Highly anisotropic distributions of parametrically excited waves in near-isotropic media

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The distribution function of parametric spin waves (PSW) in a cubic ferromagnet is determined. The PSW distribution is a set of pairs of wave packets whose angular dimensions are determined by two competing mechanisms: PSW scattering, which tends to broaden the distribution and the anisotropy of the increment of the parametric instability, which leads to a narrowing of the packet. The stability conditions for, and the collective-oscillation frequencies of, the obtained highly anisotropic PSW distributions are determined.

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Recently, there has been an upsurge in interest in the problem of the structure of the distribution function, $n_{\mathbf{k}}$, of parametrically excited spin waves (PSW) in k space. It has been found that, in the self-consistent field approximation, i.e., within the framework of the so-called S theory,[1,2] $n_{\mathbf{k}}$ is nonzero on the parametric-resonance surface:

$$n_{\mathbf{k}}\sim N_\Omega\delta(\tilde{\omega}_k-\omega_p/2), \quad (1)$$

where $\tilde{\omega}_{\mathbf{k}}$ is the wave dispersion law with allowance for the wave-wave interaction, ω_p is the frequency of the uniform external (pump) field, and Ω is a solid angle on the resonance surface. The distribution, N_Ω, over the angles should, in accordance with the S theory, be highly anisotropic and be determined by the symmetry of the system. For example, in the idealized case of an isotropic ferromagnet there exists only one preferred direction: the direction of magnetization $\mathbf{M}$. It is natural to suppose that the distribution in this case will also be axially symmetric.

Having made this assumption, Zakharov, Starobinets, and one of the present authors computed in the S-theory approximation a variety of integrated characteristics of the PSW system—the nonlinear susceptibilities χ' and χ'', the collective-oscillation frequencies, the creation threshold for a second group of pairs—and ascertained the qualitative and quantitative agreement between the theory and experiments on parametric spin-wave excitation in cubic ferromagnets by the method of parallel pumping.[2] This, however, does not mean that the distribution N_Ω for the spin waves in this case is in fact axially symmetric. The point is that the integrated characteristics of a PSW system is not very sensitive to the dependence of N_Ω on the azimuthal angle φ, and the coefficients, $S_{\mathbf{kk}'}$, of the wave-wave interaction Hamiltonian are such that the equations of the S theory in the stationary case determine an integral number, N, of PSW, imposing in the process the only limitation on the form of the dependence of $N_{\theta,\varphi}$ on φ:

$$\int N_{\theta,\varphi}e^{2i\varphi}d\varphi=0. \quad (2)$$

Thus, within the framework of the S theory, highly anisotropic—in φ—distributions right up to singular distributions, e.g., in the form of a cross made up of two pairs:

$$N^+_{\theta,\varphi}\sim[\delta(\varphi)+\delta(\varphi-\pi/2)+\delta(\varphi-\pi)+\delta(\varphi-3\pi/2)], \quad (3)$$

or a regular six-pointed star:

$$N^*_{\theta,\varphi}\sim\sum_{j=1}^{6}\delta(\varphi-\pi j/3), \quad (4)$$

are admissible.

Furthermore, there is experimental evidence showing that the PSW distribution exhibits in some cases strong anisotropy in φ, and the question arises: why does this happen? How can we, without recourse to experiment, know which of the distributions, $N_{\theta,\varphi}$, satisfying the integral condition (2) is actually realized? This problem has been investigated by Bakaĭ.[3] Remaining within the framework of the self-consistent field approximation, i.e., the S theory, he investigated the stability of singular distributions in the form of a finite number of pairs, and showed that, for certain relations between the coefficients of the Hamiltonian, the isotropic—in φ—distribution, N_φ, is unstable, while a distribution of the type (3) is stable, and therefore should be realized in experiment.

It can, however, be shown that the stationary distributions, (3) and (4), for a small number of pairs and also the single-pair state are unstable against long-period modulations of the pair amplitudes and phases within the framework of the total Hamiltonian of the problem.[4] Consequently, the problem of the fine structure of the angular distribution $N_{\theta,\varphi}$ cannot be solved in the S-theory approximation.

From our point of view, the angular distribution, $N_{\theta,\varphi}$, of the parametric spin waves over the azimuthal angle φ is determined by the competition between two weak factors that are neglected in the S theory for the axially symmetric situation. On the one hand, we have the weak crystallographic anisotropy in cubic ferromagnets, which tends to contract the PSW distribution N_φ to a singular state in the form of the cross (3) if the magnetization $\mathbf{M}$ is oriented along a fourfold symmetry axis ($\mathbf{M}\|\langle 100\rangle$), or to the six-pointed star (4) if $\mathbf{M}$ is oriented along a threefold axis ($\mathbf{M}\|\langle 111\rangle$). On the other hand, there are the processes of scattering of the PSW by each other and by small random inhomogeneities and defects, which tend to smear out and make the parametric-wave distribution N_φ isotropic. Here, as before, the total PSW number, N, is determined by the S-theory relation

$$N=\gamma(p-1)^{1/2}/|S_0|, \quad (5)$$

where γ is the PSW-damping constant, $p = h^2/h_{cr}^2$ is the pump power expressed in units of its threshold value, and S_0 is determined by the relation (7) for $m = 0$.

In the present paper, in §2, we determine the PSW distribution function $N_{\theta,\varphi}$ in homogeneous cubic ferromagnets in which the dominant isotropicizing factor is the scattering of the PSW on each other. We show that, for $\mathbf{M}\|\langle 100\rangle$, the PSW distribution has the form of a smeared cross made up of two pairs, (2.5), with finite θ and φ widths:

$$\Delta\theta^+\approx\left[d\frac{\gamma}{kv}(p-1)\right]^{1/3},\quad \Delta\varphi^+\approx\frac{\Delta\theta^+}{\delta}. \qquad (6)$$

Here the numerical coefficient d is determined in terms of the Fourier harmonics of the coefficients of the pair-interaction Hamiltonian, $S(\varphi-\varphi')$:

$$d=\left(\frac{S_2-S_{-2}}{S_0}\right)^2,\quad S_m=\frac{1}{2\pi}\int_0^{2\pi}S(\varphi)e^{-i(m-2)\varphi}d\varphi, \qquad (7)$$

$\mathbf{k}$ is the wave vector of the PSW, $v=\partial\omega_k/\partial k$ is their group velocity, and the small parameter δ characterizes the deviation of the symmetry of the problem from the axial symmetry in cubic ferromagnets.

