Damping of monochromatic waves in a nonlinear medium

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The dependence of the damping decrement γ_k of a monochromatic wave on its amplitude A is calculated for the three-wave processes of fusion, $\omega_k+\omega_{k'}=\omega_{k+k''}$ and disintegration, $\omega_k=\omega_{k'}=\omega_{k-k'}$. For $A\to 0$ the expressions for $\gamma_k(A)$ go over to the usual expressions that follow from the kinetic equation. For $A\neq 0$ the distribution function of waves in the vicinity of the resonance surface deviates from thermodynamic equilibrium; as a result, in fusion processes γ_k decreases with increase of A (at large A, $\gamma_k\propto 1/A$), and in disintegration processes it increases. The characteristic amplitude A_0 at which the damping has decreased by a factor 2 is estimated for sound in dielectrics and for spin waves in antiferromagnets; nonlinear ferromagnetic resonance is treated with allowance for the calculated amplitude dependence of the damping of uniform precession of the magnetization.

In calculating the decrement for the attenuation of waves that results from their interaction with each other, it is traditional to use the kinetic equation for waves (see, for example, [1-3]). In various physical situations, for example for sound in crystals [1-2] and for spin waves in magnetically ordered dielectrics [3], three-wave relaxation mechanisms are found to be the most important.

We notice that the laws of conservation of energy and momentum for these processes,

$$\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}=\omega_{\mathbf{k}+\mathbf{k}'}, \tag{1}$$

$$\omega_{\mathbf{k}}=\omega_{\mathbf{k}'}+\omega_{\mathbf{k}-\mathbf{k}'} \tag{2}$$

allow participation in the relaxation of a wave with frequency $\omega_{\mathbf{k}}$ only by waves with wave vectors $\mathbf{k}'$ and $\mathbf{k}+\mathbf{k}'$ lying in a thin layer close to the "resonance" surface (1) or (2). Therefore even at a comparatively small amplitude A of the original monochromatic wave, the energy being dissipated by it may cause the distribution function $n_{\mathbf{k}'}$ of waves in this layer to deviate appreciably from its thermodynamic equilibrium value. As a result, $n_{\mathbf{k}'}$ acquires a sharp maximum (or minimum). To describe such narrow wave packets by means of the kinetic equation, which is based on the hypothesis of almost random phases, is obviously not permissible. It is therefore necessary to develop a different method, which will explicitly take into account the possibility of strong phase correlation in each pair of waves, with wave vectors $\mathbf{k}'$ and $\mathbf{k}''=\mathbf{k}-\mathbf{k}'$ (or $\mathbf{k}''=\mathbf{k}+\mathbf{k}'$), that participate in the relaxation processes.

We have done this in Sec. 1 in the simplest and most graphic form. Namely, we have started from the classical dynamic equations of motion for the amplitudes of waves near the resonance surface, and we have described their interaction with all the remaining waves by a phenomenological method—Langevin's method of random forces.

As a result, simple formulas have been obtained that determine the dependence of the damping decrement $\gamma_{\mathbf{k}}$ of a monochromatic wave on its amplitude. For fusion processes (1),

$$\gamma_{\mathbf{k}}=\pi\sum_{\mathbf{k}''-\mathbf{k}'=\mathbf{k}}\frac{|V(\mathbf{k}\mathbf{k}'|\mathbf{k}'')|^2(n_{\mathbf{k}'}^0-n_{\mathbf{k}''}^0)\delta(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})}{[1+|V(\mathbf{k}\mathbf{k}'|\mathbf{k}'')A|^2/\gamma_{\mathbf{k}'}\gamma_{\mathbf{k}''}]^{1/2}}. \tag{3}$$

For disintegration processes (2),

$$\gamma_{\mathbf{k}}=\frac{\pi}{2}\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}}\frac{|V(\mathbf{k}'\mathbf{k}''|\mathbf{k})|^2(n_{\mathbf{k}'}^0+n_{\mathbf{k}''}^0)\delta(\omega_{\mathbf{k}'}+\omega_{\mathbf{k}''}-\omega_{\mathbf{k}})}{[1-|V(\mathbf{k}'\mathbf{k}''|\mathbf{k})A|^2/\gamma_{\mathbf{k}}\gamma_{\mathbf{k}''}]^{1/2}}. \tag{4}$$

Here $V(\mathbf{k}_1\mathbf{k}_2|\mathbf{k}_3)$ are the matrix elements of the three-wave interaction Hamiltonian (8), and $n_{\mathbf{k}}^0$ are the thermodynamic equilibrium values of the occupation numbers $n_{\mathbf{k}}$.

