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ON DEVELOPED HYDRODYNAMIC TURBULENCE SPECTRA

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New results in the theory of the developed hydrodynamic turbulence spectra are reviewed. Within the limits of the hypothesis of interaction locality it is shown that the series of equations for the moments has a scale-invariant solution with the Kolmogorov index values. With the help of the Wyld diagram technique the equations in the Direct Interaction Approximation are formulated which accurately take into account the transfer effect and have the precise solution in the form of the Kolmogorov spectrum. In the framework of these equations the corrections to the Kolmogorov spectrum due to gyrotropy and compressibility are found.

INTRODUCTION

The notion of the turbulence spectrum or the energy distribution plays an important role in the turbulence theory. The present concept of a developed hydrodynamic turbulence spectrum is based, to a great extent, on A.N. Kolmogorov ideas [1], according to which the small-scale developed hydrodynamic turbulence is isotropic and homogeneous, and within the inertial interval its spectrum is defined by the single dimensional parameter P, the energy flux. This spectrum $\mathcal{J}_K$ is determined up to some dimensionless factor c :

$$\mathcal{J}_K = c\,(P/\rho)^{2/3} K^{-11/3}. \tag{0.1}$$

Here

$$< V_K^{\alpha}\ V_{K'}^{\beta} > = \mathcal{J}_K \delta_{K+K'} \Delta_K^{\alpha\beta},$$

$$\Delta_K^{\alpha\beta} = \delta_{\alpha\beta} - K_\alpha K_\beta / K^2.$$

In a spherical normalization this spectrum corresponds to the well-known Kolmogorov-Obukhov "five-thirds" law

$$\varepsilon_K = 4\pi K^2 \mathcal{J}_K \backsim K^{-5/3}$$

confirmed, to a good accuracy, by the observations of the atmospheric turbulence, inertial interval of which changes by five orders of magnitude [2].

Kolmogorov's hypotheses also allow to determine the characteristic frequency ω_K of the vortices of the scale K:

$$\omega_K \approx (P/\rho)^{1/3} K^{2/3} \qquad (0.2)$$

and behaviour of the higher moments

$$J^{\alpha_1,\ldots,\alpha_N}_{\lambda K_1,\ldots,\lambda K_N} = \lambda^{\gamma_N} J^{\alpha_1,\ldots,\alpha_N}_{K_1,\ldots,K_N} \qquad (0.3)$$

where $\gamma_N = -10N/3+3.$

Later on L.D. Landau and E.M. Lifshits [3] paid attention to the fact that the energy flux P is actually a fluctuating value, so that P is not the only dimensional constant on which the spectrum may depend. Therefore A.N. Kolmogorov and A.M. Obukhov introduced some phenomenological changes to the theory, being essentially, a refusal from the idea of complete locality and acknowledgement of the fact that a new dimensionless parameter (kL) can be introduced into the expression for Jk, to some unknown, but not very large power [2]. Thus, the phenomenological approach to the hydrodynamical turbulence description did not permit to determine the spectrum and so far there were many attempts to construct the microscopic theory based directly on the Euler equations:

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V}\nabla)\vec{V} + \nabla p = 0, \quad \operatorname{div} \vec{V} = 0. \qquad (0.4)$$

It should be noted that in a number of works these equations were investigated by means of various hypotheses of the higher moments of the velocity field, for example, the Milliontshikov hypothesis of representation of the fourth moments through second moments [2]. In fact, the nonlinearity in the Euler equations is extremely strong and, apparently, there are no severe reasons for breaking the sequence of equations for the moments.

A regular procedure for investigation of the Euler equations is the diagram technique suggested by Wyld [4]: The random Gaussian force $f_{K\omega}$ is introduced in the right-hand side of equations (0.4) and the solution for $V_{K\omega}$ is presented as a power series in f. The series for $J_{K\omega}$ can be given in a form which a nonzero result is obtained also within the limit $f^2_{K\omega} \to 0$. The peculiarity of the Wyld technique (and also of any technique for strongly nonequilibrium systems [5]) is the appearance of two diagram series: for Green function $G_{K\omega}$ and second moment $J_{K\omega}$, which are connected by the universal relation only in the thermodynamic equilibrium. Wyld has in particular shown that the Direct Interaction Approximation (DIA) formulated by Kraichnan [6] corresponds to the approximation, where the vertices are not renormalized. Generally speaking, this renormalization is rather significant and thus, in the strong hydrodynamic turbulence theory as well as in the phase transition theory, one should take into account the whole diagram series renormalizing the interaction. With the account of this fact G.A. Kuzmin and A.Z. Patashinski [7] studied the ways of agreement of various scaling hypotheses with the diagram series of the Wyld technique.