For $\mathbf{M}\|\langle 111\rangle$, the PSW distribution has the form of a smeared star formed by three pairs [see (2.10) below], the expressions for $\Delta\theta=\Delta\theta^*$ and $\Delta\varphi=\Delta\varphi^*$ being different from (6):

$$\Delta\theta^*\approx\left[d\frac{\gamma\delta^2(p-1)}{kv}\right]^{1/3},\quad \Delta\varphi^*\approx\left[\frac{d\gamma(p-1)}{\delta^3 kv}\right]^{2/3}. \qquad (8)$$

In the case when the magnetization is oriented along a twofold symmetry axis ($\mathbf{M}\|\langle 110\rangle$), there gets excited beyond the parametric instability threshold one smeared-out pair with widths, $\Delta\theta^-\approx\Delta\theta^+$ and $\Delta\varphi^-\approx\Delta\varphi^+$, given by the formulas (6). Such a state does not satisfy the condition (2), obtained in the S theory for isotropic ferromagnets; as a result, it exists in a narrow supercriticality region determined by the cubic anisotropy:

$$p-1<p^--1\approx(\delta/d)^2. \qquad (9)$$

At $p>p^-$ there is excited in the equatorial plane (i.e., in the $\theta=\pi/2$ plane) a second group of pairs oriented at right angles to the first group, i.e., there is excited a cross with characteristic widths given by (6). Estimates of (6) show that, for high-quality yttrium-garnet (YIG) single crystals with a Q-factor $kv/\gamma\sim10^3$ for spin waves, in the 3-6-dB supercriticality range, which is a typical range for experiment, and for $\mathbf{M}\|\langle 100\rangle$, the smearing of the wave packets with respect to the polar angle is no longer so small: $\Delta\theta\approx10^{-1}$, which is 5-10°. It is more difficult to estimate the width of the distribution over φ; for the attenuation anisotropy is not known. For $\mathbf{M}\|\langle 100\rangle$, we have from symmetry arguments

$$\gamma_\varphi=\gamma_0+\gamma_1\cos4\varphi,\quad \delta=\left(\frac{\partial^2\gamma_\varphi}{\partial\varphi^2}\Big/\gamma\right)^{1/2}=4\left(\frac{\gamma_1}{\gamma}\right)^{1/2}.$$

To the quantity γ_1 may contribute, for example, the Kasuya-LeCroy processes,[5] which do not disappear in the long-wave limit. It may be inferred that the ratio γ_1/γ_0 is roughly estimated by the dimensionless parameter characterizing the cubic anisotropy, i.e., by $H_a/4\pi M\approx0.05$. Then it is not surprising that δ turns out to be not too small. while $\Delta\varphi$ turns out to be comparable to $\Delta\theta$. These estimates agree qualitatively with the experimental results obtained by Bakaĭ *et al.*[6] in high quality YIG single crystals: $\Delta\theta\approx5°$ and $\Delta\varphi\approx5°$.

In §3 we show that PSW scattering by random inhomogeneities gives rise to additional broadening of the packets to the values

$$\Delta\theta\approx\frac{4z\gamma_{def}(1-K)}{\delta\gamma},\quad \Delta\varphi\approx\frac{4z\gamma_{def}(1-K)}{\delta^2\gamma}. \qquad (10)$$

Here $\gamma_{\rm def}$ [see (3.2) below] is the characteristic frequency of scattering of the spin waves on the defects, $z=2$ (3) is the number of wave pairs in the cross (star), and $K<1$ is a factor characterizing the difference in the scattering of a normal and an anomalous correlator. For a single pair ($z=1$) the angular dimensions of the packet are smaller than (10) by a factor of $\gamma_{\rm def}/\gamma$ [see (3.8) below].

It is significant that for

$$\gamma_{def}\gg\delta^2\gamma \qquad (11)$$

the distribution N_φ is almost isotropic along the equator. The means that, in the case of (11), when the decrement $\gamma_{\rm def}$ is still small compared to γ, the cubic anisotropy can be neglected, and the behavior of the PSW is fully described by the isotropic-ferromagnet approximation used in previous investigations.[1,2,7-12]

In §4 we determine the spectrum of the collective oscillations of the found highly anisotropic states of a PSW system with near-random wave phases within a packet. It is shown that the character of the collective oscillations is insensitive to the ansiotropy of the PSW distribution. In particular, the expressions for the collective-oscillation frequencies

$$\Omega_\lambda^\pm=-i\gamma\pm(4S[\lambda](2T[\lambda]+S[\lambda])N^2-\gamma^2)^{1/2}, \qquad (12)$$

$$\Omega_\lambda=-\frac{iN}{\pi\langle N/v^2\rangle\gamma}\frac{S[\lambda]}{2T[\lambda]+S[\lambda]} \qquad (13)$$

(λ is the collective-oscillation mode number) coincide in form with the corresponding expressions obtained earlier[2,7,8] for the isotropic—with respect to the axial angle—PSW distributions. The difference lies only in the expressions of $S[\lambda]$ and $T[\lambda]$ in terms of the matrix elements $S_{\varphi\varphi'}$ and $T_{\varphi\varphi'}$ [see (4.3) below]. Thus, if the condition

$$S[\lambda](2T[\lambda]+S[\lambda])>0 \qquad (14)$$

is fulfilled, then $\mathrm{Im}\Omega_\lambda^\pm<0$, $\mathrm{Im}\Omega_\lambda<0$, and the anisotropic—with respect to φ—PSW distributions obtained in §2.3 are stable.

