Finite elasticity and incompressibility

In this TA we'll be looking at non-linear elasticity, a.k.a finite elasticity. We'll also discuss how we handle incompressible systems.

1 Elastic cavitation

Consider a spherical cavity of initial radius L inside an elastic material loaded by a radially symmetric tensile stress far away, σ^{∞}. The symmetry of the problem suggests that all quantities are functions of r alone and that $\sigma_{\phi\phi} = \sigma_{\theta\theta}$. The force balance equation reads

$$\partial_r \sigma_{rr} + 2\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0 \ . \tag{1}$$

Integrating this equation from the deformed radius of the cavity ℓ to r we obtain

$$\sigma_{rr}(r) = -2\int_{\ell}^{r} \frac{\sigma_{rr} - \sigma_{\theta\theta}}{\tilde{r}} d\tilde{r} \ , \tag{2}$$

where we used the traction-free boundary condition $\sigma_{rr}(r = \ell) = 0$ and $\tilde{r}$ is a dummy integration variable. Denote then $r' = \tilde{r}/\ell$ and focus on $r \to \infty$, we obtain

$$\sigma_{rr}(\infty) = -2\int_{1}^{\infty} \frac{g(r', L/\ell)}{r'} dr' \ , \tag{3}$$

where $\sigma_{rr} - \sigma_{\theta\theta} = g(r', L/\ell)$ is a property of the solution (which involves also the constitutive relation). From our previous analysis we know that the existence of the cavity amplifies the (circumferential) stress at the surface as compared to the applied stress σ^{∞} (for a cylindrical cavity we calculated the amplification factor to be 2 and for a sphere is it 3/2). If we keep on increasing the applied stress an ordinary material will simply break near the cavity surface. However, in soft materials something else can happen (the same can happen is an elasto-plastic material, to be discussed later). We can ask ourselves whether the cavity can grow (elastically!) without bound under the application of a finite stress at infinity. To mathematically formulate the question take the $\ell \to \infty$ limit in Eq. (3) and define

$$\sigma_c = -2\lim_{\ell\to\infty}\int_{1}^{\infty} \frac{g(r', L/\ell)}{r'} dr' \ . \tag{4}$$

Therefore, if the integral above converges, then for any $\sigma^{\infty} > \sigma_c$ the cavity will grow indefinitely. The critical stress σ_c is called the cavitation threshold. σ_c is finite if $g(r', L/\ell) = \sigma_{rr} - \sigma_{\theta\theta} \to 0$ as $r \to \infty$, which is the typical situation.

Let us see how this works in a concrete example, where the goal is to find $\sigma_{rr} - \sigma_{\theta\theta} = g(r', L/\ell)$ and then evaluate the integral in Eq. (4). Consider an incompressible elastic material. As above, the initial radius of the cavity is L and the radial coordinate is denoted as R. The deformed radius is ℓ and the coordinate of the deformed configuration is r. Incompressibility implies that the volume of any material piece in the reference configuration is conserved in the deformed one, in particular we have

$$\frac{4\pi}{3}\left(R^3 - L^3\right) = \frac{4\pi}{3}\left(r^3 - \ell^3\right) \quad \Longrightarrow \quad R(r) = (r^3 + L^3 - \ell^3)^{1/3} \ . \tag{5}$$

The non-radial stretches take the form

$$\lambda_\phi = \lambda_\theta = \frac{r}{R} \ . \tag{6}$$

Incompressibility implies

$$\lambda_r \equiv \lambda \quad \Longrightarrow \quad \lambda_\phi = \lambda_\theta = \lambda^{-1/2} \tag{7}$$

which leads to

$$\lambda^{-1/2} = \frac{r}{R} \quad \Longrightarrow \quad \lambda = \left(\frac{R}{r}\right)^2 \ . \tag{8}$$

