ON THE KOLMOGOROV TURBULENT SPECTRUM IN THE DIRECT INTERACTION MODEL

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In the Direct Interaction Approximation [1] equations have been formulated, which completely take into account the transfer effect and have a precise solution in the form of the Kolmogorov spectrum

An important place in the statistical theory of hydrodynamic turbulence comes to the self-consistent equations, formulated by Kraichnan [1], which take into account interaction of velocity fluctuations in first order – the so called Direct Interaction Approximation (DIA). DIA equations describe reasonably well the homogeneous turbulence within the energy containing range of the wavenumber k. However, within the inertial wavenumber interval this approximation overstates the role of interaction with long-wave fluctuations [2], which is reduced to a simple transfer of some vortices by others. Thus the energy spectrum is $I_k \sim k^{-7/2}$ [1] in contradiction with experiment. An approximate scheme of turbulence description in Lagrangian variables without these difficulties is suggested in subsequent works of Kraichnan [3]. However, the degree of its precision is unclear [4].

In the present work the equations are formulated with the help of the Wyld diagram technique [5] which exactly take into account the kinematic transfer effect and dynamic interaction in the DIA approximation. It is shown that these equations have a scale-invariant solution with Kolmogorov indices $I_k \sim k^{-11/3}$ [6].

Let us consider ideal incompressible hydrodynamics equations for a velocity v_k:

$$\frac{\partial v_k^\alpha}{\partial t} = -\tfrac{1}{2}\mathrm{i}\int \Gamma_k^\alpha|_{k_1k_2}^{\beta\gamma} v_{k_1}^\beta v_{k_2}^\gamma \delta_{k-k_1-k_2}\,\mathrm{d}k_1\,\mathrm{d}k_2, \quad (1)$$

where the vertex Γ is a homogeneous function of the first order in k_i and satisfies the Jacoby identity, expressing energy conservation [7]:

$$(\Gamma_k^\alpha|_{k_1k_2}^{\beta\gamma} + \Gamma_{k_2}^\gamma|_{kk_1}^{\alpha\beta} + \Gamma_{k_1}^\beta|_{k_2k}^{\gamma\alpha})\delta_{k+k_1+k_2} = 0. \quad (2)$$

In the case of homogeneous isotropic turbulence it follows from eq. (1) that

$$\frac{\partial I_k}{\partial t} = \frac{1}{2}\mathrm{Im}\int \Gamma_k^\alpha|_{k_1k_2}^{\beta\gamma} I_{kk_1k_2}^{\alpha\beta\gamma}\delta_{k+k_1+k_2}\,\mathrm{d}k_1\,\mathrm{d}k_2. \quad (3)$$

This equation describes the evolution of the energy spectral density I_k. One of the regular methods of its investigation in the stationary case is the Wyld diagram technique [4] which allows to express a triple correlator $I_{kk_1k_2}$ as a series in powers of $I_{k\omega}$ and the Green function $G_{k\omega}$ representing a linear response of the system.

The diagram series for $I_{k\omega}$ and $G_{k\omega}$ contain the infrared divergencies related to the kinematic transfer effect of small-scale vortices by large ones. The difference between their scales allows to extract a "weaker" dynamic vortex interaction. For this purpose, following the work of one of the authors [8], we summarize the diagram series, taking into account the first order in the dynamic interaction and transfer effect accurately. The results of such transformation can be represented as follows:

$$G_q = \langle \widetilde{G}_{k,\omega-kv}\rangle_v, \qquad \langle \widetilde{I}_{k,\omega-kv}\rangle_v; \quad (4)$$

$$\widetilde{G}_q = (\omega - \Sigma_q)^{-1}, \qquad \widetilde{I}_q = |\widetilde{G}_q|\Phi_q; \quad (5)$$

$$\Sigma_q = \int \Gamma_k^\alpha|_{k_1,-(k+k_1)}^{\beta\gamma}\Gamma_{k_1}^\beta|_{k,-(k+k_1)}^{\alpha\gamma}\widetilde{G}_{q_1}^* I_{q_2} \times (\delta_{q+q_1+q_2} - \delta_{q+q_1})\,\mathrm{d}q_1\,\mathrm{d}q_2,$$

$$\Phi_q = \int [\Gamma_k^\alpha|_{k_1,-(k+k_1)}^{\beta\gamma}]^2 \widetilde{I}_{q_1}\widetilde{I}_{q_2} \times (\tfrac{1}{2}\delta_{q+q_1+q_2} - \delta_{q+q_1})\,\mathrm{d}q_1\,\mathrm{d}q_2. \quad (6)$$