§1. THE BASIC EQUATIONS

The classical Hamiltonian function of the problem,[2,9]

$$H=\int\omega_k a_k a_k^*\,d\mathbf{k}+H_p+H_{def}+H_{int}, \qquad (1.1)$$

includes the interaction of the waves with the pump,

$$H_p=\frac{1}{2}\int(h\exp(-i\omega_p t)V_k a_k^* a_{-k}^*+\text{c.c.})\,d\mathbf{k}, \qquad (1.2)$$

their interaction with the static inhomogeneities (defects, impurities, etc.),

$$H_{def}=\int g_{\mathbf{kk}'}a_{\mathbf{k}}^{*}a_{\mathbf{k}'}b_{\mathbf{k}''}\delta(\mathbf{k}-\mathbf{k}'-\mathbf{k}'')\,d\mathbf{k}\,d\mathbf{k}'\,d\mathbf{k}'', \tag{1.3}$$

and their interaction with each other

$$H_{int}=\frac{1}{2}\int T_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4}a_{\mathbf{k}_1}^{*}a_{\mathbf{k}_2}^{*}a_{\mathbf{k}_3}a_{\mathbf{k}_4}\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4)\,d\mathbf{k}_1\,d\mathbf{k}_2\,d\mathbf{k}_3\,d\mathbf{k}_4. \tag{1.4}$$

For the study of the fine structure of the PSW distribution, we use a system of equations obtained from (1.1)–(1.4) with the aid of the diagram techniques of Refs. 9 and 10, and generalizing the basic equations of the S theory:

$$\begin{gathered} N(\Omega)=\frac{\pi k_\Omega^2}{v_\Omega}\frac{\Gamma_\Omega}{\nu_\Omega^3}[\Gamma_\Omega\Phi_\Omega+\operatorname{Im}(\Pi_\Omega^*\Psi_\Omega)], \\ \Gamma_\Omega\Sigma(\Omega)+i\Pi_\Omega N(\Omega)=0. \end{gathered} \tag{1.5}$$

Here $N(\Omega)$ and $\Sigma(\Omega)$ are the values of the normal, $n_{\mathbf{k}\omega}$, and anomalous, $\sigma_{\mathbf{k}\omega}$, correlators, integrated over the frequency and the modulus of the wave vector:

$$N(\Omega)=\int n_{\mathbf{k}\omega}k^2\,dk\,d\omega,\quad \Sigma(\Omega)=\int\sigma_{\mathbf{k}\omega}k^2\,dk\,d\omega.$$

All the remaining quantities in (1.5)—the group velocity, v_Ω, the decrement Γ_Ω, and the mass operators Π_Ω, Φ_Ω, and Ψ_Ω are evaluated at the point $\Omega\equiv(\theta,\varphi)$ on the resonance surface

$$\omega_p/2=\tilde{\omega}_{\mathbf{k}}=\omega_{\mathbf{k}}+\operatorname{Re}\Sigma_\Omega.$$

The total PSW-damping constant Γ_Ω is expressible in the usual manner in terms of Σ_Ω:

$$\Gamma_\Omega=-\operatorname{Im}\Sigma_\Omega=\gamma_\Omega+\gamma_{def}(\Omega)+\gamma_{int}(\Omega), \tag{1.6}$$

where γ_Ω is the decrement of the intrinsic PSW damping, arising as a result of the interaction of the PSW with the thermostat formed by the thermal SW, the phonons, etc., $\gamma_{\text{def}}(\Omega)$ is the two-magnon damping constant, and $\gamma_{\text{int}}(\Omega)$ is the contribution to the decrement from the PSW-PSW scattering processes.

The quantity Π_Ω has the meaning of a total PSW pump:

$$\Pi_\Omega=P_\Omega+\Pi_{def}(\Omega)+\Pi_{int}(\Omega). \tag{1.7}$$

Here P_Ω is the self-consistent part of the pump, and is the part that figures in the S theory[1,2]:

$$P_\Omega=hV_\Omega+\int S_{\Omega\Omega'}\Sigma(\Omega')\,d\Omega'.$$

The quantity ν_Ω in (1.5) characterizes that portion of the damping not compensated for by the pump:

$$\nu_\Omega=(\Gamma_\Omega^2-|\Pi_\Omega|^2)^{1/2}. \tag{1.8}$$

The mass operators Φ_Ω and Ψ_Ω are reminiscent of the arrival term in a kinetic equation; they consist of two parts:

$$\Phi_\Omega=\Phi_{def}(\Omega)+\Phi_{int}(\Omega),\qquad \Psi_\Omega=\Psi_{def}(\Omega)+\Psi_{int}(\Omega). \tag{1.9}$$

In the S-theory approximation $\Phi_\Omega=\Psi_\Omega=0$. For not too high supercriticalities and not too intense scattering by the inhomogeneities, we can neglect the renormalization of the vertices in the diagrams for the mass operators, and thereby close Eqs. (1.5):

$$\begin{gathered} \gamma_{def}(\Omega)=c\int|g_{\Omega\Omega'}|^2\frac{\pi k_{\Omega'}^2\Gamma_{\Omega'}}{v_{\Omega'}\nu_{\Omega'}}d\Omega',\quad \Pi_{def}(\Omega)=c\int g_{\Omega\Omega'}g_{\bar{\Omega}'\bar{\Omega}}\frac{\pi k_{\Omega'}^2\Pi_{\Omega'}d\Omega'}{v_{\Omega'}\nu_{\Omega'}}, \\ \Phi_{def}(\Omega)=c\int|g_{\Omega\Omega'}|^2N_{\Omega'}\,d\Omega',\quad \Psi_{def}(\Omega)=c\int g_{\Omega\Omega'}g_{\bar{\Omega}'\bar{\Omega}}\Sigma(\Omega')\,d\Omega'; \end{gathered} \tag{1.10}$$