In this paper we present the review of the new results in this field, restricting ourselves to the Authors' works mainly. In the Section I we consider some properties of the Euler equations: the conservation laws and relations for the equation coefficients resulting from the laws Besides, topological aspects of the vortex liquid motion are discussed, Hamiltonian description is given with the help of the Klebsch variables. In the second Section the equation sequence for the velocity field moments is analyzed by means of conformal transformation similar to the transformation of the kinetic wave equation suggested by V.E. Zakharov [8]. This allows to find an additional, as compared to the phase transition theory, scaling index appearing in turbulence theory in the absence of the thermodynamic equilibrium. It is in the limits Kolmogorov hypothesis of the interaction locality that the sequence of equations for the moments is shown to have actually the solution (0.3) with the Kolmogorov indices. Thus, the center of the problem is transferred to the study of interaction locality in the developed hydrodynamic turbulence. The third Section based on the results by one of the authors [9] is devoted to this subject. In this Section diagram divergencies in the Wyld technique are analyzed. It is shown that the most diverging sequence of the diagrams describes the transfer of small vortices as a whole by random large-scale motions. This sequence is summed to the form:

$$G_{K\omega} = \left\langle \left[\omega - \vec{K}\vec{V} + i\delta \right]^{-1} \right\rangle_V \qquad (0.5)$$

where $\langle \ldots \rangle$ denotes averaging over turbulent ensemble. Thus, the form of $G_{K\omega}$ in the inertial interval is not universal as it is mainly defined by the motion properties in the energy interval, which depend on the way of turbulence arising.

Kolmogorov frequency (0.2) characterizing the speed of the energy exchange between similar scale vortices is $KV_T(KL)^{-1/3}$ and less than the characteristic frequency of the Green function (0.5) $G^{-1}_{K\omega} \simeq KV_T$, describing the transfer (V_T is a root-mean--square velocity). To exclude the kinetic transfer effect impeding the description of a relatively weaker interaction of vortices of the same scale, attempts were made to describe the small--scale vortices in the system of reference moving at each point together with all large-scale vortices [10,11]. For this purpose a dividing scale $K'(L^{-1} < K' < K_0)$ was introduced in the equation describing the temporal evolution for the vortices with the scale K_0 and the motion with $K < K'$ was excluded by the transition to the moving system of reference. In the paper [12] the interaction representation was proposed, which also permits to exclude the motion with the scale less than the dividing scale $K' = K/\lambda$ $(\lambda \gg 1)$, and consecutively to take into account the influence of the large-scale velocity gradient. Unfortunately, in this procedure the transfer of $(1/K)$ vortices by much more larger vortices of the scale $1/K'$ takes place. This Section describes the method of separation of the kinematic effect and the dynamic interaction using no dividing scale. Taking the expression (0.5) for $G_{K\omega}$ into account describing the transfer only, it is proposed to seek the Green function in the form

$$G_{K\omega} = \left\langle \left[\omega - \vec{K}\vec{V} - \tilde{\Sigma}_{K,\,\omega - \vec{K}\vec{V}} \right]^{-1} \right\rangle_V \qquad (0.6)$$

Unlike the series for Σ , the diagram series for $\tilde{\Sigma}$ contain subtractions excluding the transfer. As a result, the convergence sufficiently improves and a whole series of rather symple diagrams converges. Unfortunately, in some diagrams of high order for Σ, the divergence remains even after subtractions. It is caused by the interference between the transfer effect and the dynamic interaction. So this does not permit to finally solve the problem of Kolmogorov turbulence spectra. However, if we restrict ourselves to the DIA, a number of results can be obtained, admitting the experimental test. In the simplest version suggested by Kraichnan 6 , this approximation, describing reasonably well the turbulence within the energy interval, overstates the role of interaction with the long-wave fluctuations [10,11] and, thus, leads to the energy spectrum $\mathcal{J}_k \sim k^{-7/2}$ [6] which contradicts with the experiment.