These formulas differ from those that follow from the kinetic equation (or, equivalently, from quantum-mechanical time-dependent perturbation theory) only by the presence of the radical in the denominator. In the case of fusion processes, the attenuation $\gamma_{\mathbf{k}}$ decreases with increase of A; for large A, $\gamma_{\mathbf{k}}\propto 1/A$. For disintegration processes, on the contrary, $\partial\gamma_{\mathbf{k}}/\partial A>0$. This difference is easy to understand: in each fusion event (1), a quantum of the wave $\omega_{\mathbf{k}'}$ disappears, and a quantum of $\omega_{\mathbf{k}''}$ is created. Therefore with increase of A, $n_{\mathbf{k}'}$ decreases, whereas $n_{\mathbf{k}''}$ increases. As a result the difference $n_{\mathbf{k}'}-n_{\mathbf{k}''}$, and with it the attenuation caused by fusion of waves, decreases. On the other hand, the attenuation in disintegration processes is proportional to the sum $n_{\mathbf{k}'}+n_{\mathbf{k}''}$, each term of which, as is easily understood, increases with increase of A. The value of wave amplitude

$$A_1=\min[(\gamma_{\mathbf{k}'}\gamma_{\mathbf{k}''})^{1/2}/|V(\mathbf{k}'\mathbf{k}''|\mathbf{k})|], \tag{5}$$

for which the radical in (4) vanishes, corresponds to the threshold of disintegration instability of a monochromatic wave. For $A>A_1$, the amplitudes of the waves $\mathbf{k}'$ and $\mathbf{k}''=\mathbf{k}-\mathbf{k}'$ increase exponentially in time, and formula (4), which was obtained in an approximation linear with respect to these amplitudes, is not valid.

It must be mentioned that formulas (3) and (4) describe asymptotic values of $\gamma_{\mathbf{k}}$. In fact, at the instant of turning on the monochromatic wave at t = 0, the waves in the medium are unable to deviate from thermodynamic equilibrium, and the attenuation will be determined by the usual formulas (without the radical). And only after a time t larger than $1/\gamma_{\mathbf{k}'}$ and $1/\gamma_{\mathbf{k}''}$ do the waves near the resonance surface relax to a new stationary state, in which $\gamma_{\mathbf{k}}$ is determined by formulas (3) and (4).

We add also that the decrement of four-wave attenuation of a monochromatic wave is independent of its amplitude and is determined by the usual formulas (see, for example, [2,3]). In fact, waves from a finite volume of $\mathbf{k}$-space participate in these processes, and therefore their amplitudes do not easily deviate from equilibrium.

In Sec. 2, examples are considered of physical situations in which the attenuation of a monochromatic wave is determined by fusion processes—sound in dielectrics at low temperatures, and uniform precession (UP) of the magnetization in antiferromagnets with anisotropy of the "easy plane" type. A characteristic amplitude A_0 of the wave is estimated, at which its damping has decreased by a factor 2.

In the last section, ferromagnetic resonance is studied at high levels of microwave power, under conditions that permit processes of disintegration (2) of UP into standing spin waves. It is shown that in pure ferromagnets, the increase of the attenuation of UP with increase of its amplitude changes the behavior of UP at resonance and can exert an appreciable influence on the operation of ferrite power limiters.