Here $\langle ... \rangle_v$ means averaging over the turbulent velocity ensemble $v(r, t)$ at an arbitrary point r, t with the help of the Wyld procedure, and $q = (k, \omega)$.

In such a formulation of the statistical hydrodynamics equations their Galilean invariance becomes evident. Eqs. (4)–(6) represent a better DIA; unlike the Kraichnan equations [1] they account for the kinematic transfer effect in all orders. The integrals in eq. (6) converge and thus the dynamic vortex interaction appears to be local, that is, only the vortices of the same scale interact.

Let us find solution of eqs. (4)–(6) in a scale-invariant form

$$\widetilde{G}_q = (1/k^s) g(\omega/k^s), \qquad \widetilde{I}_q = (1/k^{s+p}) f(\omega/k^s).$$

The first relation is a scaling relation, as follows from eqs. (4) and (5): $2s + p = 5$. One more relation between the s and p indices can be obtained by solving the stationary eq. (3). With the help of eqs. (4)–(6) it can be reduced to

$$\mathrm{Im} \int \mathrm{d}\omega \, \mathrm{d}\omega_1 \, \mathrm{d}\omega_2 \, \mathrm{d}k_1 \, \mathrm{d}k_2 \, \delta_{q+q_1+q_2} \Gamma^{\alpha|\beta\gamma}_{k|k_1k_2}$$
$$\times \{ \Gamma^{\alpha|\beta\gamma}_{k|k_1k_2} \widetilde{G}_q \widetilde{I}_{q_1} \widetilde{I}_{q_2} + \Gamma^{\gamma}_{k_2}{}^{|\alpha\beta}_{|kk_1} \widetilde{G}_{q_2} \widetilde{I}_q \widetilde{I}_{q_1} \qquad (7)$$
$$+ \Gamma^{\beta}_{k_1}{}^{|\gamma\alpha}_{|k_2k} \widetilde{G}_{q_1} \widetilde{I}_{q_2} \widetilde{I}_q \} = 0.$$

Note that this equation is analogous to the stationary kinetic equation for weak turbulence [3]. Its simplest solution due to eq. (2), $I_q = (T/\pi) \, \mathrm{Im} \, G_q$, corresponds to thermodynamic equilibrium. To find another solution, i.e. a scale-invariant one, corresponding to a constant energy flow, one can perform a conformal transformation [9] in the second term of eq. (7), a generalization of the Zakharov transformation [10]:

$$k = k''(k/k''), \qquad k_1 = k'(k/k''), \qquad k_2 = k(k/k''),$$

$$\omega = \omega''(k/k'')^s, \quad \omega_1 = \omega'(k/k'')^s, \quad \omega_2 = \omega(k/k'')^s.$$

The third term is transformed in the same way by the change $q \to q_1$. As a result the integrand becomes:

$$\Gamma^{\alpha|\beta\gamma}_{k|k_1k_2} \widetilde{G}_q \widetilde{I}_{q_1} \widetilde{I}_{q_2} \left\{ \Gamma^{\alpha|\beta\gamma}_{k|k_1k_2} \right.$$
$$\left. + \left(\frac{k}{k_1}\right)^x \Gamma^{\gamma}_{k_2}{}^{|\alpha\beta}_{|kk_1} + \left(\frac{k}{k_2}\right)^x \Gamma^{\beta}_{k_1}{}^{|\gamma\alpha}_{|k_2k} \right\} \delta_{k+k_1+k_2},$$

where $x = 8 - s - 2p$. Due to identity (2) it vanishes when $s + 2p = 8$. From this and the scaling relation it follows that $s = 2/3$ and $p = 11/3$.

Thus the system of eqs. (4)–(6) has a scale-invariant solution with Kolmogorov values for the indices.

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