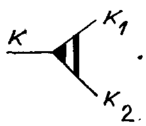
In Section 4, based on authors' work [13], statistical equations in improved DIA are formulated with the accurate account of the transfer effect and dynamic interaction in the first order. Besides the thermodynamically equilibrium solution these equations are shown to have the scale-invariant solution with the Kolmogorov indices. Sections 5 and 6 are devoted to calculation of the additions to the Kolmogorov spectrum. In the Section 5 the gyrotropic turbulence spectrum is determined with the help of improved DIA in weak gyrotropy limits [14]. The spectrum correction due to liquid compressibility is investigated in the Section 6.

1. EULER EQUATION PROPERTIES

A. In k - space equation (0.4) is as follows:

$$\frac{\partial V_k^\alpha}{\partial t} = -\frac{i}{2}\int \Gamma_k^\alpha\Big|_{k_1 k_2}^{\beta\gamma} V_{k_1}^{\beta *} V_{k_2}^{\gamma *}\,\delta_{k+k_1+k_2}\, d\vec{k}_1\, d\vec{k}_2 \qquad (1.1)$$

where Γ is the homogeneous function of the first order of k_i :

$$\Gamma_k^\alpha\Big|_{k_1 k_2}^{\beta\gamma} = \left[\Delta_k^{\alpha\alpha_1} k_{\beta_1} + \Delta_k^{\alpha\beta_1} k_{\alpha_1}\right]\Delta_{k_1}^{\alpha_1\beta}\Delta_{k_2}^{\beta_1\gamma} =$$ 

The vertices Γ satisfy Jacoby identity [16]

$$\left(\Gamma_k^\alpha\Big|_{k_1 k_2}^{\beta\gamma} + \Gamma_{k_1}^\beta\Big|_{k_2 k}^{\gamma\alpha} + \Gamma_{k_2}^\gamma\Big|_{k\, k_1}^{\alpha\beta}\right)\delta_{k+k_1+k_2} = 0 \qquad (1.2)$$

which expresses the fact of the energy conservation:

$$E = \int \frac{V^2}{2}\, d\vec{r} = \int \frac{|V_k|^2}{2}\, d\vec{k}.$$

B. The energy E becomes the Hamiltonian H in the Klebsch variables λ and μ [17], and the Euler equations (0.4) have the canonical form

$$\frac{\partial\lambda}{\partial t} = \frac{\delta H}{\delta\mu} = -\operatorname{div}\lambda\vec{V}, \quad \frac{\partial\mu}{\partial t} = -\frac{\delta H}{\delta\lambda} = -(\vec{V}\nabla)\mu, \qquad (1.3)$$

where

$$\operatorname{rot}\vec{V} = [\,\nabla\lambda,\ \nabla\mu\,]. \qquad (1.4)$$

From the Klebsch variables λ and μ it is convenient to pass on to the complex variables a_K, a_K^*:

$$a_K = (\lambda_K + i\mu_K)/\sqrt{2}$$

for which canonical equations of motion (1.3) and Hamiltonian take the form

$$\frac{\partial a_K}{\partial t} = -i\,\frac{\delta H_t}{\delta a_K^*}, \tag{1.5}$$

$$H_t = \frac{1}{4}\int T_{12,34}\, a_1^* a_2^* a_3 a_4\, \delta_{K_1+K_2-K_3-K_4} \prod_i dK_i \tag{1.6}$$

where

$$T_{12,34} = \rho\left[(\vec{\varphi}_{13}\,\vec{\varphi}_{24}) + (\vec{\varphi}_{14}\,\vec{\varphi}_{23})\right] = \tag{1.7}$$

$$\varphi_{12} = \frac{1}{2\rho\,(2\pi)^{3/2}}\left[\vec{K}_1 + \vec{K}_2 - (\vec{K}_1 - \vec{K}_2)\frac{K_1^2 - K_2^2}{|\vec{K}_1 - \vec{K}_2|}\right]$$

and

$$\vec{V}_K = -i\int \vec{\varphi}_{12}\, a_1 a_2^*\, \delta_{K-K_1+K_2}\, dK_1\, dK_2,$$
$$a_1 \equiv a_{K_1},\quad a_2 \equiv a_{K_2}, \dots \tag{1.8}$$

In the canonical formulation it is especially clearly seen that the problem of turbulent spectrum is the problem of strong interaction: Hamiltonian (1.6) contains no quadratic part but consists of the Hamiltonian of interaction only.

