Strong turbulence and self-focusing of parametrically excited spin waves

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An analysis is made of the nonlinear behavior of a pair of spin waves under parametric excitation conditions. It is shown that the amplitude and phase of such a pair are randomly modulated in the transverse direction. The average level of turbulence, the characteristic scale of modulation (which is much longer than the wavelength), and the characteristic frequency of motion are determined. Deep transverse modulations of the pair are unstable in the case of formation of collapsible filaments in which the amplitude is very high and is limited by nonlinear damping. Under these conditions the energy is dissipated very rapidly. The situation is then similar to the formation of foci in the self-focusing of light in dielectrics and it provides an additional mechanism which limits the amplitude of the parametrically excited spin waves. It is shown that practically every amplitude maximum is involved in the collapse process if the pumping amplitude h exceeds a certain critical value h_c. The properties of such turbulence are analyzed and the possibilities of its experimental observation are considered.

In parametric excitation of spin waves in ferromagnets[1] the spin-wave amplitude is frequently such that the behavior of the system as a whole is governed by the interaction between spin waves. The physical phenomena which then arise are of great variety and this is why the parametric excitation of spin waves has attracted so much attention. The variety of these phenomena is responsible for the difficulties encountered in studies of such excitation. Several workers[2-4] have considered a situation in which the parametric instability threshold is minimal for pairs of waves with vectors ±k which occupy a line or fill a surface in the k space. The sum of the phases in a pair is a dynamic variable and the separate phases of the waves are quite random. This reduces the interaction Hamiltonian to the form diagonal in respect of pairs of waves, i.e., the problem simplifies considerably.[3] The effect associated with the correlation of the individual phases can be allowed for using the perturbation theory.[5]

The present paper is concerned with the situation in which the excitation threshold is minimal for a single pair $\pm k_0$, for example, in the case of perpendicular pumping of spin waves in cubic ferromagnets in the presence of a second-order instability ($\omega_k \simeq \omega_p$) or in the case of parallel pumping of uniaxial ferromagnets with the "easy plane" anisotropy. The principal feature of the problem is the narrowness of the wave packets excited in the k space. This means that we cannot use a statistical description, as in ref. 3, but we can simplify the exact interaction Hamiltonian using the narrowness of the packets. This can be done by formulating the problem in terms of envelope waves, as is done in Sec. 1. The initial stage of the development of a parametric instability produces a monochromatic standing wave $\pm k_0$, which is one of the steady-state solutions of Eq. (4) for the envelope waves. This state is practically always unstable in the presence of wave interactions of the type

$$2\omega_{k_0}=\omega_{k_0+\varkappa}+\omega_{k_0-\varkappa},\quad \omega_{k_0}+\omega_{-k_0}=\omega_{k_0+\varkappa}+\omega_{-k_0-\varkappa}.$$

In several important case the instability increment is positive only for the directions which are almost perpendicular to[6] k_0 and it can be shown[7] that the phases and amplitudes of the envelope waves are equal even in the nonlinear stage. We shall use this to simplify the problem further in Sec. 2 and to obtain two-dimensional equations (6) for the amplitude A and the phase Φ of the envelope of a pair.

The simplest variant of the nonlinear behavior of such a system, which is its transition to a steady state other than a plane standing wave A = const and Φ = const, is considered in ref. 7. States of this kind, called "domains" in ref. 7, are periodic waves of the modulation of the amplitude and phase of a pair $A(r - Vt)$ and $\Phi(r - Vt)$, which are either at rest or move at a constant velocity V. There is every reason to assume[7] that these domains are unstable and have an increment which is usually larger than that of a plane wave. Therefore, the nonlinear behavior of the system is essentially unstable. The average characteristics of a turbulent state of this kind are discussed in Sec. 2. It is found that the state can be represented by a stochastic modulation of the amplitude and phase of a pair of spin waves $\pm k_0$ and such modulation occurs mainly at right angles to k_0. The depth of modulation is of the order of unity and its characteristic length is much greater than the wavelength. The modulation pattern changes appreciably in a time interval of the order of the reciprocal of the parametric instability increment.