$$\begin{gathered} \Phi_{int}(\Omega)=2\int\{|T_{\Omega1,23}|^2N(\Omega_1)N(\Omega_2)N(\Omega_3) \\ +T_{\Omega1,23}T^{*}_{\Omega\bar{2},\bar{1}3}\Sigma^{*}(\Omega_1)[N(\Omega_2)\Sigma(\Omega_3)+\Sigma(\Omega_2)N(\Omega_3)]\} \\ \times\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\Omega_1\,d\Omega_2\,d\Omega_3, \\ \Psi_{int}(\Omega)=2\int\{T_{\Omega1,23}\Sigma^{*}(\Omega_1)\Sigma(\Omega_2)\Sigma(\Omega_3) \\ +T_{\Omega1,23}T_{\Omega\bar{2},\bar{1}3}N(\Omega_1)[\Sigma(\Omega_2)N(\Omega_3)+N(\Omega_2)\Sigma(\Omega_3)]\} \\ \times\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,d\Omega_1\,d\Omega_2\,d\Omega_3. \end{gathered} \tag{1.11}$$

Here c is the defect concentration, $g_{\mathbf{kk}'}$ characterizes the two-magnon scattering on a single defect in accordance with (1.3), Ω_i and $\bar{\Omega}_i$ are the angular coordinates of $\mathbf{k}_i$ and $-\mathbf{k}_i$ on the resonance surface, and the $T_{12,34}$ are the coefficients of the interaction Hamiltonian (1.4).

As can be seen from (1.5), the PSW are concentrated in the region of the minimum of ν_Ω. Near the minimum the quantity ν_Ω^2 can be expanded in a series:

$$\nu_\Omega^2=\nu_0^2+\nu_1^2\varphi^2+\nu_2^2\cos^2\theta=\nu_\varphi^2+\nu_2^2\cos^2\theta. \tag{1.12}$$

In cubic ferromagnets

$$V_{\mathbf{k}}\sim\sin^2\theta,\quad S_{\mathbf{kk}'}\sim\sin^2\theta\sin^2\theta'$$

and, consequently, $\nu_2\approx\gamma$. We shall assume the anisotropy with respect to φ to be slight (i.e., that $\nu_1/\gamma\equiv\delta\ll1$), but important: $\nu_1\gg\nu_0$. The case $\nu_1\ll\nu_0$ is not of particular interest, since in this case the anisotropy leads only to the modulation of the previously-studied isotropic solution with a small modulation index of the order of ν_1^2/ν_0^2. It can be seen from the expressions (1.5) and (1.12) that, in all the cases in which Φ_Ω and Ψ_Ω can be regarded as constants within the limits of the width of the PSW packet, the angular dimensions of the packets with respect to θ and φ are given by the formulas

$$\Delta\theta\approx\nu_0/\gamma,\qquad \Delta\varphi\approx\nu_0/\nu_1\approx\Delta\theta/\delta. \tag{1.13}$$

In the following section we investigate the PSW distribution in a homogeneous ferromagnet ($\Phi_{\text{int}}\gg\Phi_{\text{def}}$), when the scattering of the PSW on each other predominates.

§2. THE BROADENING OF THE DISTRIBUTION OF THE PARAMETRIC WAVES AS A RESULT OF THEIR SCATTERING ON EACH OTHER

In the case of slight anisotropy the PSW distribution on the resonance surface comprises of a set of long strips with $\Delta\varphi\gg\Delta\theta$, stretched along the equator. Therefore, in analyzing the distribution over θ, we can assume the distribution over φ to be isotropic. The θ distribution is analyzed under this assumption by one of the present authors in Ref. 10, where it is shown that

$$\Delta\theta\approx\frac{\nu_0}{\gamma}\approx\left[\frac{\gamma d}{kv}(p-1)\right]^{1/3}. \tag{2.1}$$

Integrating (1.5) over θ with allowance for (2.1), we obtain an equation for the determination of N_φ:

$$N_\varphi=\int N_{\theta,\varphi}\,d\cos\theta,\quad N_\varphi=\frac{4\pi\gamma^2N_\varphi}{kv\nu_\varphi^3}\int\frac{(\operatorname{Im}S_{\varphi\varphi'})^2N_{\varphi'}^2\,d\varphi'}{|\sin(\varphi-\varphi')|}, \tag{2.2}$$

$$\nu_\varphi^2=\tilde{\nu}_0^2+\nu_1^2\sum_j\sin^2(\varphi-\varphi_j). \tag{2.3}$$

The quantity $\tilde{\nu}_0$ is slightly greater (by not more than a factor of two) than ν_0, and depends on the specific structure of the θ distribution. In deriving Eq. (2.2), we used, as before,[11] the fact that all the PSW lie in a narrow belt around the equator, and that $|\Sigma(\Omega)| \approx N(\Omega)$. In cubic ferromagnets only the coefficients S_0 and $S_{\pm 2}$ of the Fourier-series expansion, (7), of $S(\varphi - \varphi')$ are nonzero. Therefore,

$$\operatorname{Im} S_{\varphi\varphi'} = \Delta S \sin 2(\varphi - \varphi').$$

For M‖⟨100⟩ we seek the solution to Eq. (2.3) in the form of a smeared-out cross:

$$N_\varphi = \frac{N}{4} \sum_{j=1}^{4} F(\varphi - \pi j/2), \tag{2.4}$$

$$\int_0^{2\pi} F(\varphi)\, d\varphi = 1. \tag{2.5}$$

The function $F(\varphi)$ satisfies an equation that follows from (2.2), (2.4):

$$F_\varphi = \frac{2\pi\gamma^2(\Delta SN)^2 F_\varphi}{kv v_\varphi^3} \int_{-\alpha}^{\alpha} |\varphi - \varphi'| F_{\varphi'}^2\, d\varphi'. \tag{2.6}$$

