{u} = \boldsymbol{f}(z)e^{ikx-i\omega t} \ . \tag{8}$$

If $\boldsymbol{u}$ satisfies the wave equation $\left(\frac{1}{c_i^2}\partial_{tt} - \nabla^2\right)\boldsymbol{u} = 0$, with $c_i = c_s$ or c_d, we have

$$\partial_{zz}\boldsymbol{f} = \left(k^2 - \frac{\omega^2}{c_i^2}\right)\boldsymbol{f} \ .$$

If $k^2 > \frac{\omega^2}{c_i^2}$ this gives a damped wave in the bulk. We denote

$$\boldsymbol{f}(z) = \boldsymbol{\gamma}^{(i)}e^{\eta_i z}, \qquad \eta_i = \sqrt{k^2 - \frac{\omega^2}{c_i^2}}, \quad i = s, d \ .$$

As stated above, Rayleigh waves are modes which mix dilational and shear waves. We therefore guess the ansatz

$$\boldsymbol{u} = \boldsymbol{u}^{(d)} + \boldsymbol{u}^{(s)} \ , \tag{9}$$

$$\boldsymbol{u}^{(i)} = \left(\gamma_x^{(i)}\hat{\boldsymbol{x}} + \gamma_z^{(i)}\hat{\boldsymbol{z}}\right)e^{\eta_i z + ikx - i\omega t} \ , \tag{10}$$

where $\boldsymbol{u}^{(d)}, \boldsymbol{u}^{(s)}$ are dilatational and shear waves, and $\gamma_j^{(i)}$ are constants. That is, each of $\boldsymbol{u}^{(d)}, \boldsymbol{u}^{(s)}$ satisfies its own wave equation,

$$(\partial_{tt} - c_s^2\nabla^2)\boldsymbol{u}^{(s)} = 0 \qquad (\partial_{tt} - c_d^2\nabla^2)\boldsymbol{u}^{(d)} = 0 \ . \tag{11}$$

They both oscillate with the same frequency ω (the ω of Eq. (8)). Of course, the $\boldsymbol{u}^{(i)}$ are not exactly bulk modes, because they decay exponentially with z, each over over a different length-scale η_i.

Following the discussion about the different polarizations of the different types of waves, note that we should demand

$$\vec{\nabla} \cdot \boldsymbol{u}^{(s)} = \vec{\nabla} \times \boldsymbol{u}^{(d)} = 0 \ . \tag{12}$$

Plugging the ansatz into equation (12) yields

$$\frac{\partial u_x^{(s)}}{\partial x} + \frac{\partial u_z^{(s)}}{\partial z} = \left(ik\gamma_x^{(s)} + \eta_s \gamma_z^{(s)}\right) e^{\cdots} = 0 \qquad \Rightarrow \qquad \frac{\gamma_z^{(s)}}{\gamma_x^{(s)}} = -i\frac{k}{\eta_s} \ , \tag{13}$$

$$\frac{\partial u_x^{(d)}}{\partial z} - \frac{\partial u_z^{(d)}}{\partial x} = \left(\eta_d \gamma_x^{(d)} - ik\gamma_z^{(d)}\right) e^{\cdots} = 0 \qquad \Rightarrow \qquad \frac{\gamma_z^{(d)}}{\gamma_x^{(d)}} = -i\frac{\eta_d}{k} \ . \tag{14}$$

So we write

$$\boldsymbol{u}^{(s)} = A\left(\eta_s \hat{\boldsymbol{x}} - ik\hat{\boldsymbol{z}}\right) e^{\eta_s z + ikx - i\omega t} \qquad A \in \mathbb{C} \ , \tag{15}$$

$$\boldsymbol{u}^{(d)} = B\left(ik\hat{\boldsymbol{x}} + \eta_d \hat{\boldsymbol{z}}\right) e^{\eta_d z + ikx - i\omega t} \qquad B \in \mathbb{C} \ , \tag{16}$$