An interesting and important phenomenon, which is the collapse of the envelopes of a standing spin wave, occurs against the background of such turbulence. It is found that deep modulations do not disperse, but contract rapidly in such a way that the amplitude at the center increases strongly and is limited by the nonlinear damping to a level considerably higher than the average level of turbulence. The results of a computer study of this phenomenon are presented in Secs. 3 and 4.[1)] The conditions for the collapse are determined, the dependence of the probability of collapse on the pumping amplitude is obtained, and the dissipation of energy in the collapse process is computed. The possibility of experimental observation of strong turbulence and collapse of spin waves is considered.

1. EQUATION FOR SLOWLY VARYING AMPLITUDES

If we use the canonical equation of motion (I.5) and the Hamiltonian (I.1)[2)] and assume that narrow wave packets

$$a_k=[A\,(k-k_0)+B\,(k+k_0)]$$

are excited in the k space, we find – in the usual manner[8] – the equation for the envelopes A(r) and B(r) of the Fourier components $A(\varkappa)$ and $B(\varkappa)$:

$$\left.\begin{aligned}&\left[i\left(\frac{\partial}{\partial t}+\mathbf{V}\nabla\right)+\frac{1}{2}\hat{L}\right]A=-i\gamma A+hVB^{*}\\&+\left[\omega_{k_0}-\frac{\omega_p}{2}+T\,|A|^2+2S\,|B|^2\right]A,\\&\left[i\left(\frac{\partial}{\partial t}-\mathbf{V}\nabla\right)+\frac{1}{2}\hat{L}\right]B=-i\gamma B+hVA^{*}\\&+\left[\omega_{k_0}-\frac{\omega_p}{2}+T\,|B|^2+2S\,|A|^2\right]B,\end{aligned}\right\}\qquad(1)$$

where

$$\mathbf{V}=\frac{\partial\omega}{\partial\mathbf{k}},\quad L=\sum_{\alpha\beta}\frac{\partial^2\omega}{\partial k_\alpha\partial k_\beta}\frac{\partial^2}{\partial x_\alpha\partial x_\beta},\quad T=T_{\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0\mathbf{k}_0},\quad S=T_{\mathbf{k}_0,\,-\mathbf{k}_0,\,\mathbf{k}_0,\,-\mathbf{k}_0}$$

[these are the coefficients of the wave interaction Hamiltonian (I.1)].

The system (1) has the trivial solution A(r) = B(r) = A_0, where

$$\left.\begin{aligned}&A(r)=|A|\,e^{-i\psi_1},\quad B(r)=|B|\,e^{-i\psi_2},\\&2SA_0^2=\sqrt{(hV)^2-\gamma^2},\quad \sin(\psi_1+\psi_2)=\frac{\gamma}{hV},\quad \psi_1=\psi_2=\frac{\Phi}{2},\end{aligned}\right\}\qquad(2)$$

and this solution corresponds to the excitation of a standing spin wave $\pm k_0$. It is shown in I that the solution (2) is practically always unstable in the case of growth of the modulation of the amplitudes and phases of the envelope waves:

$$\delta A(r,t)\propto\exp(i\varkappa\mathbf{r}+\nu(\varkappa)t),$$
$$\delta B(r,t)\propto\exp(i\varkappa\mathbf{r}+\nu(\varkappa)t).$$

The nature of the development of an instability depends strongly on the parameters of the Hamiltonian (I.1). With the exception of the case T > 0 and S > 0, which we shall not discuss further, the instability increment is maximal on the surface $\varkappa\perp k_0$:

$$[\gamma+\nu(\varkappa)]^2-\gamma^2=\begin{cases}-\frac{1}{4}L\varkappa^2(L\varkappa^2+4TA_0^2),\\-8S(2S+T)A_0^4\\-\frac{1}{4}L\varkappa^2(L\varkappa^2+4(4S+T)A_0^2),\end{cases}\qquad(3)$$

and it decreases rapidly away from this surface. Consequently, important properties of the nonlinear stage of the development of an instability can be determined using two-dimensional equations, in which A and B depend only on the coordinates x and y, which are orthogonal to **V**. The first and second rows in Eq. (3) correspond to perturbations of the type $\delta A(r,t)=\pm\delta B(r,t)$. If T > 0 and S < 0, we see that the perturbation $\delta A=-\delta B$ has a margin of stability and, as shown in ref. 7, in this case we have A(r, t) = B(r, t) even in the nonlinear stage of motion. For simplicity we shall consider the case $\omega''>0$ because in this case a change in scale can be used to transform Eq. (3) to

$$i\left(\frac{\partial}{\partial t}+\gamma\right)A-hVA^{*}+\frac{1}{2}\Delta A=[(2S+T)\,|A|^2-TA_0^2]\,A.\qquad(4)$$

Here, we define $\omega(k_0)$ by the external instability condition[3]

$$\omega(k_0)-\omega_p/2+TA_0^2=0.$$

In this way we select the most stable standing wave for which, as shown in ref. 6, the positive-increment region in the k space is limited ($\varkappa\ll k_0$).