Discuss now the meaning of the λ and μ variables. From (1.4) it is seen that the vortex lines coinside with intersection of two surfaces $\lambda(r)=$ const and $\mu(r)=$ const, stratifying the space. Thus, the Klebsch variables do not describe topologically more complex flows for which the vortex lines make knots.

C. As it is known [18], the liquid flow topology in ideal hydrodynamics is characterized by a preserved value

$$I = \int (\vec{V},\ \mathrm{rot}\,\vec{V}) \tag{1.9}$$

having the meaning of the Hopf invariant [19]. The fact of conservation (1.9) results in additional ratio for vertex Γ :

$$\left(R_K^{\alpha}\Big|_{K_1K_2}^{\beta\gamma} + R_{K_1}^{\beta}\Big|_{K_2K}^{\gamma\alpha} + R_{K_2}^{\gamma}\Big|_{KK_1}^{\alpha\beta}\right)\delta_{K+K_1+K_2} = 0 \tag{1.10}$$

where

$$R_K^{\alpha}\Big|_{K_1K_2}^{\beta\gamma} = \varepsilon_{\alpha\alpha_1\alpha_2} K_{\alpha_1} \Gamma_K^{\alpha_2}\Big|_{K_1K_2}^{\beta\gamma}$$

It should be noted that for the finite flows determined through λ and μ variables, integral (1.9) is identically equal to zero, and thus the Klebsch variables do not describe the knotted flows.

D. To describe barotropic liquid flows with the account of compressibility, a continuity equation for the density ρ is added to the Euler equations, and a pair of canonical variables ρ and velocity potential Φ are added to the λ and μ Klebsch variables; for this case

$$\vec{V} = \frac{\lambda}{\rho} \nabla\mu + \nabla\Phi,$$

and the Hamiltonian coincides with the total energy. After transition to the turbulent complex variables a, a^* and sound ones b, b^* we obtain equations of motion

$$\frac{\partial a_K}{\partial t} = -i \frac{\delta H}{\delta a_K^*}, \quad \frac{\partial b_K}{\partial t} = -i \frac{\delta H}{\delta b_K^*} \tag{1.11}$$

with the Hamiltonian H, containing "turbulent Hamiltonian" H_t (1.6), sound Hamiltonian H_S of usual form

$$H_S = \int \Omega_K b_K b_K^* dK + \ldots \qquad (\Omega_K = K c_S)$$

and Hamiltonian of interaction between sound and turbulence [15]:

$$H_{st} = \int S_{12,34} a_1^* a_2 b_3^* b_4 \delta_{K_1-K_2+K_3-K_4} \prod_i dK_i + \tag{1.12}$$

$$+ \frac{1}{4} \int W_{K12,34}(b_K + b_K^*) a_1^* a_2^* a_3 a_4 \delta_{K-K_1-K_2+K_3+K_4} \prod_i dK_i$$

where

$$W_{K12,34} = \frac{1}{(2\pi)^{3/2}} \left(\frac{2\rho_0 K}{c_S}\right)^{1/2} \left[(\vec{\varphi}_{12} \vec{n}_K)(\vec{\varphi}_{34} \vec{n}_K) + (\vec{\varphi}_{14} \vec{n}_K)(\vec{\varphi}_{23} \vec{n}_K)\right] \tag{1.13}$$

$$\vec{n}_K = \vec{K}/K \quad \text{and} \quad S_{12,34} \simeq K_1 K_3 / \rho_0$$

2. MOMENT EQUATIONS ANALYSIS

Now consider statistical description of the Euler equations (1.1) for correlation functions determined as follows:

$$\langle V_{K_1}^{\alpha_1} \ldots V_{K_N}^{\alpha_N} \rangle = J_{K_1,\ldots,K_N}^{\alpha_1,\ldots\alpha_N} \delta_{K_1+\ldots+K_N}$$

These values obey the equations:

$$\frac{\partial}{\partial t} J_{K_1,\ldots,K_N}^{\alpha_1,\ldots,\alpha_N} = \frac{i}{2} \sum_{\substack{\text{cycle} \\ \{K_1,\ldots,K_N\}}} \int \Gamma_{K_1}^{\alpha_1} \Big|_{K'K''}^{\alpha'\alpha''} J_{K_2,\ldots,K_N,K',K''}^{\alpha_2,\ldots\alpha_N,\alpha',\alpha''} \delta_{K-K'-K''} dK' dK'' \tag{2.1}$$

particularly, for the second moment:

$$\frac{\partial}{\partial t} J_K = \frac{1}{2} \operatorname{Im} \int \Gamma_K^{\alpha} \Big|_{K_1K_2}^{\beta\gamma} J_{KK_1K_2}^{\alpha\beta\gamma} \delta_{K+K_1+K_2} dK_1 dK_2 \tag{2.2}$$

This equation is an analogue of the kinetic equation describing the spectrum evolution in the inertial interval due to energy flux.