Finally, this leads to

$$\lambda = \left(\frac{r^3 + L^3 - \ell^3}{r^3}\right)^{2/3} = \left[\frac{r'^3 + (L/\ell)^3 - 1}{r'^3}\right]^{2/3} \ , \tag{9}$$

with $r' \equiv r/\ell$. Consider then the stress state. It is triaxial and contains only the diagonal components $(\sigma_{rr}, \sigma_{\phi\phi}, \sigma_{\theta\theta})$, with $\sigma_{\phi\phi} = \sigma_{\theta\theta}$. However, since the material is incompressible we can superimpose on this stress state a hydrostatic stress tensor of the form $-\sigma_{\theta\theta}\boldsymbol{I}$ without affecting the deformation state, resulting in $(\sigma_{rr} - \sigma_{\theta\theta}, 0, 0)$, which is a uniaxial stress state in the radial direction. Therefore, the constitutive relation takes the form $\sigma_{rr} - \sigma_{\theta\theta} = g(\lambda_r)$. Focus then on a neo-Hookean material for which $g(\lambda) = \mu(\lambda^2 - \lambda^{-1})$ and evaluate the integral in Eq. (4)

$$\sigma_c = -2\lim_{\ell\to\infty}\int_1^\infty \frac{g[\lambda(r', L/\ell)]}{r'}dr' = -2\mu\int_1^\infty\left[\frac{(1 - r'^{-3})^{4/3} - (1 - r'^{-3})^{-2/3}}{r'}\right]dr' \ . \tag{10}$$

This integral can be readily evaluated (just use $x \equiv 1 - r'^{-3}$ and $dx = 3r'^{-4}dr'$), yielding

$$\sigma_c = \frac{5\mu}{2} \ . \tag{11}$$

This result, which was verified experimentally (see, for example, J. Appl. Phys. **40**, 2520 (1969)), clearly demonstrates the striking difference between ordinary and "soft" solids. The *ideal* strength of ordinary solids is about $\mu/10$. The actual strength is *much* smaller (see later in the course). However, "soft" solids can sustain stresses larger than μ without breaking (though, as we have just shown, they can experience unique instabilities such as elastic cavitation).

2 2D plane-stress

We'll now consider a 2D plane-stress problem of an incompressible neo-Hookean material. The Neo-Hookean energy functional is (Eq. (7.10) in Eran's notes)

$$u(\boldsymbol{F}_{3D}) = \frac{\mu}{2}\left[\mathrm{tr}(\boldsymbol{F}_{3D}^T\boldsymbol{F}_{3D}) - 3\right] , \tag{12}$$

together with the requirement that $J_{3D} \equiv \det \boldsymbol{F}_{3D} = 1$. We want to consider now the case where the stresses are only in-plane. If we consider now λ_i, the principle values of $\boldsymbol{F}_{3D}$, we may write

$$u = \frac{\mu}{2}\left[\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3\right] \underbrace{=}_{\lambda_1\lambda_2\lambda_3=1} \frac{\mu}{2}\left[\lambda_1^2 + \lambda_2^2 + \frac{1}{\lambda_1^2\lambda_2^2} - 3\right] . \tag{13}$$

If we now consider the two-dimensional deformation gradient tensor $\boldsymbol{F}_{2D}$, we find that the Neo-Hookean energy functional for plane-stress is

$$u(\boldsymbol{F}_{2D}) = \frac{\mu}{2}\left[\mathrm{tr}(\boldsymbol{F}_{2D}^T\boldsymbol{F}_{2D}) + (\det \boldsymbol{F}_{2D})^{-2} - 3\right] . \tag{14}$$

Note that $\det \boldsymbol{F}_{2D}$ appears in the elastic energy functional due to the incompressibility condition. For the rest of the section I'll drop the subscript and just write $\boldsymbol{F}$, as we will be dealing with 2D problems, but keep in mind that it is $\boldsymbol{F}_{2D}$.