As pointed out at the beginning of this paper, the system (4) has many steady-state periodic solutions of the A(r – Vt) type, but they are unstable.[7] The initial nonlinear stage of the development of this instability is also considered in ref. 7 with special attention to the case in which the increment is anomalously small. This case is of interest because the nonlinear interaction begins to play an important role at very low amplitudes. It is found that this does not restrict the growth of the initial perturbation but simply slows it down greatly. The amplitude then rises as $\sqrt{t}$. In this way a packet of waves with $\varkappa\sim\varkappa_0$, $\varkappa_0\ll k_0$, and $\Delta\varkappa\sim\varkappa_0$ is formed in the k space. This state is strongly turbulent and we shall consider it in the next section.

2. AVERAGE CHARACTERISTICS OF STRONG TURBULENCE

We shall estimate first the width of the excited region in the k space for an arbitrary pumping level. It is shown in refs. 3 and 4 that in the case when V_k is maximal at the point k_0, a packet of parametrically excited waves relaxes to a standing monochromatic wave with $k=k_0$ if the individual phases of the waves can be regarded as random. The phases are random if two waves in a packet become different in phase by an amount of the order of unity in a time shorter than the characteristic nonlinear interaction time. This occurs in a packet with $(\Delta k)^2\gg\varkappa_0^2=SA_0^2/\omega''_k$. Therefore, a packet with $\Delta k\gg\varkappa_0$ reduces in width to $\sim\varkappa_0$ and its average amplitude relaxes to $\sim A_0$. However, at $\Delta k\ll\varkappa_0$ such a packet is unstable and is characterized by the increment (3) in the presence of perturbation with $\varkappa\sim\varkappa_0$ and, consequently, it expands to $\Delta k\sim\varkappa_0\sim(SA_0^2/\omega'')^{1/2}$. We must stress that throughout the turbulent motion region, the instability increment (3) is positive in a narrow range $\varkappa_\parallel V\leqslant SA_0^2$ near the plane $\varkappa\perp\mathbf{V}$, i.e., the turbulence we are discussing is almost two-dimensional:

$$\left(\frac{\varkappa_\parallel}{\varkappa_\perp}\right)^2\sim\frac{SA_0^2}{\omega_k}\ll1.$$

The average turbulence level A^2 cannot differ greatly from A_0^2 defined by Eq. (2). In fact, as mentioned in Sec. 1, it is shown in I that a monochromatic plane wave is stable under the action of perturbations of relatively short wavelengths $\varkappa\sim k_0$ if the wave amplitude is A_0. Such an instability is also exhibited by a pair modulated with $\varkappa\ll k_0$ because if A_0^2 differs considerably from A^2, short-wavelength modulations are excited and this is in conflict with the foregoing conclusion about the narrowness of the packet.

Thus, the development of a plane-wave instability gives rise to a strong quasi-two-dimensional turbulence of the modulation waves A(r, t) with an average level

$$\overline{A^2}\simeq\overline{A_0^2}=\frac{(hV^2-\gamma^2)^{1/2}}{2\,|S|},$$

a depth of modulation of the order of unity, a characteristic frequency $hV-\gamma$, and a characteristic scale in the coordinate space $r_\perp \sim \varkappa_0^{-1} \sim k_0^{-1}(\omega_k/SA_0^2)^{1/2}$. In the wave approach we can say that a dynamic domain structure with a coherence length of the order of the domain size, i.e., $A_0^{-1}(\omega''/S)^{1/2}$, is generated and this length varies considerably in space in a time $(hV - \gamma)^{-1}$.

3. COLLAPSE OF LARGE-AMPLITUDE WAVES

In those parts of a domain structure where the amplitude A during turbulent motion is anomalously large, $A \gg A_0$, we can ignore the damping and the pumping in the equations of motion because the system cannot exchange a significant amount of energy with the thermostat and the pumping source in the characteristic time of the problem, which is $(SA^2)^{-1}$.