We now want to demand that the boundary is traction-free. That is, we want to impose $\sigma_{ij}\big|_{z=0} n_j = 0$, where n_j is the local normal to the deformed surface. In principle, $\hat{n}$ also changes because the surface deforms. However, since $\boldsymbol{\sigma}$ is already first-order in the deformation, we are allowed to take the zeroth order of $\hat{n}$, that is, we can take $\hat{n} = \hat{z}$. Therefore, imposing the traction-free boundary conditions means $\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$ on $z = 0$. This translates via Hooke's law to

$$\sigma_{xz} = 2\mu\varepsilon_{xz} = \mu\left(\partial_z u_x + \partial_x u_z\right) = 0 \ , \tag{17}$$

$$\sigma_{zz} = (2\mu + \lambda)\varepsilon_{zz} + \lambda\varepsilon_{xx} = (2\mu + \lambda)\partial_z u_z + \lambda\partial_x u_x = \frac{c_d^2 \partial_z u_z + \left(c_d^2 - 2c_s^2\right)\partial_x u_x}{\rho} = 0 \ . \tag{18}$$

We now plug Eqs. (15)-(16) into (17)-(18). This is some uninteresting but necessary algebra. Eq. (17) is relatively simple:

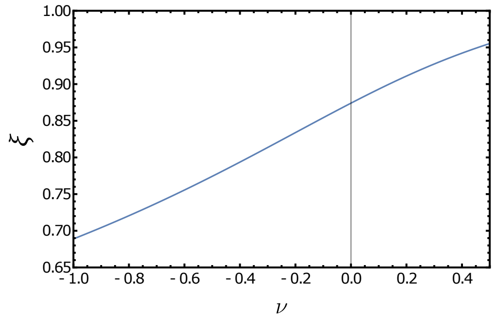
$$0 = \partial_z u_x + \partial_x u_z = \left(\eta_s^2 A + i\eta_d k B\right) + \left(k^2 A + i\eta_d k B\right) = \left(\eta_s^2 + k^2\right) A + 2i\eta_d k B \ , \tag{19}$$

Eq. (18) requires some simplification in order to be sensible:

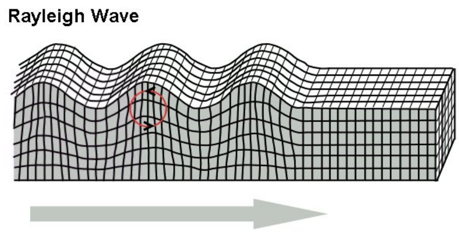
$$\begin{aligned} 0 &= c_d^2 \partial_z u_z + \left(c_d^2 - 2c_s^2\right)\partial_x u_x \\ &= c_d^2\left(\eta_d^2 B - ik\eta_s A\right) - \left(c_d^2 - 2c_s^2\right)\left(ik\eta_s A - k^2 B\right) \\ &= \left(\frac{c_d^2}{c_s^2}\left(\eta_d^2 - k^2\right) + 2k^2\right) B - 2ik\eta_s A \end{aligned} \tag{20}$$

But since $(k^2 - \eta_i^2)c_i^2 = \omega^2$ is the same for both i's, we can replace $\left(\eta_d^2 - k^2\right) c_d^2$ in the last equation by $\left(\eta_s^2 - k^2\right) c_s^2$ and get

$$\left(\eta_s^2 + k^2\right) B - 2ik\eta_s A = 0 \ , \tag{21}$$



(a) Numerical solution for z.



(b) A Rayleigh wave (from wikipedia).

Figure 2: Stuff about Rayleigh waves.

Eq. (19) together with (21) form a linear set of equations:

$$\begin{pmatrix} k^2+\eta_s^2 & 2ik\eta_d \\ -2i\eta_s k & k^2+\eta_s^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

The condition for a non-trivial solution to exist is det $=0$, that is $\left(k^2+\eta_s^2\right)^2 = 4k^2\eta_s\eta_d$. Plugging in $\eta_i^2 = k^2 - \left(\frac{\omega}{c_i}\right)^2$ and squaring, this gives

$$\left(2k^2 - \frac{\omega^2}{c_s^2}\right)^4 = 16\,k^4 \left(k^2 - \frac{\omega^2}{c_s^2}\right)\left(k^2 - \frac{\omega^2}{c_d^2}\right) \quad . \tag{22}$$

This is the dispersion relation for Rayleigh waves (also called Rayleigh equation). It is actually a linear dispersion relation. To realize this we divide both sides by k^8 to get

$$\left(2 - \left(\frac{\omega}{kc_s}\right)^2\right)^4 = 16\left(1 - \left(\frac{\omega}{kc_s}\right)^2\right)\left(1 - \left(\frac{\omega}{kc_s}\right)^2\left(\frac{c_s}{c_d}\right)^2\right) \quad .$$