It is different from zero in the interval $-\alpha \leqslant \varphi \leqslant \alpha$ and normalized in it to unity. In this interval we can cancel out F_φ in Eq. (26) and differentiate the resulting equation twice with respect to φ. As a result, we obtain

$$F_\varphi^2 = \frac{kv}{\pi\gamma^2(\Delta SN)^2} \frac{\partial^2}{\partial \varphi^2} v_\varphi^3 = \frac{3v_1^2 \tilde{v}_0 kv}{\pi\gamma^2(\Delta SN)^2} \frac{1 + 2v_1^2\varphi^2/\tilde{v}_0^2}{(1 + v_1^2\varphi^2/\tilde{v}_0^2)^{1/2}}. \tag{2.7}$$

Substituting this function into (2.6), we find α:

$$\alpha^2 = \tilde{v}_0^2 / 2v_1^2. \tag{2.8}$$

It can be seen that in the region $|\varphi| < \alpha$ the function F_φ changes by not more than 10%. For $|\varphi| > \alpha$ the function F_φ falls abruptly to zero. The quantity $\tilde{\nu}_0$, which determines through (2.8) the width of the broadening of the PSW packet with respect to φ, can be determined from the normalization condition, (2.5), for F_φ:

$$\frac{\tilde{v}_0}{\gamma} = \left[\frac{\pi}{6} \frac{\gamma}{kv} d(p-1)\right]^{1/3}. \tag{2.9}$$

This expression coincides up to an unimportant numerical coefficient with the expression (2.1), which was obtained for the isotropic—with respect to φ—situation. The expressions (1.13), (2.8), and (2.9) determine the distribution function's θ and φ widths referred to in the Introduction—the formulas (6).

A somewhat different situation obtains in the M‖⟨111⟩ case, when the distribution has the shape of a smeared-out six-pointed star:

$$N_\varphi = \frac{N}{6} \sum_{j=1}^{6} F\left(\varphi - \frac{\pi j}{3}\right). \tag{2.10}$$

In this case, for the function $F(\varphi)$ we obtain from (2.2), (2.3), and (2.10) an equation different from (2.6):

$$F_\varphi = \frac{8\pi}{9} \frac{\gamma^2(\Delta SN)^2}{kv v_\varphi^3} F_\varphi \int_{-\alpha}^{\alpha} \left(|\varphi - \varphi'| + \frac{\sqrt{3}}{4}\right) F_{\varphi'}^2\, d\varphi'. \tag{2.11}$$

Here the dominant contribution is made by the scattering through the angles $|\varphi - \varphi'| \approx \pi/3$; $2\pi/3$, while the scattering through the angles $|\varphi - \varphi'| \approx 0$; π determines the shape of the function F_φ.

As before, cancelling out F_φ in (2.11), and differentiating the resulting equation twice, we obtain

$$F_\varphi^2 = \frac{27 v_1^2 \tilde{v}_0 kv}{16\pi\gamma^2(\Delta SN)^2} \frac{1 + 2v_1^2\varphi^2/\tilde{v}_0^2}{(1 + v_1^2\varphi^2/\tilde{v}_0^2)^{1/2}}. \tag{2.12}$$

However, the span, α^*, of the sector in which the solution N_φ is nonzero is equal to

$$\alpha^* = 4v_0^2/3\sqrt{3}v_1^2, \tag{2.13}$$

and the function F_φ is close in shape to a rectangle of height $1/2\alpha^*$. From the normalization condition for the expression (2.12) for F_φ, we have

$$\frac{\tilde{v}_0}{\gamma} = \left[\frac{3\pi}{4} d \frac{v_1^2}{\gamma kv}(p-1)\right]^{1/3} \approx \Delta\theta^*. \tag{2.14}$$

From this relation and (2.13) we obtain the expressions (8) for the angular dimensions of a packet:

$$\Delta\varphi^* = 2\alpha^*, \quad \Delta\theta^* \approx \tilde{v}_0/\gamma.$$

Notice that in the considered case $\Delta\varphi^*$ and $\Delta\theta^*$ are not connected by the standard relation (1.13), to wit, $\Delta\varphi^* \ll \Delta\theta^*/\delta$. It should be said that the expressions (1.11) for Φ_{int} and Ψ_{int} were derived under the assumption that the renormalization of the vertices was negligible, i.e., that

$$\varepsilon = \left|\frac{\Delta T}{T}\right| \sim \frac{\gamma^2(p-1)^{1/3}}{kv\tilde{v}_0\alpha^2} \sim \frac{\delta^2}{p-1} \ll 1. \tag{2.15}$$

This same parameter characterizes the deviation of the quaternary correlator from the product of the pair ones. In other words, the phases of the waves in a packet are almost random phases. Therefore, it is to be expected that the totally coherent PSW state studied earlier[4] can be realized only in a highly anisotropic medium, or in the case of low supercriticality ($\varepsilon \gg 1$).

Thus, in the case of a slight anisotropy ($\delta \ll 1$) the PSW distribution consists of a set of pairs of wave packets with near-random phases and sharp cutoffs with respect to the axial angle φ. Notice that the weak scattering on the inhomogeneities and the small corrections $\delta\Phi_{\text{int}}$ raising the accuracy of the approximation (2.2),

$$\delta\Phi_{int} \sim \tilde{v}_0^2/\gamma^2 \Phi_{int},$$

lead to the smearing out of the discontinuities in N_φ.