In the $\gamma = 0$, $h = 0$ approximation the system of equations (1) describes the transient behavior of a pair of waves in a conservative medium. The behavior of a single almost monochromatic wave in a nonlinear conservative medium has been investigated very intensively by experimental, theoretical, and computer approaches in view of the importance of this behavior in nonlinear optics,[9] plasma physics, and fluid dynamics.[10] The phenomenon of self-focusing of light has been established[11] and it has then been shown that a self-focused light beam is unstable[12] and in some cases this instability results in a collapse of the beam in a finite time.[13,14]

We shall now show that similar phenomena occur also in the case of a pair of waves which we are considering. Direct calculations demonstrate that the system (1), subject to the condition $\gamma = h = 0$, has the following integrals of motion:

The total energy of the system is

$$H = \omega_{k_0}(N_A + N_B) + \mathbf{PV} + I;$$

the "number" of waves of each kind is given by

$$N_A = \int |A|^2 d\mathbf{r}, \quad N_B = \int |B|^2 d\mathbf{r};$$

the total momentum of the system is

$$\mathbf{P} = \frac{i}{2}\int (A^*\nabla A - B^*\nabla B - \text{c. c.})\, d\mathbf{r};$$

and the integral I which occurs in the above equations is

$$I = \frac{\omega''}{2}\int (|\nabla A|^2 + |\nabla B|^2)\, d\mathbf{r} + \frac{T}{2}\int (|A|^4 + |B|^4)\, d\mathbf{r} + 2S\int |A|^2 |B|^2 d\mathbf{r}. \tag{5}$$

We shall show that the sign of this integral has a strong influence on the evolution of the system. We shall do this by considering the second derivative of an essentially positive quantity R,

$$R = \frac{1}{2\omega''}\int r_\perp^2 (|A|^2 + |B|^2)\, d\mathbf{r} > 0. \tag{6}$$

Direct calculations based on Eq. (1), in which the condition $(SA_0^2/\omega)^{1/2} \ll 1$ allows us to ignore the second derivatives with respect to z, demonstrate (see ref. 14) that $d^2R/dt^2 = 2I$ and hence

$$R(t) = It^2 + 2\alpha t + \beta, \tag{7}$$

where α and β are the constants of integration. We can see that if $I < 0$, the quantity R(t) become negative in a finite time, but this contradicts Eq. (6). Therefore, it follows that the solution of the system (1) "collapses" in a finite time, i.e., a singularity is observed.

We shall consider this effect in greater detail. We shall employ the two-dimensional equation (4) – where $A(x, y) = B(x, y)$, as shown in Sec. 2 – and analyze, on the basis of computer calculations, the evolution of the initial axially symmetric distribution:

$$A(r, 0) = A_0\left[1 + k(a - r)\exp\left(-\frac{2r^2}{\sqrt{\pi}a^2}\right)\right], \tag{8}$$

which simulates a local increase in the amplitude arising spontaneously in the course of turbulent motion. Equation (4) can be integrated numerically using a method described in ref. 15. The distribution (8) corresponds to an average turbulence level A_0, defined by

$$A(r, 0) \to A_0 \text{ when } r \to \infty, \quad \int_0^\infty [A(r, 0) - A_0]\, r dr = 0.$$

A natural choice for the characteristic length a_0, in which $A(r, 0) - A_0$ vanishes, is the quarter of the wavelength of the envelope wave, corresponding to the maximum instability increment of the original pair:

$$a^2 = \frac{\pi^2 S^2 \omega''}{(4S + T)^2 (hV^2 - \gamma^2)^{1/2}}.$$

Figures 1-3 give the results of a numerical experiment: the evolution of the amplitude $|A(0, t)|$ (Fig. 1) and of the phase (Fig. 2) at the center of a packet, and the amplitude distribution $|A(r, t)|$ at certain characteristic times (Fig. 3). We can see that there is a critical modula-

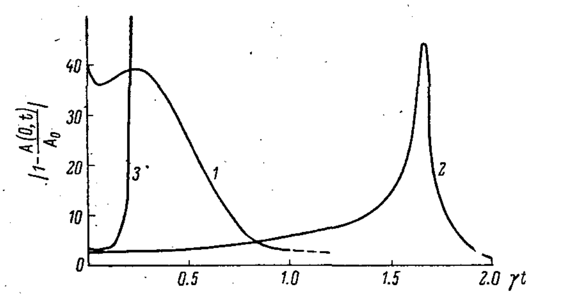

Fig. 1. Evolution of the amplitude A(0, t) at the center of a packet for $hV - 2\gamma$, $T = -S > 0$. Curves 1, 2, and 3 correspond to k = 2, 3, and 3.5, respectively. For the sake of clarity the vertical scale for curve 3 is increased by a factor of 20.