Denoting the dimensionless phase velocity $z = \frac{\omega}{kc_s} = \frac{c_{ph}}{c_s}$ and remembering our definition $\beta = \frac{c_s}{c_d}$ (cf. Eq. (4)), this turns to

$$\left(2-z^2\right)^4 - 16\left(1-z^2\right)\left(1-\beta^2 z^2\right) = 0 \ ,$$

or,

$$z^6 - 8z^4 + 8\left(3-2\beta^2\right)z^2 + 16\left(\beta^2-1\right) = 0$$

So knowing β, which is a material parameter that equals $\sqrt{\frac{1-2\nu}{2(1-\nu)}}$ gives the (physically unique) solution for z and thus completely defines the **linear** dispersion relation $\omega = zc_s k$. The solution is shown in Fig. 2a, and it is seen that the wave speed, zc_s, is somewhat slower than c_s.

1.2.1 Some remarks regarding Rayleigh waves

- Dilational and shear waves travel at two different speeds. Nevertheless, Rayleigh waves couple the two (!) to create a different mode that travels at a third speed (!!), and all this is within a linear theory (!!!).

- The coupling comes from the traction-free boundary condition.
- A single Rayleigh mode with k, ω is a combination of two evanescent bulk modes with the same ω, but different k.
- The bulk modes are evanescent because the velocity of the Rayleigh mode is slower than c_s and c_d. This makes η_s, η_d real. Otherwise, the modes will not be localized on the surface.
- Rayleigh waves are surface waves. Therefore, their magnitude decreases only as $1/\sqrt{r}$ rather than the bulk $1/r$. In large earthquakes, some Rayleigh waves circle the earth a few times before dissipating.
- They are confined to propagate on the surface and decay exponentially with depth. Therefore, the amplitude of earthquake-generated Rayleigh waves is generally a decreasing function of the depth of the earthquake's hypocenter (origin/focus).
- The particle trajectories in a Rayleigh wave are elliptic, much like in ocean surface waves.

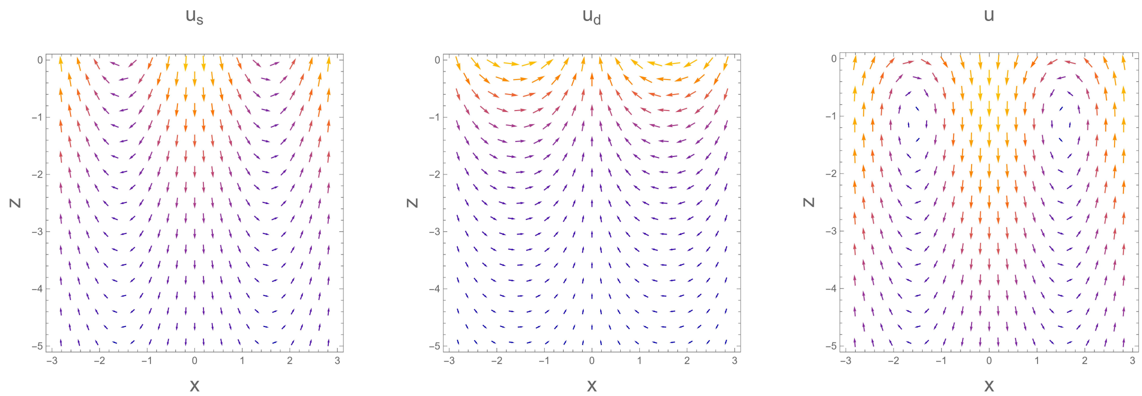


Figure 3: The real parts of u_s, u_d and $u = u_s + u_d$ of the obtained solutions to the Rayleigh derivation. This was produced with $\rho = 1000\text{kg/m}^3$, $\mu = 50\text{GPa}$, and $\nu = \frac{1}{3}$ (resulting in $\beta = \frac{1}{2}$).

2 Thermo-Elasticity

In 1993, Yuse & Sano published a remarkable paper regarding instabilities of thermally induced fractures (Yuse & Sano, Nature (362) 1993). They consider a strip of material which is pulled out of an oven **at a constant velocity** and cools down as it moves. The gradients of the temperature field induce fracture, as is seen in Figure 4.