§3. THE BROADENING OF THE DISTRIBUTION OF THE PSW AS A RESULT OF THEIR BEING SCATTERED BY THE INHOMOGENEITIES

We shall restrict ourselves to the consideration of sufficiently small-scale inhomogeneities, the characteristic angles of scattering on which are large compared to the angular dimensions of the PSW packet. In this case it can be assumed that $\Phi_{\text{def}}(\Omega)$ and $\Psi_{\text{def}}(\Omega)$ are constants in the vicinity of the ν_Ω minimum. Of interest is the case when the characteristic frequency of scattering of the spin waves by the defects γ_{def}, (3.2), is low ($\gamma_{\text{def}} \ll \gamma$), since in the opposite case ($\gamma_{\text{def}} \gg \gamma$) the PSW distribution on the resonance surface is isotropic.[9]

With allowance for the assumptions made, we obtain from (1.5) the expression

$$N_\varphi = \frac{1}{\pi} \frac{\gamma_{def}\,\gamma(1-K)}{v_\varphi^2} N, \tag{3.1}$$

where

$$\gamma_{def}=\frac{4\pi k^2}{v}\frac{\gamma}{v_2}\frac{1}{z}\sum_{j=1}^{z}|g(\varphi_j)|^2, \tag{3.2}$$

$$K=\sum_{j=1}^{z}g^2(\varphi_j)\exp(2i\varphi_j)\Big/\sum_{j=1}^{z}|g(\varphi_j)|^2;$$
$$g(\varphi-\varphi')=g_{\Omega\Omega'}\quad \text{for}\quad \theta=\theta'=\pi/2, \tag{3.3}$$

and ν_Ω^2 is given by the expression (2.3). It can be seen that the distribution N_φ has the shape of a Lorentz function with width $\Delta\varphi=\tilde{\nu}_0/\nu_1$. The quantity $\tilde{\nu}_0$, which determines $\Delta\varphi$ and $\Delta\theta\approx\tilde{\nu}_0/\gamma$, can be found by integrating (3.1) over φ. For $\tilde{\nu}_0\ll\tilde{\nu}_1$, we obtain

$$\tilde{\nu}_0=\gamma\gamma_{def}(1-K)/\nu_1=\gamma_{def}(1-K)/\delta. \tag{3.4}$$

In the opposite case, i.e., when the condition (11) is fulfilled, N_φ is almost a constant along the equator of the resonance surface.

It should be noted that, in accordance with (3.3), for a single wave pair, $K=1$ and $\tilde{\nu}_0=0$. Consequently, this case should be considered separately, retaining in (1.5) the dependence on φ not only in ν_φ, but also in the expressions for $\Phi_{def}(\Omega)$ and $\Psi_{def}(\Omega)$. Taking this fact into account, we obtain in place of (3.1) the more accurate equation

$$N_\varphi=\frac{\gamma_{def}\gamma}{\tilde{\nu}_0^2+\nu_1^2\sin^2\varphi}\int\frac{|g(\varphi-\varphi')|^2}{|g(0)|^2}[1-e^{2i(\varphi-\varphi')}]N_{\varphi'}\,d\varphi', \tag{3.5}$$

Here we have used the fact that for the wave pair

$$\nu_\Omega^2=\tilde{\nu}_0^2+\nu_1^2\sin^2\varphi+\nu_2^2\cos^2\theta.$$

In (3.5) the entire φ-integration domain turns out to be important; therefore, it is necessary to give the explicit form of $g(\varphi-\varphi')$. Assuming, for simplicity, that $g(\varphi)=\text{const}$, we obtain for N_φ an integral equation with a degenerate kernel:

$$N_\varphi=\gamma_{def}\gamma(N-e^{2i\varphi}N_1)/(\tilde{\nu}_0^2+\nu_1^2\sin^2\varphi),$$
$$N=\int N_\varphi d\varphi,\quad N_1=\int N_\varphi e^{-2i\varphi}d\varphi. \tag{3.6}$$

Integrating (3.6) over φ with the weights 1 and $e^{-2i\varphi}$, we obtain for N and N_1 the system of linear equations:

$$N(1-a)+N_1b=0,\quad -Nb^*+(1+a)N_1=0; \tag{3.7}$$

$$a=\gamma_{def}\gamma\int\frac{d\varphi}{\tilde{\nu}_0^2+\nu_1^2\sin^2\varphi},\quad b=\gamma_{def}\gamma\int\frac{e^{2i\varphi}d\varphi}{\tilde{\nu}_0^2+\nu_1^2\sin^2\varphi}, \tag{3.8}$$

the solvability condition for which yields the relation

$$1=a^2-|b|^2.$$

Hence for $\nu_0\ll\gamma$ we obtain for ν_0 the expression

$$\tilde{\nu}_0=4\pi^2\gamma_{def}/\delta\gamma. \tag{3.9}$$

With the same accuracy we have

$$N_\varphi=\frac{N}{\pi}\frac{\Delta\varphi\cos2\varphi+\varphi^{1/2}\delta^{-1}\sin^2\varphi}{(\Delta\varphi)^2+\sin^2\varphi},\quad \Delta\varphi=\frac{\tilde{\nu}_0}{\gamma\delta}=\left(\frac{2\pi\gamma_{def}}{\delta\gamma}\right)^2. \tag{3.10}$$

The expression for N_φ is a Lorentz function with half-width $\Delta\varphi$ for $|\varphi-\varphi_j|<\varphi_0$, where

$$\varphi_0=\delta^{1/2}(\Delta\varphi)^{1/2}\approx(2\pi\gamma_{def}/\gamma)^{1/2}, \tag{3.11}$$

and a constant for $|\varphi-\varphi_j|>\varphi_0$.

By comparing (3.10) and (10), we can verify that the broadening of the distribution function N_φ turns out in the case of one pair, (3.10), to be significantly less than for two and three pairs, (10). The Eqs. (3.1) and (3.5), which determine the PSW distribution, are linear and homogeneous in the integrated amplitude N. The amplitude N is itself determined in the case of weak scattering ($\gamma_{def}\ll\gamma$) from the condition for the compensation of the self-consistent contribution to ν_Ω, (1.12), in accordance with the S-theory formula (5).

Notice that, in deriving the expressions (1.10) for Φ_{def} and Ψ_{def}, we neglected multiple scattering. It can be shown that the higher-order diagrams for Φ_{def} and Ψ_{def} corresponding to these processes have small values if

$$\left(\frac{\delta\gamma}{\gamma_{def}}\right)^2\frac{\gamma}{kv}\ll1. \tag{3.12}$$

As is well known, in one-dimensional systems multiple scattering is not weak as compared to single scattering. The criterion (3.12), written in the form

$$\Delta\varphi>(\gamma/kv)^{1/4},$$

is the condition for the PSW distribution to be non-one-dimensional.