1. DAMPING DECREMENT OF A MONOCHROMATIC WAVE IN THREE-WAVE PROCESSES

We consider for definiteness the fusion process (1), and we write the dynamic equations of motion for the canonical wave amplitudes $a_{\mathbf{k}'}$ and $a_{\mathbf{k}''}$ near the resonance surface (1):

$$\left(\frac{d}{dt}+i\omega_{\mathbf{k}'}+\gamma_{\mathbf{k}'}\right)a_{\mathbf{k}'}+iV(\mathbf{k}_0\mathbf{k}'|\mathbf{k}'')\exp(i\omega_{\mathbf{k}}t)A^*a_{\mathbf{k}''}=f_{\mathbf{k}'}(t),$$
$$\left(\frac{d}{dt}+i\omega_{\mathbf{k}''}+\gamma_{\mathbf{k}''}\right)a_{\mathbf{k}''}+iV^*(\mathbf{k}_0\mathbf{k}'|\mathbf{k}'')A_k\exp(i\omega_k t)a_{\mathbf{k}'}=f_{\mathbf{k}''}(t). \qquad (6)$$

Here A is the amplitude of the monochromatic wave,

$$a_{\mathbf{k}}(t)=A\Delta(\mathbf{k}-\mathbf{k}_0)\exp(-i\omega_{\mathbf{k}}t), \qquad (7)$$

$V(\mathbf{kk}'|\mathbf{k}'')$ are the matrix elements of the interaction Hamiltonian

$$\mathscr{H}_{int}=\frac{1}{2}\sum_{\mathbf{kk}'\mathbf{k}''}(V^*(\mathbf{kk}'|\mathbf{k}'')a_{\mathbf{k}}a_{\mathbf{k}'}a^*_{\mathbf{k}''}+\text{c.c.})\Delta(\mathbf{k}+\mathbf{k}'-\mathbf{k}''), \qquad (8)$$

and $f_{\mathbf{k}}(t)$ is the Langevin random force, with correlator

$$(f^2)_{\mathbf{k}\omega}=\gamma_{\mathbf{k}}n_{\mathbf{k}}^0/\pi, \qquad (9)$$

in which $n^0_{\mathbf{k}}$ is the thermodynamic equilibrium value of $n_{\mathbf{k}}=\langle a_{\mathbf{k}}a^*_{\mathbf{k}}\rangle$.

In the writing of Eq. (6), important use has been made of the fact that the system of waves $a_{\mathbf{k}'}$ and $a_{\mathbf{k}''}$ of interest to us is concentrated in a thin layer, and that the energy contained in it is much smaller than the total energy of the remaining waves. Therefore the remaining waves (it is natural to call them a thermostat) can be considered to be in thermodynamic equilibrium, and their phases almost random. This permitted us in equations (6) effectively to describe the interaction of a separate system $a_{\mathbf{k}'}$ and $a_{\mathbf{k}''}$ with a thermostat: the terms $\gamma_{\mathbf{k}'}a_{\mathbf{k}'}$ and $\gamma_{\mathbf{k}''}a_{\mathbf{k}''}$ describe "friction of the waves against the thermostat", that is loss of energy to the thermostat. The random forces $f_{\mathbf{k}'}(t)$ and $f_{\mathbf{k}''}(t)$ simulate disorderly shocks from the thermostat, which lead on the average to an increase of the energy of the waves $a_{\mathbf{k}'}$ and $a_{\mathbf{k}''}$. As a result, for A = 0 the correlators $n_{\mathbf{k}'}$ and $n_{\mathbf{k}''}$ relax, as they should, not to zero but to the thermodynamic equilibrium values $n^0_{\mathbf{k}'}$ and $n^0_{\mathbf{k}''}$.