To model the phenomenon, we consider an infinite (in the x direction) 2D strip of width $2b$. The strip is subjected to a y-independent temperature distribution $T(x)$, and is free of tractions at its boundaries $y = \pm b$. Fracture will be considered later in this course. For now, we'll limit ourselves to finding an expression for the stretching component σ_{yy} along the strip's symmetry axis $y = 0$. This is the driving force that induces fracture.

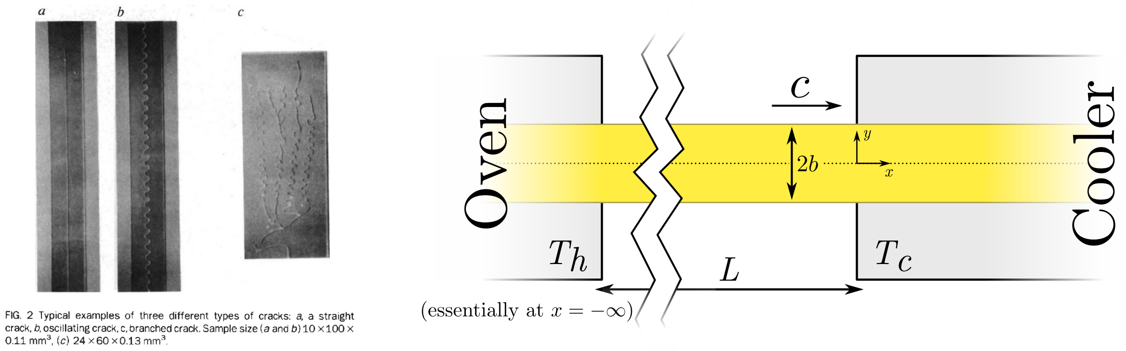


Figure 4: Left: Thermally induced fracture (from the paper). Right: the simplified model. Note the position of the origin of axes, and that the plot **is not to scale**: we assume $L \gg b$.

2.1 Temperature distribution

We begin by finding the temperature distribution. In the material coordinates $\{X, Y, \tau\}$, the diffusion equation is

$$\partial_\tau T = D\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right) T \ . \tag{23}$$

The change of variables from the material coordinates to the lab coordinates $\{x, y, t\}$ is defined by

$$x = X + c\tau, \quad t = \tau, \quad y = Y \ , \tag{24}$$

and we find that the differentiation operators transform as

$$\begin{aligned}
\frac{\partial}{\partial \tau} &= \frac{\partial t}{\partial \tau}\frac{\partial}{\partial t} + \frac{\partial x}{\partial \tau}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \tau}\frac{\partial}{\partial y} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x} \ , \\
\frac{\partial}{\partial X} &= \frac{\partial t}{\partial X}\frac{\partial}{\partial t} + \frac{\partial x}{\partial X}\frac{\partial}{\partial x} + \frac{\partial y}{\partial X}\frac{\partial}{\partial y} = \frac{\partial}{\partial x} \ , \\
\frac{\partial}{\partial Y} &= \frac{\partial t}{\partial Y}\frac{\partial}{\partial t} + \frac{\partial x}{\partial Y}\frac{\partial}{\partial x} + \frac{\partial y}{\partial Y}\frac{\partial}{\partial y} = \frac{\partial}{\partial y} \ ,
\end{aligned} \tag{25}$$

therefore, the diffusion equation takes the form

$$\partial_t T = D\nabla^2_{\boldsymbol{x}} T - c\,\partial_x T \ . \tag{26}$$

Notice that we have *not* changed to the deformed coordinates. We have just moved from a FOR traveling with the strip to an FOR of the oven and cooler. As seen, this affects the heat equation, but as we are still considering linear thermo-elasticity, and hence small deformation, we need not move to the deformed coordinates. Why do we want to consider these coordinates?

To better understand the coordinate systems, imagine that instead of a solid strip we were to examine a fluid instead. The material coordinate system describe a frame of reference following one of the particles — imaging dying one particle, and sticking to it through the dynamics. Its neighbors will not change during the dynamics, but the temperature defined for this particle would vary as it moves from the hot to the cold reservoir. In this sense, we are moving with the particle.