Let us assume that in the inertial interval the turbulence is isotropic, and solution of equations (2.1) has the scaling form:

$$\mathcal{J}_{\lambda k_1,\dots,\lambda k^N} = \lambda^{\beta_N}\mathcal{J}_{k_1,\dots,k_N}$$

Assuming the interaction locality we show that stationary equations (2.1) permit such solutions with the indices coinciding with Kolmogorov ones γ_N (0.3).

At first, determine the triple correlator index. For this purpose make konformal transformations [8.20], preserving the integrating area in the equation

$$\operatorname{Im}\int \Gamma_{k}^{\alpha}\Big|_{k_1k_2}^{\beta\gamma}\mathcal{J}_{kk_1k_2}^{\alpha\beta\gamma}\delta_{k+k_1+k_2}\,dk_1\,dk_2 = 0. \tag{2.3}$$

In the arbitrary plane determined by the external vector $\vec{k}$ introduce the complex value $W = k_x + ik_y$ via which this transformation is written as:

$$W = W'\,\frac{W}{W'},\quad W_1 = W\,\frac{W}{W'},\quad W_2 = W''\,\frac{W}{W'} \tag{2.4}$$

As a result the vertex Γ "turns" in (2.3):

$\Gamma_{k}^{\alpha}\Big|_{k_1k_2}^{\beta\gamma} \to \Gamma_{k_1}^{\beta}\Big|_{k_2}^{\alpha\gamma}$ and an additionall multiplier $(k/k_1)^{\beta_3+7}$ appears in the integrand. Similarly, making the turn $k_2\to k$, $k\to k_2$, $k_1\to k_1$ and summing the three expressions (expression (2.3) plus two "turned" ones), obtain the equation

$$\operatorname{Im}\int\Big[\Gamma_{k}^{\alpha}\Big|_{k_1k_2}^{\beta\gamma} + \left(\frac{k}{k_1}\right)^{\beta_3+7}\Gamma_{k_1}^{\beta}\Big|_{k_2k}^{\gamma\alpha} + \left(\frac{k}{k_2}\right)^{\beta_3+7}\Gamma_{k_2}^{\gamma}\Big|_{kk_1}^{\alpha\beta}\Big]$$

$$\mathcal{J}_{kk_1k_2}^{\alpha\beta\gamma}\delta_{k+k_1+k_2}\,dk_1\,dk_2 = 0$$

Due to the identity (1.2) this expression is equal to zero for $\beta_3 + 7 = 0$, or $\beta_3 = \gamma_3 = -7$

Proceed now to the solution of the next equation for the fourth moment which can be presented as a sum of the cumulant $S^{(4)}$ and products of the second moments. From (2.1) for $S^{(4)}$ obtain the inhomogeneous equation:

$$2\Big[\Gamma_{k}^{\alpha}\Big|_{k_1k_2}^{\beta\gamma}\mathcal{J}_{k_1}\mathcal{J}_{k_2} + \Gamma_{k_1}^{\beta}\Big|_{k_2k}^{\gamma\alpha}\mathcal{J}_{k}\mathcal{J}_{k_2} + \Gamma_{k_2}^{\gamma}\Big|_{kk_1}^{\alpha\beta}\mathcal{J}_{k}\mathcal{J}_{k_1}\Big] +$$

$$+\sum_{\substack{\text{cycle}\\ \{\alpha\beta\gamma\} \\ \{k,k_1,k_2\}}}\int \Gamma_{k}^{\alpha}\Big|_{k'k''}^{\alpha'\alpha''} S_{k_1k_2k'k''}^{\beta\gamma\alpha'\alpha''}\delta_{k-k'-k''}\,dk'\,dk'' = 0$$