In order to calculate the damping decrement $\gamma_{\mathbf{k}}$ of a monochromatic wave in the process (2), we substitute (7) in the expression (8) for $\mathscr{H}_{int}$; and on using the fact that the energy flow P into the medium is equal on the one hand to $\partial\langle\mathscr{H}_{int}\rangle/\partial t$ and on the other to $2\omega_{\mathbf{k}}\gamma_{\mathbf{k}}|A|^2$, we get

$$\gamma_{\mathbf{k}}=\frac{1}{A}\operatorname{Im}\sum_{\mathbf{k}'}V(\mathbf{kk}'|\mathbf{k}+\mathbf{k}')\langle e^{-i\omega_{\mathbf{k}}t}a_{\mathbf{k}'}a^*_{\mathbf{k}+\mathbf{k}'}\rangle$$
$$=\frac{1}{A}\operatorname{Im}\sum_{\mathbf{k}'}\int d\omega' d\omega'' V(\mathbf{kk}'|\mathbf{k}+\mathbf{k}')\exp[-i(\omega_{\mathbf{k}}+\omega'-\omega'')t]\langle a_{\mathbf{k}'\omega'}a^*_{\mathbf{k}+\mathbf{k}',\omega''}\rangle. \qquad (10)$$

On expanding $a_{\mathbf{k}'}(t)$ and $a_{\mathbf{k}''}(t)$ in (6) as Fourier integrals, it is easy to express $a_{\mathbf{k}'\omega'}$ and $a_{\mathbf{k}''\omega''}$ in terms of the Fourier components of the random forces $f_{\mathbf{k}'\omega'}$ and $f_{\mathbf{k}''\omega''}$ and to substitute the result in (10). On taking into account that

$$\langle f_{\mathbf{k}\omega}f^*_{\mathbf{k}'\omega'}\rangle=(f^2)_{\mathbf{k}\omega}\Delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')$$

and using the expression (9) for the correlator of the random forces, we get

$$\gamma_{\mathbf{k}}=\frac{1}{\pi}\int d\omega'\sum_{\mathbf{k}'}\gamma_{\mathbf{k}'}\gamma_{\mathbf{k}''}(n_{\mathbf{k}'}^0-n^0_{\mathbf{k}''})|V(\mathbf{kk}'|\mathbf{k}'')|^2$$
$$\times|[i(\omega_{\mathbf{k}'}-\omega')+\gamma_{\mathbf{k}'}][i(\omega_{\mathbf{k}''}-\omega'')+\gamma_{\mathbf{k}''}]+|V(\mathbf{kk}'|\mathbf{k}'')|^2A^2|^{-2},$$

where $\mathbf{k}''\equiv\mathbf{k}+\mathbf{k}''$, $\omega''\equiv\omega_{\mathbf{k}}+\omega'$. Hence after integration over ω' we find

$$\gamma_{\mathbf{k}}=\sum_{\mathbf{k}'}|V(\mathbf{kk}'|\mathbf{k}'')|^2(n_{\mathbf{k}'}^0-n^0_{\mathbf{k}''})(\gamma_{\mathbf{k}'}+\gamma_{\mathbf{k}''})$$
$$\times\left[(\omega_{\mathbf{k}}+\omega_{\mathbf{k}'}-\omega_{\mathbf{k}''})^2+(\gamma_{\mathbf{k}'}+\gamma_{\mathbf{k}''})^2\left(1+\frac{|V(\mathbf{kk}'|\mathbf{k}'')A|^2}{\gamma_{\mathbf{k}'}\gamma_{\mathbf{k}''}}\right)\right]^{-1}. \qquad (11)$$

For A = 0 this result differs from the kinetic equation by a finite resonance width, determined by the sum of the attenuations of waves $\gamma_{\mathbf{k}'}+\gamma_{\mathbf{k}''}$. Formula (11) for A = 0 was obtained earlier by means of selective summation of diagrams for the Green temperature functions [4,5].

It is obvious that when the fusion processes are allowed by the conservation laws (1), formula (11) can be further simplified to the form (3).