2.1 The first Piola-Kirchhoff stress tensor

To calculate $\boldsymbol{P} \equiv \frac{\partial u}{\partial \boldsymbol{F}}$ note that

$$\frac{\partial\, \mathrm{tr}(\boldsymbol{F}^T\boldsymbol{F})}{\partial \boldsymbol{F}} = 2\boldsymbol{F}, \quad \frac{\partial(\det \boldsymbol{F})^{-2}}{\partial \boldsymbol{F}} = -2(\det \boldsymbol{F})^{-3}\left(\mathrm{tr}\,\boldsymbol{F}\ \boldsymbol{I} - \boldsymbol{F}^T\right) , \tag{15}$$

where the latter can be easily obtained using the identity (valid in 2D only)

$$\det \boldsymbol{F} = \frac{1}{2}\left[(\mathrm{tr}\,\boldsymbol{F})^2 - \mathrm{tr}\,\boldsymbol{F}^2\right] . \tag{16}$$

Therefore, we have

$$\boldsymbol{P} = \frac{\partial u}{\partial \boldsymbol{F}} = \mu\left[\boldsymbol{F} - (\det \boldsymbol{F})^{-3}\left(\mathrm{tr}\,\boldsymbol{F}\ \boldsymbol{I} - \boldsymbol{F}^T\right)\right] , \tag{17}$$

$$\boldsymbol{P} = \mu\left[\begin{pmatrix} \partial_X\varphi_x & \partial_Y\varphi_x \\ \partial_X\varphi_y & \partial_Y\varphi_y \end{pmatrix} - J^{-3}\begin{pmatrix} \partial_Y\varphi_y & -\partial_X\varphi_y \\ -\partial_Y\varphi_x & \partial_X\varphi_x \end{pmatrix}\right] , \tag{18}$$

where $J \equiv \det \boldsymbol{F} = \partial_X\varphi_x(X,Y)\partial_Y\varphi_y(X,Y) - \partial_Y\varphi_x(X,Y)\partial_X\varphi_y(X,Y)$.

2.2 Linearized energy functional

Before going fully non-linear, let's examine the linearized version our equations to see if we get something that we recognize. Assume for simplicity that the axes are chosen in parallel to the the principal stretches, i.e.

$$\boldsymbol{F} = \begin{pmatrix} 1+\varepsilon_x & 0 \\ 0 & 1+\varepsilon_y \end{pmatrix} .$$

The energy density is then

$$u = \frac{\mu}{2}\left[(1+\varepsilon_x)^2 + (1+\varepsilon_y)^2 + \frac{1}{(1+\varepsilon_x+\varepsilon_y+\varepsilon_x\varepsilon_y)^2} - 3\right] . \tag{19}$$

Expanding to second order in the ε_i (we need second order because we develop the energy, which has quadratic terms in the stretch) we have

$$\begin{aligned} (1+\varepsilon_x)^2 &= 1 + 2\varepsilon_x + \varepsilon_x^2 \qquad , \qquad (1+\varepsilon_y)^2 = 1 + 2\varepsilon_y + \varepsilon_y^2 \ , \\ \frac{1}{(1+\varepsilon_x+\varepsilon_y+\varepsilon_x\varepsilon_y)^2} &= 1 - 2(\varepsilon_x+\varepsilon_y) + 3(\varepsilon_x^2+\varepsilon_y^2) + 4\varepsilon_x\varepsilon_y + \mathcal{O}(\varepsilon^3) \ , \end{aligned} \tag{20}$$

all in all we get

$$u = \mu\left(\varepsilon_x^2 + \varepsilon_y^2 + (\varepsilon_x+\varepsilon_y)^2\right) = \mu \operatorname{tr} \boldsymbol{\varepsilon}^2 + \mu \operatorname{tr}^2 \boldsymbol{\varepsilon} \ . \tag{21}$$