Our transformed coordinates allow us to fix our frame of reference, and stay stationary as the fluid (or solid) moves. Using the analogue of a fluid again, measuring the temperature here would not follow a single particle, but would be more like sticking a thermometer inside the fluid at a fixed place.

In the traveling coordinates, it is clear that any element isn't in steady state, as it cools down and stresses develop with time. But in the lab frame we do expect a steady state to emerge, leading to the assumption that $\partial_t = 0$.

Assuming T is y-independent and the boundary conditions $T(x = 0) = T_c$, $T(x = -L) = T_h$, the steady-state solution for $-L \leq x \leq 0$ is

$$T = \frac{\left(e^{\frac{c}{D}(L+x)} - 1\right) T_c - e^{\frac{cL}{D}} \left(e^{\frac{c}{D}x} - 1\right) T_h}{e^{\frac{cL}{D}} - 1} \ . \tag{27}$$

We now define the diffusive length $d = D/c$, and use the fact that $L \gg d$ to simplify the above solution,

$$T \approx T_h + e^{\frac{x}{d}} \left(T_c - T_h\right) \ . \tag{28}$$

We set the zero of temperature at T_c, and denote $\Delta T \equiv T_h - T_c$. Incorporating the fact that $\Delta T = 0$ for $x > 0$ we obtain the temperature distribution

$$T(x) = \Delta T \left(1 - e^{\frac{x}{d}}\right) \Theta(-x) \ . \tag{29}$$

Θ is Heaviside's theta function. Note that we don't care that T is not exactly constant for $x < -L$, because the correction is roughly $e^{-L/d}$, which we assume is a very small number.

2.2 Thermo-elastic plane-stress

The thermo-elastic Hooke's law was derived in class and it reads

$$\boldsymbol{\sigma} = \lambda \operatorname{tr}(\boldsymbol{\varepsilon})\boldsymbol{I} + 2\mu\boldsymbol{\varepsilon} - \alpha K T \boldsymbol{I} \ . \tag{30}$$

We'd next like to invert this relation, so we want to write everything in matrix form. Also, in the end we'd like to have relations of the form $\boldsymbol{\varepsilon}(\boldsymbol{\sigma})$ (compliance form), so it is more natural to work with ν and E, rather than μ and λ. Using the conversions between $\mu, \lambda \to E, \nu$, Eq. (30) is written as:

$$\begin{pmatrix} \sigma_{xx} + \alpha K T \\ \sigma_{yy} + \alpha K T \\ \sigma_{zz} + \alpha K T \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} = \frac{2\mu}{1-2\nu} \begin{pmatrix} 1-\nu & \nu & \nu & & & \\ \nu & 1-\nu & \nu & & & \\ \nu & \nu & 1-\nu & & & \\ & & & 1-2\nu & & \\ & & & & 1-2\nu & \\ & & & & & 1-2\nu \end{pmatrix} \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{pmatrix}, \tag{31}$$

inverting, we get

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \\ \varepsilon_{xy} \\ \varepsilon_{xz} \\ \varepsilon_{yz} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & & \\ -\nu & -\nu & 1 & & & \\ & & & 1+\nu & & \\ & & & & 1+\nu & \\ & & & & & 1+\nu \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{xy} \\ \sigma_{xz} \\ \sigma_{yz} \end{pmatrix} + \frac{\alpha T}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ , \tag{32}$$

where we used the fact $K = \frac{E}{3(1-2\nu)}$. The assumptions of plane stress are that $\sigma_{zz}=\sigma_{xz}=\sigma_{yz}=0$. Thus, if we delete the 3rd, 5th and 6th rows of this equation, it reduces to

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{pmatrix} = \frac{1}{E} \begin{pmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 1+\nu \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{pmatrix} + \frac{\alpha T}{3} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} . \tag{33}$$