In the study of the attenuation of a monochromatic wave in disintegration processes (3), we started from equations of motion analogous to (6):

$$\left(\frac{d}{dt}+i\omega_{\mathbf{k}'}+\gamma_{\mathbf{k}'}\right)a_{\mathbf{k}'}+iV(\mathbf{k}'\mathbf{k}''|\mathbf{k}_0)Ae^{-i\omega_{\mathbf{k}}t}a^*_{\mathbf{k}''}=f_{\mathbf{k}'}(t),$$
$$\left(\frac{d}{dt}-i\omega_{\mathbf{k}''}+\gamma_{\mathbf{k}''}\right)a^*_{\mathbf{k}''}-iV^*(\mathbf{k}'\mathbf{k}''|\mathbf{k}_0)A^*e^{i\omega_{\mathbf{k}}t}a_{\mathbf{k}'}=f^*_{\mathbf{k}''}(t) \qquad (12)$$

and hence obtained in similar fashion the result (4).

2. BLEACHING OF THE MEDIUM

In the preceding section we described the effect of "bleaching of the medium": decrease of the attenuation of a monochromatic wave with increase of its amplitude in the case when fusion processes predominate. Here we shall estimate for concrete situations the amplitude A_0 at which bleaching becomes noticeable:

$$|V(\bar{\mathbf{k}}\mathbf{k}|\mathbf{k}+\bar{\mathbf{k}})A_0|\approx\sqrt{\gamma_{\bar{\mathbf{k}}}\gamma_{\mathbf{k}+\bar{\mathbf{k}}}} \qquad (13)$$

($\bar{\mathbf{k}}$ corresponds to the waves that make the largest contribution to the attenuation (3)). Before doing this, we recall that formula (3) was obtained by a classical path and that therefore in it $n^0_{\mathbf{k}}$ is the Rayleigh-Jeans distribution: $n_{\mathbf{k}}=T/\omega_{\mathbf{k}}$. It is obvious that our simple procedure is easily generalized to the quantum-mechanical case by supposing that in the initial equations (6) $a_{\mathbf{k}}$ are Bose operators. We shall then again arrive at a formula, in which $n^0_{\mathbf{k}}$ will be the Planck distribution:

$$n_{\mathbf{k}}^0=\hbar[\exp(\hbar\omega_{\mathbf{k}}/T)-1]^{-1}.$$

We shall now estimate A_0 in the problem of sound attenuation in crystals at $T\ll T_D$. We shall suppose that the sound frequency $\omega_{\mathbf{k}}$ is larger than $\gamma_{\bar{\mathbf{k}}}$; in this situation, the largest contribution to the attenuation is made by phonons with energy $\hbar\omega_{\bar{\mathbf{k}}}\approx 3T$ [5], and therefore for estimation of $\gamma_{\mathbf{k}}$ it is possible to use the result of Slonimskiĭ, who calculated $\gamma_{\mathbf{k}}$ for $\hbar\omega_{\mathbf{k}}\gg T$ [6]:

$$\gamma_{\bar{\mathbf{k}}}\approx\omega_{\bar{\mathbf{k}}}\left(\frac{\hbar\omega_{\bar{\mathbf{k}}}}{T_D}\right)^3\frac{\hbar\omega_{\mathbf{k}}}{ms^2}\approx10^2\omega_{\bar{\mathbf{k}}}\left(\frac{T}{T_D}\right)^3\frac{T}{ms^2} \qquad (14)$$

(m is the mass of an elementary cell, s is the velocity of sound, and T_D is the Debye temperature).

For the matrix element V we shall use the result of Landau and Rumer [1] (see also [2,5]), writing it for $\mathbf{k} \parallel \mathbf{k}_1' \parallel \mathbf{k}''$:

$$V(\mathbf{kk}'|\mathbf{k}'')=\frac{6(P+Q+R)\,(\omega_\mathbf{k}\omega_{\mathbf{k}'}\omega_{\mathbf{k}''})^{1/2}}{V^{1/2}(2\rho)^{3/2}s^3}. \qquad (15)$$

Here V and ρ are the volume and density of the crystal, and P, Q, R are the coefficients of anharmonicity, with the dimensions of energy density, which can be estimated thus [2]:

$$P\approx Q\approx R\approx(1-3)\rho s^2.$$

As a result we get from (13)—(15)

$$\omega_{\mathbf{k}_0}|A_0|^2/V\rho s^2\approx(T/T_D)^6(T/ms^2)^2. \qquad (16)$$