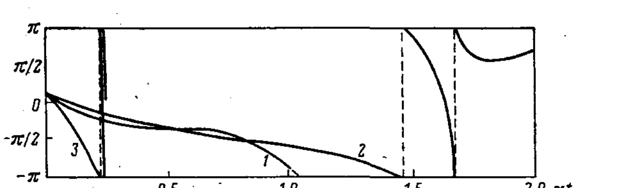

Fig. 2. Evolution of the phase $\psi(0, t)$ at the center of a packet for $hV - 2\gamma$, $T = -S > 0$. Curves 1-3 have the same meaning as in Fig. 1.

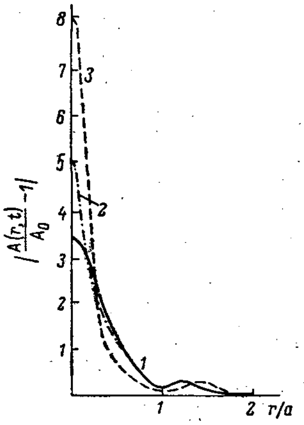

Fig. 3. Distribution $A(r, t)$, for $hV - 2\gamma$, $T = -S > 0$, $k = 3.5$. Curves 1, 2, and 3 correspond to $\gamma t = 0.05$, 0.125, and 0.2, respectively; $a = 0.8\sqrt{\omega''/\gamma}$.

tion depth k_c such that the packet collapses in a finite time if $k > k_c$ and this collapse is accompanied by a rise of the amplitude at the center of the packet to infinity. The critical values of k are listed in Table 1.

We can see that if $hV - \gamma \ll \gamma$ the critical amplitude is $k_c \gg 1$. In order to understand this point we note that the phase $\psi(r, t)$ near the focus $r = 0$ of a collapsing packet increases monotonically. This has been shown analytically in ref. 13 and it is quite clear from Fig. 2 in the range where $\gamma t > 0.2$. The rotation of the phase of the packet relative to the phase of the pumping source suppresses the energy flux in the vicinity of the focus and, therefore, the collapse occurs only if SA^2 is at least of the order of γ, which exceeds considerably the average turbulence level $A_0^2 \propto (hV^2 - \gamma^2)^{1/2}$ in the range $hV - \gamma \ll \gamma$. This result ($k_c \gg 1$ for $kV - \gamma \ll \gamma$) depends weakly on the phase of the initial distribution. For example, if we vary the phase of the initial distribution, i.e., if we assume that k in Eq. (8) is complex, we find that $|k_c|$ varies from $k_c = 22$ to $k_c = 25$ for $hV - \gamma = 0.001\gamma$.

4. SELF-FOCUSING AMPLITUDE LIMITATION MECHANISM

As pointed out earlier, the depth of modulation of the amplitude in turbulent motion is of the order of unity and, therefore, the probability of formation of regions with amplitudes exceeding greatly A_0 is exponentially small. This means (Table 1) that for a small excess ($hV - \gamma \lesssim \gamma$) hardly any collapsing regions are formed. It is clear from Fig. 1 that the modulations of amplitude $1 < k < k_c$ gradually disperse. If h is increased, the critical depth of modulation k_c decreases and there is a characteristic amplitude h_c for which $k_c \sim 1$. We note that the initial distribution (8) is, to a great extent, quite arbitrary and, therefore, we shall understand the amplitude h_c to be that value of the pumping amplitude above which practically any region with a characteristic dimension $1/\varkappa_0$ is captured in the course of collapse. It is evident from Fig. 4

TABLE 1. Dependence of Critical Initial Amplitude k_c on Excess over Threshold: $k_1 < k_c < k_2$ (for $k = k_1$ the packet still spreads, whereas for $k = k_2$ it collapses)