The compatibility relation,

$$\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} , \tag{34}$$

is a trivial identity that follows from the definition $\varepsilon_{ij} \equiv \frac{1}{2}(\partial_i u_j + \partial_j u_i)$. If we substitute ε_{ij} by Hookes law, Eq. (33)

$$\begin{aligned} 0 &= \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - 2 \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \\ &= \frac{\partial^2}{\partial y^2} \left(\frac{\sigma_{xx} - \nu \sigma_{yy}}{E} + \frac{\alpha T}{3} \right) + \frac{\partial^2}{\partial x^2} \left(\frac{\sigma_{yy} - \nu \sigma_{xx}}{E} + \frac{\alpha T}{3} \right) - 2 \frac{1+\nu}{E} \frac{\partial^2}{\partial x \partial y} \sigma_{xy} , \end{aligned} \tag{35}$$

to which we may now substitute σ_{ij} by derivatives of the Airy stress function χ we get (after multiplying by E)

$$\begin{aligned} 0 &= \frac{\partial^2}{\partial y^2} \left(\partial_{yy}\chi - \nu \partial_{xx}\chi \right) + \frac{\partial^2}{\partial x^2} \left(\partial_{xx}\chi - \nu \partial_{yy}\chi \right) + 2\left(1+\nu\right) \frac{\partial^2}{\partial y \partial x} \partial_{xy}\chi + \frac{\alpha E}{3} \nabla^2 T \\ &= (\partial_y^4 + 2\partial_{xx}\partial_{yy} + \partial_x^4)\chi + \frac{\alpha E}{3} \nabla^2 T = \nabla^2 \nabla^2 \chi + \frac{\alpha E}{3} \nabla^2 T . \end{aligned} \tag{36}$$

2.3 General solution

Next, we need to solve Eq. (36) under the conditions

$$\sigma_{yy}\big|_{y=\pm b} = \sigma_{xy}\big|_{y=\pm b} = 0 . \tag{37}$$

Now we Fourier-transform Eq. (36) with respect to x gives

$$\left(k^4 - 2k^2 \frac{\partial^2}{\partial y^2} + \frac{\partial^4}{\partial y^4} \right) \hat{\chi}(k,y) = -\frac{E\alpha}{3} k^2 \hat{T}(k) , \tag{38}$$

where k is the Fourier frequency associated with x. Note that we now use the $\hat{\bullet}$ notation to emphasize the Fourier transformation, and that the arguments of the functions are (k,y), and not (x,y).

A particular solution of this nonhomogeneous equation is clearly $\hat{\chi} = -\frac{E\alpha}{3k^2}\hat{T}(k)$. The homogeneous equation admits four independent solutions: $\hat{\chi} \propto e^{\pm ky}$, $\hat{\chi} \propto y e^{\pm ky}$. Note that the y-independence of T was used (and was crucial!). Because $\sigma_{yy} = \partial_{xx}\chi$ is an even function of y, $\hat{\chi}$ must also be even in y. So we use even combinations of the solutions we found to get

$$\hat{\chi}(k,y) = A(k)\cosh(ky) + yB(k)\sinh(ky) - \frac{E\alpha}{3k^2}\hat{T}(k) . \tag{39}$$

The boundary conditions (37) translate to

$$\partial_{xx}\chi|_{y=b} = 0 \Rightarrow \quad A\cosh(kb) + Bb\sinh(kb) - \frac{E\alpha}{3k^2}\hat{T}(k) = 0\ , \tag{40}$$

$$\partial_{xy}\chi|_{y=b} = 0 \Rightarrow \quad Ak\sinh(kb) + B\sinh(kb) + Bbk\cosh(kb) = 0\ . \tag{41}$$

This is a set of linear equations which is solved by

$$A(k) = \frac{E\alpha}{3k^2}\underbrace{2\frac{bk\cosh(bk)+\sinh(bk)}{2bk+\sinh(2bk)}}_{\equiv\hat{\Phi}(bk)}\hat{T}(k) = \frac{E\alpha}{3k^2}\hat{\Phi}(bk)\hat{T}(k)\ , \tag{42}$$

$$B(k) = -\frac{E\alpha}{3k}\frac{2\sinh(bk)}{2bk+\sinh(2bk)}\hat{T}(k)\ . \tag{43}$$