The left member is the ratio of the energy density in the monochromatic wave to ρs^2, in order of magnitude equal to $(u/s)^2$, where u is the velocity amplitude of the medium in the wave. This estimate was obtained for transverse sound. Obviously it will be correct also for longitudinal sound, if the linear attenuation of the sound in consequence of finite width of the resonance is caused by the three-phonon fusion processes (2) [5]. The critical amplitude A_0, as is seen from (16), depends strongly on temperature and is reached, for example, in the pulse experiments of de Klerk on attenuation of sound in quartz at frequency 1 MHz at helium temperatures [7].

Another interesting example is the attenuation of uniform precession (UP) of the magnetization in antiferromagnets with anisotropy of the "easy plane" type ($MnCO_3$, $CsMnF_3$, etc.). The spin-wave spectrum has two branches:

$$\Omega_\mathbf{k}=[\Omega_0^2+(s k)^2]^{1/2}, \quad \omega_\mathbf{k}=[\omega_0^2+(sk)^2]^{1/2}; \quad \omega_0\gg\Omega_0. \qquad (17)$$

As was shown by Ozhogin [8], in pure and well polished crystals the principal contribution of the lower branch to the attenuation of UP may be due to the three-magnon fusion process

$$\Omega_0+\Omega_\mathbf{k}=\omega_\mathbf{k}, \qquad (18)$$

in which spin waves with frequency $\Omega_\mathbf{k} = \omega_0^2/2\Omega_0$ participate. Thus the characteristic amplitude A_0 of UP, at which its attenuation decreases by a factor $\sqrt{2}$, is determined by the formula

$$|V(0\mathbf{k}|\mathbf{k})A_0|^2=\Gamma_\mathbf{k}\gamma_\mathbf{k}, \qquad (19)$$

in which $\Gamma_\mathbf{k}$ and $\gamma_\mathbf{k}$ are the attenuations of the spin waves in the lower and upper branch with frequency $\Omega_\mathbf{k} \approx \omega_\mathbf{k} = \omega_0^2/2\Omega_0$. The matrix element V(0k|k) has the form [8]

$$V(0\mathbf{k}|\mathbf{k})\approx(2g\omega_{ex}/M\Omega_0)^{1/2}gH_0,$$

where g is the gyromagnetic ratio[1)], M is the sublattice magnetization, $\omega_{ex} = gH_{ex}$, H_{ex} is the exchange field, and H_0 is the external magnetic field.

UP oscillations can be excited by an external microwave magnetic field $\mathbf{h}(t) = \mathbf{h}\cos\omega t$, polarized for example along the hard axis. At resonance

$$A=\frac{h}{\Gamma_0}\left(\frac{gM_0\Omega_0}{2\omega_{ex}}\right)^{1/2},$$

where Γ_0 is the damping decrement of the UP. The characteristic amplitude A_0 is reached when $h = h_0$, where

$$h/H_0=\Gamma_0\sqrt{\gamma_\mathbf{k}\Gamma_\mathbf{k}}/(gH_0)^2. \qquad (20)$$

This amplitude of the microwave field can be easily attained experimentally. By measuring the dependence of the attenuation of the UP on its amplitude, $\Gamma_0(|A|^2)$ (by the absorption of energy in the specimen, not by the width of the resonance!), it is possible to determine the attenuation of spin waves $\gamma_\mathbf{k}$, $\Gamma_\mathbf{k}$ at frequencies not accessible by direct observation.