In conclusion, let us find out when it is necessary to take the PSW scattering by the inhomogeneities into consideration. For this purpose, it is sufficient to compare the angular dimensions $\Delta\theta$ and $\Delta\varphi$ of the PSW packets, computed without allowance for one of the above-considered broadening mechanisms. As a result, we find that, when

$$\frac{\gamma_{def}}{\gamma}>\frac{\delta}{4\pi^2}\left[\frac{\gamma}{kv}d(p-1)\right]^{1/4} \tag{3.13}$$

in the case of a wave pair, or

$$(1-K)\frac{\gamma_{def}}{\gamma}>\frac{1}{\delta}\left[\frac{\gamma}{kv}d(p-1)\right]^{1/4} \tag{3.14}$$

in the case of a cross, scattering by the inhomogeneities predominates. If, on the other hand, the inequalities that are the opposites of (3.13) and (3.14) are fulfilled, then the PSW-PSW scattering, considered in §2, is more intense. The case of a star ($z=3$) is more complex. Scattering by the inhomogeneities predominates when

$$(1-K)\frac{\gamma_{def}}{\gamma}>\delta\left[\frac{\gamma}{kv}\delta d(p-1)\right]^{1/4}, \tag{3.15}$$

while the PSW-PSW scattering is dominant when

$$(1-K)\frac{\gamma_{def}}{\gamma}<\left[\frac{\gamma}{kv}\delta^2d(p-1)\right]^{3/4}. \tag{3.16}$$

In the intermediate case, it is necessary to take into account the scattering of PSW belonging to different pairs on each other and the scattering by the inhomogeneities, which determines the angular width, $\Delta\varphi$, of a packet. The simultaneous consideration of these effects yields

$$\Delta\theta^*\approx\frac{\tilde{\nu}_0}{\gamma}=\left[\frac{1}{12\sqrt{3}}\frac{\gamma}{kv}\frac{\delta^2d(p-1)}{\gamma_{def}(1-K)/\gamma}\right]^{1/3},$$
$$\Delta\varphi^*=2\sqrt{6}\delta^{-2}\gamma^{-1}\gamma_{def}(1-K). \tag{3.17}$$

The expression obtained for $\Delta\varphi^*$ in this case coincides up to a numerical coefficient with the packet width due to the scattering of the PSW by the inhomogeneities.

§4. THE STABILITY AND THE COLLECTIVE OSCILLATIONS OF ANISOTROPICALLY DISTRIBUTED PSW

Here we obtain the collective-oscillation frequencies of the highly ansiotropic PSW states investigated above. As is well known,[8] two types of collective oscillations occur in a PSW system: relatively high-frequency oscillations and low-frequency ones. The first type of oscillations can, in the case when $\gamma_{def} \ll \gamma$, be described in the S-theory approximation, while the second type is connected with the limited amount of PSW scattering, and needs to be considered outside the framework of this approximation.

1. The high-frequency collective oscillations of the correlators

$$\tilde n_{k\omega}(\Omega)=\langle a_{k\omega}a^*_{k\omega-\Omega}\rangle,\quad \tilde n_{k\omega}^+(\Omega)=\langle a_{k\omega+\Omega}a^*_{k\omega}\rangle$$

in a uniform steady-state background are described by the following system of equations[12]:

$$\begin{aligned}(i\gamma_k+\Omega)\tilde n_k(\Omega)-i\gamma_k\tilde n_k^+(\Omega)&=2n_k\Big\{\int T_{kk'}[\tilde n_{k'}(\Omega)\\&+\tilde n_{k'}^+(\Omega)]dk'+\frac{\Pi_k}{\gamma_k^2}\int S_{kk'}\Pi_{k'}^*\tilde n_{k'}(\Omega)dk'\Big\}\\ i\gamma_k\tilde n_k(\Omega)-(i\gamma_k+\Omega)\tilde n_k^+(\Omega)&=2n_k\Big\{\int T_{kk'}[\tilde n_{k'}(\Omega)\\&+\tilde n_{k'}^+(\Omega)]dk'+\frac{\Pi_k^*}{\gamma_k^2}\int S^*_{kk'}\Pi_{k'}\tilde n_k^+(\Omega)dk'\Big\}.\end{aligned}\tag{4.1}$$

Here the wave vectors $\mathbf{k},\mathbf{k}'$ lie on the resonance surface.

If the angular dimension of the excitable packets is small, then it is easy to integrate (4.1) over the angles. As a result, we obtain after a Fourier transformation the equations

$$\begin{aligned}&[i\gamma+\Omega-2(T[\lambda]+S[\lambda])N]\tilde n_\lambda-(i\gamma+2T[\lambda]N)\tilde n_\lambda^+=0,\\&(i\gamma-2T[\lambda]N)\tilde n_\lambda-[i\gamma+\Omega+2(T[\lambda]+S[\lambda])N]\tilde n_\lambda^+=0,\end{aligned}\tag{4.2}$$

where

$$\begin{aligned}&\tilde n_\lambda=\sum_{j=1}^{2z}\tilde n_j e^{-i\lambda\pi j/z},\quad \tilde n_\lambda^+=\sum_{j=1}^{2z}\tilde n_j^+e^{-i\lambda\pi j/z},\\&\tilde n_j=\int_{-\alpha}^{\alpha}\tilde n(\varphi-\pi j/z)d\varphi,\quad \tilde n_j^+=\int_{-\alpha}^{\alpha}\tilde n^+(\varphi-\pi j/z)d\varphi,\\&T[\lambda]=\frac{1}{2z}\sum_{j=1}^{2z}T\frac{\pi j}{z}e^{-i\lambda\pi j/z},\quad S[\lambda]=\frac{1}{2z}\sum_{j=1}^{2z}S\frac{\pi j}{z}e^{i(2-\lambda)\pi j/z};\\&\lambda=0,1,\dots,2z-1.\end{aligned}\tag{4.3}$$