Note the definition of Φ, the use of which will become clear immediately. σ_{yy} along the symmetry line $y=0$ is given by

$$\begin{aligned}\sigma_{yy}(x,y=0) = \partial_{xx}\chi(x,y=0) &= \mathcal{F}^{-1}\left\{-k^2\hat{\chi}(k,y=0)\right\}\\ &= \mathcal{F}^{-1}\left\{-k^2\left(A(k)-\frac{E\alpha\hat{T}(k)}{3k^2}\right)\right\}\\ &= \mathcal{F}^{-1}\left\{\frac{E\alpha}{3}\left(1-\hat{\Phi}(bk)\right)\hat{T}(k)\right\}\ ,\end{aligned} \tag{44}$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. We now define $\hat{\Psi}(bk) \equiv 1 - \hat{\Phi}(bk)$ and use the convolution theorem to finally obtain

$$\sigma_{yy}(x,y=0) = \mathcal{F}^{-1}\left\{\frac{E\alpha}{3}\hat{\Psi}(bk)\hat{T}(k)\right\} = \frac{E\alpha}{3}\Psi(x) * T(x)\ , \tag{45}$$

where $*$ denotes convolution and $\Psi(x) = \frac{1}{2\pi}\int e^{-ikx}\hat{\Psi}(bk)dk$.

In order to proceed, we non-dimensionalise our equations. We renormalize lengths by b, that is, we define $\tilde{k} \equiv bk$, $\tilde{x} \equiv x/b$, and also $\tilde{\sigma} \equiv 3\sigma/(E\alpha\Delta T)$, $\tilde{T} \equiv T/\Delta T$, and $\tilde{\Psi} \equiv b\Psi$. We thus obtain

$$\hat{\Psi}(\tilde{k}) = 1 - 2\frac{\tilde{k}\cosh(\tilde{k})+\sinh(\tilde{k})}{2\tilde{k}+\sinh(2\tilde{k})}\ , \tag{46}$$

$$\tilde{\Psi}(\tilde{x}) = \int e^{-i\tilde{k}\tilde{x}}\hat{\Psi}(\tilde{k})\frac{d\tilde{k}}{2\pi} = \delta(\tilde{x}) - \underbrace{\int 2\cos\left(\tilde{k}\tilde{x}\right)\frac{\tilde{k}\cosh(\tilde{k})+\sinh(\tilde{k})}{2\tilde{k}+\sinh(2\tilde{k})}\frac{d\tilde{k}}{2\pi}}_{=\Phi(\tilde{x})}\ , \tag{47}$$

$$\tilde{\sigma}_{yy}(\tilde{x},y=0) = \tilde{T}(\tilde{x}) * \Psi(\tilde{x}) = \tilde{T}(\tilde{x}) * [\delta(\tilde{x}) - \Phi(\tilde{x})] = T(\tilde{x}) - T(\tilde{x}) * \Phi(\tilde{x})\ , \tag{48}$$

where the convolution is done in $\tilde{x}$, rather than x. In Eq. (47) the $e^{i\tilde{k}\tilde{x}}$ was replaced with $\cos(\tilde{k}\tilde{x})$ since the integrand is even. Also, note that since $\Psi(x)$ has a δ-function singularity, the convolution is done by convolving with Φ and subtracting T, as is shown in Eq. (48).

We see that $\hat{\Psi}(\tilde{x})$ is a "universal" function that can be computed once, and then the stress field resulting from an arbitrary temperature distribution can be obtained by convolving the temperature with the kernel $\Psi(x)$ (a different terminology would be that $\Psi(x)$ is the Green's function of the problem).

Note also how useful is the non-dimensionalization: The kernel as a function of $\tilde{x}$ will not change when b or D/c change. This will come in only through the temperature profile:

$$\tilde{T}(\tilde{x}) = T\left(\frac{x}{b}\right)/\Delta T = \left(1 - e^{\frac{b}{d}\tilde{x}}\right)\theta(-\tilde{x}) \ , \tag{49}$$

so the only numeric value we have to use is the dimensionless ratio b/d. Other than that, all the functions can be pre-calculated.

A sanity check: if $T(x) = const$, then σ_{yy} should be zero. According to Eq. (45), the stress will be

$$\sigma_{yy} \propto \int \Psi(x)dx \propto \hat{\Psi}(k=0) \ ,$$

indeed, it is easily seen that $\hat{\Psi}(k=0) = 0$.

The function $\tilde{\Psi}(x)$ cannot be computed analytically. On the website there's a Mathematica notebook with very detailed explanations about the numerics. There is also a MATLAB script that does the numerics, but without explanations. The results are shown in Fig. S1.