3. NONLINEAR FERROMAGNETIC RESONANCE

We shall consider a uniformly magnetized ferromagnet placed in a powerful microwave magnetic field $\mathbf{h}(t) = \mathbf{h}\cos\omega t$ with polarization $\mathbf{h} \perp \mathbf{M}_0$, and we shall assume fulfillment of the condition for resonance with UP, $\omega = \omega_0$; the specimen shape and the value of the constant external field are to be so chosen that disintegration processes[2)]

$$\omega_0=\omega_\mathbf{k}+\omega_{-\mathbf{k}}. \qquad (21)$$

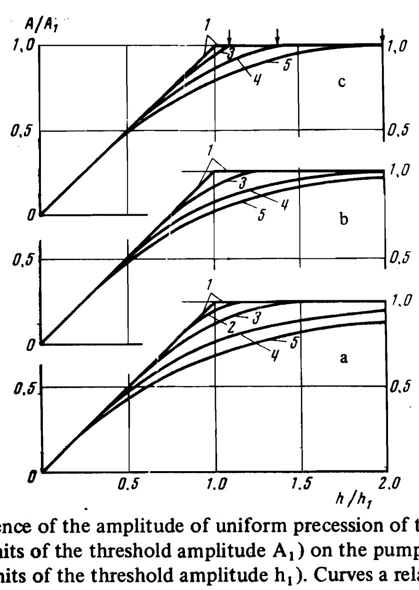
are allowed.

Beginning with the pioneering work of Suhl [9] and up to the present, it has been customary to suppose (see, for example, [10,11]) that in this situation the amplitude of the UP increases with the field h according to the linear law $\gamma_0 A = hu$ until attainment of the critical value

$$A_1=\min[|V(0|\mathbf{k},-\mathbf{k})|/\gamma_\mathbf{k}],$$

and then becomes completely "frozen" at the threshold value: $A = A_1$ for $h \geq h_1 = \gamma_0 A_1/u$. In actuality, even for $h < h_1$ an increase of the damping of the UP begins, in accordance with (4), and this leads to a slowing of the increase of A.

We shall study the dependence of A on h. Let the UP damping consist of two parts: $\gamma_0 = \gamma_1 + \gamma_2$, where γ_1 is due to disintegration processes (21) and depends on A in accordance with (4), and where γ_2 is the contribution of other processes, including scattering on defects and roughness of the specimen surface, and is independent



Dependence of the amplitude of uniform precession of the magnetization (in units of the threshold amplitude A_1) on the pumping amplitude h (in units of the threshold amplitude h_1). Curves a relate to the case of spherical symmetry of the problem, curves b to the case of axial symmetry, and curves c to symmetry lower than axial. The broken line 1 in these figures corresponds to Suhl's theory ($\delta = 1$); curves 2, 3, 4, and 5 correspond to $\delta = 0.99$, 0.9, 0.5, and 0.0. In cases b and c, curve 2 on this scale does not differ from the broken line 1. In figure c the arrows mark the supercritical points h_2 corresponding to excitation of monochromatic pairs.

of A. Then for A < A_1 we have

$$[\gamma_1(A)+\gamma_2]A=hu. \quad (22)$$

If it happens that the values of the medium-wave damping $\gamma_{\mathbf{k}}$ and of the matrix element $V(\mathbf{k}, -\mathbf{k}|0)$ are independent of the angular coordinates on the resonance surface (21), over which it is necessary to integrate in (4), then it is obvious that

$$\gamma_1(A)=\gamma_1/(1-|A/A_1|^2)^{1/2}. \quad (23)$$

In the figure, case a, are shown the A(h) dependences obtained from (22) and (23) for various values of the parameter $\delta = \gamma_2/(\gamma_1 + \gamma_2)$. It is evident that for $\gamma_1 \leq 10^{-2}\gamma_2$, curve 2 for A(h) is close to the broken line 1 given by Suhl's theory, and that it already differs radically from it for $\gamma_1 \gtrsim 0.1\,\gamma_2$ (curves 3, 4, 5). For $\gamma_2 = 0$ (curve 5),

$$A=A_1h/(h^2+h_1^2)^{1/2}. \quad (24)$$

For $h \gg h_1$, one can obtain from (22) and (23)