So we see that the linear form of the energy functional is the familiar and expected form $u = \frac{1}{2}(2\tilde{\mu} \operatorname{tr} \boldsymbol{\varepsilon}^2 + \tilde{\lambda} \operatorname{tr}(\boldsymbol{\varepsilon})^2)$. This also means that our material has $\tilde{\lambda} = 2\tilde{\mu}$ which implies

$$\tilde{\nu} = \frac{\tilde{\lambda}}{2(\tilde{\lambda}+\tilde{\mu})} = \frac{1}{3} \ . \tag{22}$$

This value of ν should come as a surprise because we started with an incompressible material, so we should expect to have $\nu = \frac{1}{2}$. What went wrong? Keep in mind that the energy functional (14) is the result of the reduction of a set of 3D equations to 2D. We have done this in detail in the linear case, and we all remember well that the elastic constants are not the same as the 3D ones, but renormalized ones (see Eq. (5.62) in the lecture notes). The relation between the renormalized elastic constants to the real ones is

$$\tilde{\mu} = \mu, \qquad \tilde{\lambda} = \frac{2\nu\mu}{1-\nu} \ , \tag{23}$$

rearranging the latter, we get

$$\nu = \frac{\tilde{\lambda}}{\tilde{\lambda} + 2\tilde{\mu}} \ . \tag{24}$$

Plugging in our result $\tilde{\lambda} = 2\tilde{\mu}$ gives

$$\nu = \frac{2\tilde{\mu}}{2\tilde{\mu} + 2\tilde{\mu}} = \frac{1}{2} \ . \tag{25}$$

What a relief. The real Poisson ratio is 1/2, which means that the material is indeed incompressible. The fact that the "apparent" 2D Poisson's ration is different that 1/2 means that in-plane compressibility is allowed. This is because the material expands in the third direction, which is unaccounted for in the 2D description.

2.3 The equations of motion in our system

We remind ourselves that in the material coordinates the equations of motion read (see the related subsection below)

$$\rho_0 \dot{\boldsymbol{V}} = \nabla_{\boldsymbol{X}} \cdot \boldsymbol{P} \ . \tag{26}$$

Plugging in our expression for $\boldsymbol{P}$, Eq. (18), we get

$$\begin{aligned}\frac{\rho_0}{\mu}\ddot{\varphi}_x &= \nabla^2\varphi_x - \frac{\partial\varphi_y}{\partial Y}\frac{\partial J^{-3}}{\partial X} + \frac{\partial\varphi_y}{\partial X}\frac{\partial J^{-3}}{\partial Y}\ ,\\ \frac{\rho_0}{\mu}\ddot{\varphi}_y &= \nabla^2\varphi_y - \frac{\partial\varphi_x}{\partial X}\frac{\partial J^{-3}}{\partial Y} + \frac{\partial\varphi_x}{\partial Y}\frac{\partial J^{-3}}{\partial X}\ .\end{aligned} \tag{27}$$

2.4 Small-on-Large waves

Consider then a homogeneously deformed body with principal stretches λ_x and λ_y. On this stretched state we superimpose a small displacement $\boldsymbol{\Delta}(\boldsymbol{X},t)$. The deformation $\boldsymbol{\varphi}(\boldsymbol{X},t)$ is thus

$$\begin{aligned}\varphi_X(\boldsymbol{X},t) &= \lambda_x X + \Delta_X(\boldsymbol{X},t)\ ,\\ \varphi_Y(\boldsymbol{X},t) &= \lambda_y Y + \Delta_Y(\boldsymbol{X},t)\ .\end{aligned} \tag{28}$$

Note that the homogeneous solution $\boldsymbol{\Delta}=0$ satisfies the equations of motion. The motion gradient reads

$$\boldsymbol{F} = \begin{pmatrix}\lambda_x + \partial_X\Delta_X & \partial_Y\Delta_X\\ \partial_X\Delta_Y & \lambda_y + \partial_Y\Delta_Y\end{pmatrix}\ . \tag{29}$$