$hV/\gamma - 1$	k_1	k_2	$hV/\gamma - 1$	k_1	k_2
10^{-3}	23	24	1	3	3.1
10^{-2}	10.5	11	2	2.6	2.7
10^{-1}	5.4	5.5	3	—	0.5
0.4	3.4	3.5			

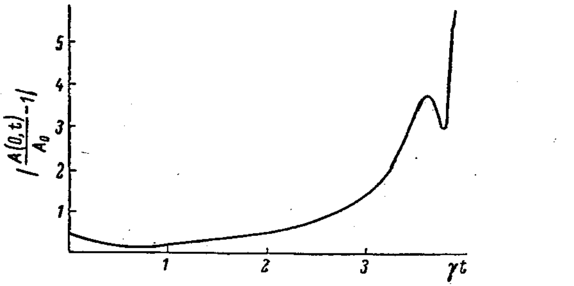

Fig. 4. Evolution of $A(0, t)$ at the center of a packet for $hV - 4\gamma$, $T = -S$.

($hV = 4\gamma$) that for a packet which initially has a small value of $k < 1$, the parametric instability mechanism increases the amplitude in the central region (characteristic increment of the order of hV) to values exceeding unity and this is followed by a rapid collapse. It means that if $h > h_c$, the nonlinearity not only does not stop the development of the instability but, quite the opposite, accelerates the growth and collapse of a packet.

We shall now consider the phenomena which occur if $h > h_c$. Obviously, in a study of the evolution of a collapsing packet we can ignore the influence of pumping and damping. Equation (4) then changes to a nonlinear parabolic expression, whose solutions have been studied in great detail in connection with the self-focusing of light. As shown in ref. 12, the amplitude at the center of a collapsing packet rises rapidly with time: $A(0, t) \propto (t - t_0)^{2/3}$. The radius of a collapsing filament decreases, so that a definite amount of energy is captured:

$$I_c = \omega \int |A|^2 d\mathbf{r} = \frac{1.86\omega''\omega}{|2S + T|}.$$

If the wave amplitude in a collapsing packet reaches a sufficiently high value, the nonlinear damping becomes important and this damping rapidly dissipates an energy $\sim I_c$ in the collapsing region. The effective nonlinear damping can be estimated from

$$\gamma_{nl} \propto \frac{I_c \varkappa_0^2}{\tau\omega A_0^2} \sim \frac{1}{\tau},$$

where τ is the time between two consecutive collapse events in a region of dimension $\sim \varkappa_0^{-2}$. In dimensional estimates we obtain $\tau \sim (hV)^{-1}$ for $h > h_c$.

Allowance for the collapse energy-dissipation mechanism shows that the average amplitude of turbulence pulsations $\bar{A}$ becomes less than A_0 and the susceptibility $\varkappa''$ does not decrease with increasing amplitude h, but reaches a plateau which is at the same level as the maximum value of $\varkappa''$.

A promising method for experimental investigation of strong turbulence of parametrically excited spin waves is the measurement of the spectral density of the electromagnetic emission from a ferromagnet at frequencies close to the pumping frequency[16] ω_p. If $h < h_c$, the collapse events are rare and the spectral density of the noise $(h^2)_\omega$ is close to the Gaussian width $\sqrt{(\omega - \omega_p)^2} \sim \gamma\sqrt{(P/P_{th}) - 1}$. If $h > h_c$ considerable contribution to such emission is made by the collapsing parts of the sample, where the phase of a pair $\psi(r, t)$ becomes "decoupled" from the pumping phase and begins to rotate rapidly in accordance with $\psi \propto (t - t_c)^{-1/3}$ (see ref. 13). The moment of decoupling and the first few turns of the phase can be seen in Fig. 2. The rapid rotation of the phase broadens considerably the emission spectrum $(h^2)_\omega$, which can be used to study the collapse events. Using the results given in ref. 13, we can show that $(h^2)_\omega \propto (\omega - \omega_p)^{-7/4}$. The nonlinear damping, which limits the amplitude in a collapse to $A < A_{max}$, should lead to a cutoff of the noise emission at the frequency

$$SA^2_{\max} \simeq |\omega_{\max} - \omega_p|.$$

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1) We used a BÉSM-6 computer at the Computing Center of the Academy of Sciences of the USSR.

2) The references to the equations in ref. 6 will be identified by I and the notation, identical with that employed in I, will not be explained.

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