$$A_1-A=A_1\frac{\gamma_1}{\gamma_1+\gamma_2}\frac{h_1}{h}. \quad (25)$$

There will be an entirely different asymptotic behavior in the case of axial symmetry, when $\gamma_{\mathbf{k}}$ and $V(\mathbf{k}, -\mathbf{k}|0)$ on the resonance surface depend only on the polar angle[3)]. It is then evident that for $A \to A_1$ the integral in (4) diverges logarithmically, and instead of the power-law asymptotic behavior (25) there will be an exponential one. The specific $\gamma_1(A)$ dependence is determined by the details of the behavior of $\gamma(\theta)$ and $V(\theta)$; it can be simulated qualitatively by the function

$$\gamma_1(A)=\gamma_1\frac{A_1}{2A}\ln\frac{A_1+A}{A_1-A}. \quad (26)$$

The A(h) dependence that follows from (22) and (26), for various δ's, is shown in the figure, case b. For $h \gg h_1$, we have instead of (25)

$$A_1-A=2A_1\exp\left(-\frac{2h_1}{h}\frac{\gamma_1+\gamma_2}{\gamma_1}\right). \quad (27)$$

For $\gamma_2 = 0$ (curve 5),

$$A=A_1\,\mathrm{th}(h/h_1). \quad (28)$$

We consider finally the case in which the symmetry of the problem is lower than axial. Then for $A \to A_1$ the radical in (4) vanishes at a single (or at several symmetric) pair(s) of points of the resonance surface $\pm\mathbf{k}_1, \pm\mathbf{k}_2, \ldots$ Therefore for $A = A_1$ the integral (4) converges and the damping $\gamma_1(A_1)$ remains finite. Consequently, the critical UP amplitude A_1 is attained at a finite value $h = h_2 = h_1\gamma_0(A_1)_1/\gamma_0(\theta)$. For $h > h_2$, pairs of waves are parametrically excited at the points $\pm\mathbf{k}_1, \pm\mathbf{k}_2, \ldots$, and their converse effect on the UP freezes its amplitude at the threshold level A_1. The distribution function of the spin waves in the neighborhood of these points has the form

$$n_{\mathbf{k}}-n_{\mathbf{k}}^0=\frac{a}{\alpha(\mathbf{k}-\mathbf{k}_j)_{\parallel}^2+\beta(\mathbf{k}-\mathbf{k}_j)_{\perp}^2}+b\delta(\mathbf{k}-\mathbf{k}_j). \quad (29)$$

The first term describes thermal waves "heated" by the uniform precession, because of which the UP damping has increased from $\gamma_0(0)$ to $\gamma_0(A_1)$; the second describes the parametric waves, whose total amplitude b is proportional to $h - h_2$. From a formal point of view, (29) is the general solution of the inhomogeneous linear system of equations (12) written in the ω-representation. The first term is a particular solution of the inhomogeneous system (12); the second is the general solution of the homogeneous system, which is concentrated at the points $\pm\mathbf{k}_1, \pm\mathbf{k}_2, \ldots$ at which the determinant of the system vanishes for $A = A_1$[4)].

Characteristic A(h) dependences in the case just considered are shown in the figure by curves c. By comparing the cases a—c in the figure, one can see that the difference between our results and the simple theory of Suhl decreases, first on going to lower symmetry, and second when the contribution $\gamma_1(A)$ of the disintegration process to the total UP damping $\gamma_0 = \gamma_1(A) + \gamma_2$ becomes insignificant.

In closing, we note that ferrite power limiters are so constructed that the UP is "revved up" by the microwave input signal h, while the amplitude of the output signal is proportional to the amplitude A of the UP[11]. Therefore the A(h) dependence studied here is essentially the dynamic characteristic $h_{out} = f(h_{in})$ of these devices.

It is a pleasure to express my sincere thanks to V. Zakharov and S. Starobinets for their interest in the research and for useful discussions.

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1)Equal approximately to 2.8 MHz/kOe.

2)Sometimes this situation is called "coincidence of uniform and secondary resonances."

3)This corresponds, for example, to the case of a cubic ferromagnet with **M** ∥ [111] or **M** ∥ [100].

4)The structure of the distribution function of the parametric spin waves at T ≠ 0 and the dependence on the symmetry of the problem have been studied in detail in [12].

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