We want to look at small perturbations on the stretched state, that is, we want to expand the equations of motion to first order in $\boldsymbol{\Delta}(\boldsymbol{X},t)$. First, we calculate

$$\det\boldsymbol{F} \simeq (\lambda_x + \partial_X\Delta_X)(\lambda_y + \partial_Y\Delta_Y) \approx \lambda_x\lambda_y\left(1 + \frac{\partial_X\Delta_X}{\lambda_x} + \frac{\partial_Y\Delta_Y}{\lambda_y}\right) + \mathcal{O}\left(\boldsymbol{\Delta}^2\right) \tag{30}$$

$$(\det\boldsymbol{F})^{-3} \simeq \frac{1}{\lambda_x^3\lambda_y^3}\left(1 - 3\frac{\partial_X\Delta_X}{\lambda_x} - 3\frac{\partial_Y\Delta_Y}{\lambda_y}\right) + \mathcal{O}\left(\boldsymbol{\Delta}^2\right)\ . \tag{31}$$

The equations of motions are thus, to linear order,

$$\begin{aligned}\nabla^2\Delta_X + 3\frac{\partial_{XX}\Delta_X}{\lambda_x^4\lambda_y^2} + 3\frac{\partial_{XY}\Delta_Y}{\lambda_x^3\lambda_y^3} &= \frac{\rho}{\mu}\ddot{\Delta}_X = c_s^{-2}\ddot{\Delta}_X\ ,\\ \nabla^2\Delta_Y + 3\frac{\partial_{YY}\Delta_Y}{\lambda_x^2\lambda_y^4} + 3\frac{\partial_{XY}\Delta_X}{\lambda_x^3\lambda_y^3} &= \frac{\rho}{\mu}\ddot{\Delta}_Y = c_s^{-2}\ddot{\Delta}_Y\ ,\end{aligned} \tag{32}$$

where $c_s \equiv \sqrt{\frac{\mu}{\rho}}$. Assume then a solution in the form of plane waves

$$\begin{aligned}\Delta_X(\boldsymbol{X},t) &= a_X e^{iK(\boldsymbol{N}\cdot\boldsymbol{X}-ct)}\ ,\\ \Delta_Y(\boldsymbol{X},t) &= a_Y e^{iK(\boldsymbol{N}\cdot\boldsymbol{X}-ct)}\ ,\end{aligned} \tag{33}$$

where $\boldsymbol{N} = (\cos\theta, \sin\theta)$ is the direction of propagation in the undeformed coordinates, K is the wavenumber in the undeformed coordinates, and c is the (yet unknown) speed. What kind of waves are there in the system? what is (are) the wavespeed(s)?

Plugging in the ansatz (33) into the equations of motion (32) we get

$$\begin{aligned} a_X + 3\frac{\cos^2\theta}{\lambda_x^4\lambda_y^2}a_X + 3\frac{\sin\theta\cos\theta}{\lambda_x^3\lambda_y^3}a_Y - \frac{c^2}{c_s^2}a_X = 0 \ , \\ a_Y + 3\frac{\sin^2\theta}{\lambda_x^2\lambda_y^4}a_Y + 3\frac{\sin\theta\cos\theta}{\lambda_x^3\lambda_y^3}a_X - \frac{c^2}{c_s^2}a_Y = 0 \ , \end{aligned} \tag{34}$$

which is more concisely written as

$$\underbrace{\begin{pmatrix} 1 + \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2} - \frac{c^2}{c_s^2} & \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} \\ \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} & 1 + \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} - \frac{c^2}{c_s^2} \end{pmatrix}}_{\equiv \boldsymbol{M}} \begin{pmatrix} a_X \\ a_Y \end{pmatrix} = 0 \ . \tag{35}$$