Here we have restricted ourselves to the oscillations of the integrated amplitudes $\tilde n_k$ and $\tilde n_{k'}^+$ inside each narrow sector $\varphi\approx\pi j/z$. As to the oscillations of the packet shape that occur without changes in the integrated amplitude of the packets, for them the right member of (4.1) vanishes, and such modes attenuate with the frequency $\Omega=-2i\gamma$. As a result, we obtain for the system (2.2) the spectrum (12) given in the Introduction. This spectrum coincides in form with the collective-oscillation frequencies obtained earlier in the axially symmetric case.[2,8] Notice that, on account of the property $S(\varphi+\pi)=S(\varphi)$, all the $S[\lambda]$ and, together with them, $\Omega_\lambda^\pm$ vanish when λ is odd. The quantities $T[\lambda]$ and $S[\lambda]$ can be expressed in terms of the Fourier harmonics of the matrix elements $T(\varphi)$ and $S(\varphi)$:

$$T_m=\frac{1}{2\pi}\int_0^{2\pi}T(\varphi)e^{-im\varphi}d\varphi,\quad S_m=\frac{1}{2\pi}\int_0^{2\pi}S(\varphi)e^{-i(m-2)\varphi}d\varphi,$$

as follows:

For a wave pair ($z=1$), only two modes, $\Omega_0^\pm$, remain, where

$$S[0]=S(0)=S_0+S_2+S_{-2},\qquad T[0]=T(0)=T_0+2T_2.$$

For a cross ($z=2$), there exist two pairs of modes, $\Omega_0^\pm$ and $\Omega_2^\pm$, in which

$$S[0]=S_0,\quad T[0]=T_0;$$
$$S[2]=S_2+S_{-2},\quad T[2]=2T_2.$$

For a six-pointed star ($z=3$), there are three pairs of modes, $\Omega_0^\pm$, $\Omega_2^\pm$, and $\Omega_4^\pm$, with frequencies coinciding with the frequencies for the axially symmetric case (the $\Omega_4^\pm$ modes go over into the $\Omega_{-2}^\pm$ modes):

$$S[0]=S_0,\quad T[0]=T_0;$$
$$S[2]=S_2,\quad T[2]=T_2;$$
$$S[4]=S_{-2},\quad T[4]=T_{-2}.$$

Here we have used the specific forms that $S(\varphi)$ and $T(\varphi)$ have in ferromagnets[2]:

$$S(\varphi)=(S_0+S_2e^{2i\varphi}+S_{-2}e^{-2i\varphi})e^{-2i\varphi},\quad T(\varphi)=T_0+T_1(e^{i\varphi}+e^{-i\varphi})+T_2(e^{2i\varphi}+e^{-2i\varphi}).$$

It should be noted that our result (12) differs essentially from the collective-oscillation frequencies obtained by Bakaĭ in Ref. 3. The difference is explained by the fact that, in Ref. 3, it is assumed that only a finite number, namely, 4, of PSW are excited in the system. We, on the other hand, showed in the preceding section that infinitely many degrees of freedom are excited in a continuous medium even in the presence of slight anisotropy.

2. The low-frequency collective oscillations of PSW in an axially symmetric system were studied earlier by us.[8] In our case, when scattering by the inhomogeneities is ignored, these oscillations are described by the equations

$$\begin{aligned}&\tilde N_\varphi=\tilde n_\varphi(\Omega)+\tilde n_\varphi^+(\Omega)=\frac{3}{2\pi}\frac{N_\varphi}{v_\varphi^2}\gamma\int\tilde S_{\varphi\varphi'}\tilde M_{\varphi'}d\varphi',\\&\tilde M_\varphi=\frac{1}{\gamma}[\Pi_\varphi^*\tilde\sigma_\varphi(\Omega)+\Pi_\varphi\tilde\sigma_\varphi^+(\Omega)]=\frac{3}{4\pi}\frac{N_\varphi}{v_\varphi^2}\\&\times\Big\{2i\Omega\int T_{\varphi\varphi'}\tilde N_{\varphi'}d\varphi'-\Big(\frac{v_\varphi^2}{\gamma}-i\Omega\Big)\int\tilde S_{\varphi\varphi'}\tilde N_{\varphi'}d\varphi'\Big\},\end{aligned}\tag{4.4}$$

where

$$\tilde\sigma_{k\omega}(\Omega)=\langle a_{k\omega+\Omega}a_{-k,\omega_p-\omega}\rangle,\quad \tilde\sigma_{k\omega}^+(\Omega)=\langle a_{k\omega}a_{-k\omega_p-\omega+\Omega}\rangle.$$

Using the fact that the distribution N_φ is narrow and the fact that

$$\tilde S_{\varphi\varphi'}=e^{2i(\varphi-\varphi')}S(\varphi-\varphi'),\quad T_{\varphi\varphi'}=T(\varphi-\varphi'),$$

we can find, after a Fourier transformation in (4.4), that

$$\Omega_\lambda=-\frac{iN}{\langle N/v^2\rangle\gamma}\frac{S[\lambda]}{2T[\lambda]+S[\lambda]};\quad \langle N/v^2\rangle=\int\frac{N_\varphi}{v_\varphi^2}d\varphi.\tag{4.5}$$

From (4.5) and (2.7), we obtain finally for the most interesting case—the cross—the expression

$$\Omega_\lambda = -\frac{2i}{\pi} v_0 \delta \frac{S[\lambda]}{2T[\lambda]+S[\lambda]}. \tag{4.6}$$

Notice that, in an anisotropic system, the stability conditions for the high-frequency and low-frequency modes (12) and (13) coincide with (14), exactly as in the axially symmetric case.[8]

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