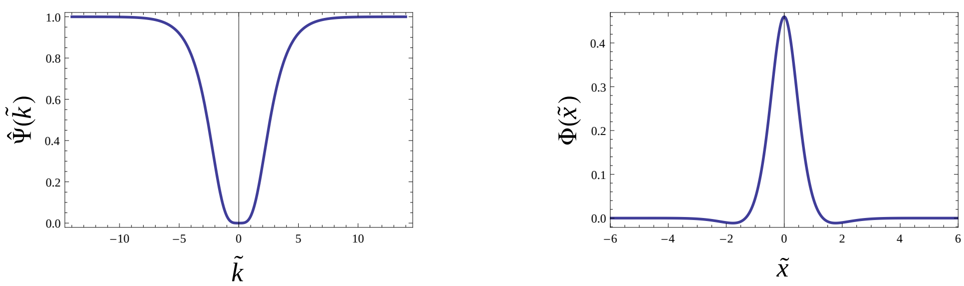
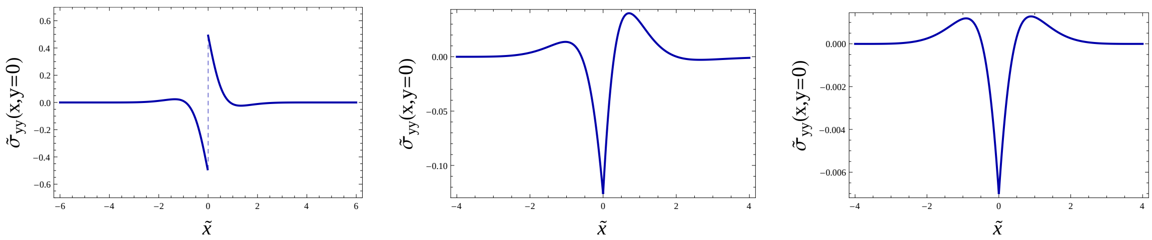


Figure S1: The functions $\hat{\Psi}(\tilde{k})$ and $\Phi(\tilde{x})$ (one can't really plot $\Psi(x)$ as it contains a δ function). Note that the width of $\Psi(\tilde{x})$ is roughly 1, or in other words, the width of $\Psi(x)$ is roughly b. The stress is numerically obtained by substracting $T(x)$ from the convolution $T(x) * \Psi(x)$

To obtain the solution for σ_{yy}, we need to convolve $\Psi(x)$ with $T(x)$. Note that the width of $T(x)$ is d, while the width of $\Psi(x)$ is b. Therefore, for $b \gg d$, $T(x)$ essentially looks like a step-function. So,

$$\sigma_{yy}(x, y=0) = \int_{-\infty}^{\infty} \Psi(\xi)T(x-\xi)d\xi \propto \int_{-\infty}^{x} \Psi(\xi)d\xi \ .$$

The numerical calculation of this integral is shown in Fig S2. Note the discontinuity at $x = 0$ which is due to the $\delta(x)$ term in Ψ. For $b \approx d$ and $b \ll d$, one needs to numerically convolve. The result is also shown in Fig S2. Note that the length scale of the stress variation is always b – the width of the kernel – and not d.



$b \gg d$ $\qquad$ $b = d$ $\qquad$ $b \ll d$

Figure S2: Numerical solution of $\sigma_{yy}(x, y = 0)$ for various conditions. Note that the stress increases as d/b decreases. This is because the stresses are roughly proportional to $\partial_{\tilde{x}\tilde{x}}T$, and when b is small, the derivative with respect to $\tilde{x}$ is large.

Let us check that our solution is at least consistent with our assumptions — what would happen if we were to reduce the y-dimension to zero, and apply the plane-stress approximation? From the three figures above, we see that as b/d decreases, the maximal value of the σ_{yy} stress reduces dramatically — this behavior would be consistent if we were to apply the plane-stress approximation in the y-direction as then we would take $\sigma_{yy} = 0$ explicitly. Indeed it seems that $b/d \to 0$ is in agreement with such an approximation.

Can we relate our results to the observations in Fig. 4? Maybe we can deduce the conditions for the initiation of fracture from $\sigma_{yy}(x, y = 0)$, but we most certainly cannot say how it behaves. The observations shown in Fig. 4 show various different phenomena — straight, oscillating, and splitting fracture — these all are dynamical processes. At most, we can predict the onset of fracture from our calculations, but we cannot commit to how such fracture will propagate. While you will see fracture mechanics towards the end of the course, here are a couple of papers examining fracture dynamics (see here, here, and here).