Similarly to what we've done with Rayleigh waves, solutions are obtained when the determinant vanishes. This condition reads

$$\det \boldsymbol{M} = \left(1 - \frac{c^2}{c_s^2}\right)\left(1 + \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} + \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2} - \frac{c^2}{c_s^2}\right) = 0 \ , \tag{36}$$

so you immediately see that there are two families of solutions,

$$c = \pm c_s \ , \qquad \text{and} \qquad c = \pm c_s\sqrt{1 + \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} + \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2}} \ . \tag{37}$$

The first family are shear-like waves and their velocity is independent on direction. In order to see that they are shear waves, note that the amplitudes a_X, a_Y can be obtained, up to a multiplicative factor, by the kernel of the matrix $\boldsymbol{M}(c = c_s)$, which is

$$(a_X, a_Y) \in \ker \begin{pmatrix} \frac{3\cos^2(\theta)}{\lambda_x^4\lambda_y^2} & \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} \\ \frac{3\cos(\theta)\sin(\theta)}{\lambda_x^3\lambda_y^3} & \frac{3\sin^2(\theta)}{\lambda_x^2\lambda_y^4} \end{pmatrix} \propto (\lambda_x \sin\theta, -\lambda_y\cos\theta) \ . \tag{38}$$

These waves are "almost transverse" because $(a_X, a_Y) \cdot \boldsymbol{N} \propto (\lambda_x - \lambda_y)\sin(2\theta)$. Therefore, they are purely transverse for $\theta = 0, \frac{\pi}{2}$ (i.e. in the X or Y directions) or when $\lambda_x = \lambda_y$. Note that this also means that the shape of the waves will depend on the direction of propagation.

The other family of solutions has a direction-dependent velocity, which is an interesting situation which is not uncommon of anisotropic systems. Following the same logic as above, the amplitudes of these waves is, up to a multiplicative factor

$$(a_X, a_Y) \propto (\lambda_y\cos\theta, \lambda_x\sin\theta) \ , \tag{39}$$

such that $(a_X, a_Y) \times \boldsymbol{N} \propto (\lambda_x - \lambda_y)\sin(2\theta)$ and again these waves are purely longitudinal for waves propagating in the X or Y direction, or for $\lambda_x = \lambda_y$.

2.5 Example

Consider a uniaxial pre-stress (applied λ_y), for which we have (due to incompressibility)

$$\lambda_x = \lambda_y^{-1/2} \; . \tag{40}$$

With this setup, the longitudinal wavespeed will be

$$c = \pm c_s \sqrt{1 + 3\left(1 + \left(\frac{1}{\lambda_y^3} - 1\right)\sin^2\theta\right)} \; . \tag{41}$$

This function is plotted in Fig. 1.

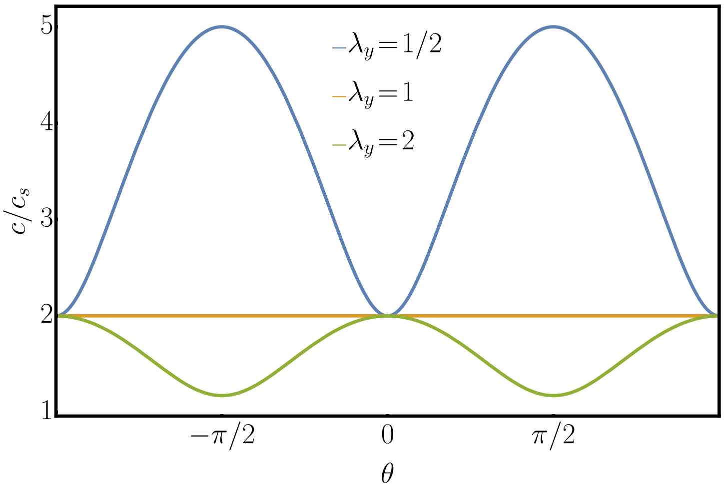

Figure 1: The longitudinal wavespeed (in units of c_s) as a function of the propagation direction θ for three values of λ_y.

Note that if you'd go to the lab and measure the wave speeds, you'll find different results, because these wave speeds are given in the *material coordinates*, and not in the deformed (lab) coordinates. Also, the absence of anisotropy in the shear wave-speed is a special case specific to this constitutive law and not a general feature of finite elasticity.

3 Divergence in spherical coordinates

In many problems, the geometry of the problem clearly suggests that spherical coordinates should be used. So let's take the opportunity to discuss how to derive the equation of motion $\nabla \cdot \boldsymbol{\sigma} = \rho \ddot{\boldsymbol{u}}$ in curvilinear coordinates. What do we mean when we write a tensor $\boldsymbol{A}$ in Cartesian coordinates as

$$\boldsymbol{A} = \begin{pmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{pmatrix} ? \tag{42}$$

This is a shorthand notation for $\boldsymbol{A} = \sum_{ij} A_{ij} \boldsymbol{e}_i \otimes \boldsymbol{e}_j$ with $i, j \in \{x, y, z\}$ and $\boldsymbol{e}_i$ is the unit vector in the i direction. Writing the tensor in, say, spherical coordinates, means to write it in terms of the unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\phi$ which are *space-dependent*. Since spherical coordinates are orthonormal, we know that *locally* the transformation from Cartesian to spherical coordinates is given by a rotation,

$$[\boldsymbol{A}]_{r,\phi,\theta} = \boldsymbol{R}(\phi, \theta)^T [\boldsymbol{A}]_{x,y,z} \boldsymbol{R}(\phi, \theta) \ , \tag{43}$$

but you have to remember that the rotation matrix $\boldsymbol{R}$ is different in different points in space.

When you calculate derivative of the tensor in curvilinear coordinates you need to keep track of the fact that not only the components of the tensor change in space, but also the unit vectors themselves change. This amounts to differentiating Eq. (43) and remembering to differentiate both copies of $\boldsymbol{R}(\phi, \theta)$, because ϕ and θ are space-dependent. Doing this properly is a long and technical calculation which we will not do here, but you should be able in principle to do it, and you should definitely understand it's algebraic structure. The bottom line is that the divergence of a tensor in spherical coordinates is

$$\begin{aligned} \nabla \cdot \boldsymbol{A} = & \\ & \left[\frac{\partial A_{rr}}{\partial r} + 2\frac{A_{rr}}{r} + \frac{1}{r}\frac{\partial A_{\theta r}}{\partial \theta} + \frac{\cot\theta}{r} A_{\theta r} + \frac{1}{r\sin\theta}\frac{\partial A_{\phi r}}{\partial \phi} - \frac{1}{r}(A_{\theta\theta} + A_{\phi\phi}) \right] \boldsymbol{e}_r \\ & + \left[\frac{\partial A_{r\theta}}{\partial r} + 2\frac{A_{r\theta}}{r} + \frac{1}{r}\frac{\partial A_{\theta\theta}}{\partial \theta} + \frac{\cot\theta}{r} A_{\theta\theta} + \frac{1}{r\sin\theta}\frac{\partial A_{\phi\theta}}{\partial \phi} + \frac{A_{\theta r}}{r} - \frac{\cot\theta}{r} A_{\phi\phi} \right] \boldsymbol{e}_\theta \\ & + \left[\frac{\partial A_{r\phi}}{\partial r} + 2\frac{A_{r\phi}}{r} + \frac{\sin\theta}{r}\frac{\partial A_{\theta\phi}}{\partial \theta} + \frac{\cos\theta}{r} A_{\theta\phi} + \frac{1}{r\sin\theta}\frac{\partial A_{\phi\phi}}{\partial \phi} + \frac{1}{r}(A_{\phi r} + A_{\phi\theta}) \right] \boldsymbol{e}_\phi \ . \end{aligned} \tag{44}$$

You can find similar expressions for other differential operators (Laplacian, gradient, material derivative, etc.) for both spherical and cylindrical coordinates on this page in Wikipedia.