Due to (1.2) this equation transforms into a homogeneous one, when $\mathcal{J}_K = \mathrm{const}$, which corresponds to thermodynamic equilibrium. For Kolmogorov values of the indices the cumulant $S^{(4)}$, characterizing the difference of the statistics from the Gausslan one, is not zero. Its index is $\beta_4 = 2\beta_2 - 3$. Similarly, the relationship between indices of higher and lower cumulants can be found. For example,

$$\beta_5 = \beta_2 + \beta_3 - 3 = \beta_2 - 10, \quad \beta_6 = 2\beta_3 - 3 = \beta_2 + \beta_4 - 3$$

From these relationships it follows that

$$\beta_2 = \gamma_2 = -11/3 \quad \text{and} \quad \beta_N = \gamma_N = 10N/3 - 3$$

It should be pointed out that the above solutions have the meaning if the integrals in equations (2.1), (2.2) are convergent. To clearify the integral convergence it is required to know the cumulant asymptotics in case when one of K_i is smaller then the others. One of the possible ways is the representation of higher cumulants via lower ones. It can be given by the Wyld diagram technique.

3. WYLD DIAGRAM TECHNIQUE

In the right side of canonical equations (1.5) introduce the random Gaussian force $f_{K\omega}$:

$$\omega a_q = \int T_{K1,23}\, a_1^* a_2 a_3\, \delta_{q+q_1-q_2-q_3} \prod_i dq_i + f_{K\omega}$$

$$a_1 \equiv a_{q_1}, \ldots, q = (\vec{K}, \omega)$$

Iterating this equations present a_q as a power series in f_q. Determine the Green function

$$G_q \delta_{q-q'} = \left\langle \frac{\delta c_q}{\delta f_{q'}} \right\rangle = \frac{\langle a_q f_{q'}^* \rangle}{f_q^2}, \quad G = \longrightarrow$$

$$\langle f_q f_{q'}^* \rangle = f_q^2 \delta_{q-q'}$$

and second moment:

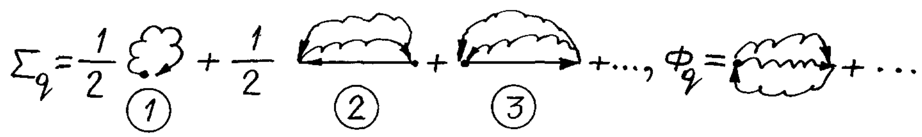
$$n_q \delta_{q-q'} = \langle a_q a_q^* \rangle, \quad n = \rightsquigarrow$$

For them it is possible to get the system of Dyson equations by usual summing of weakly connected diagrams:

$$G_q = (\omega - \Sigma_q)^{-1}, \qquad \mathrm{Im}\, G_q = |G_q|^2\, \mathrm{Im}\, \Sigma_q,$$

$$n_q = |G_q|^2 \Phi_q$$

Here Σ_q and Φ_q are the sums of compact diagrams, in Σ_q it is possible to pass from the input of graph to the output along the lines G_q. Φ_q can be cut into two parts by the only way, along the n_q lines. Present the first diagrams (without vertex renormalization):



(3.2)

It has been shown [21] that when all the integrals in Σ_q and Φ_q converge the diagram equations have the Kolmogorov type solutions:

$$G_q = \frac{1}{P^{1/2} k^{2/3}} \tilde{g}\left(\frac{\omega}{P^{1/3} k^{2/3}}\right), \quad n_q = \frac{P^{2/3}}{k^5} \tilde{f}\left(\frac{\omega}{P^{1/3} k^{2/3}}\right)$$

In fact, for these values of the indices diagrams 1,2 in (3.2) converge, and diagrams 3 and 4 diverge in the range of small k. This results in renormalization of the frequency index so that the functions G_q, n_q should be sought in the form:

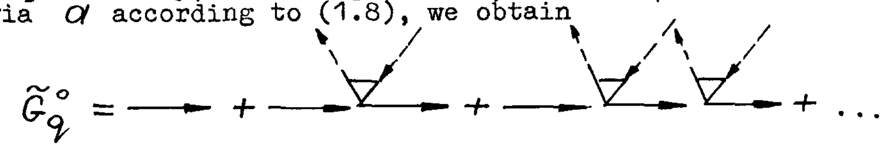
$$G_q = \frac{1}{k U_T} g\left(\frac{\omega}{k U_T}\right), \quad n_q = \frac{1}{k^{13/3+\gamma} k U_T} f\left(\frac{\omega}{k U_T}\right).$$

Substituting these expressions into the lower diagrams for Σ_q one can easily see that many of them also diverge. The greatest contribution to Σ_q is introduced by the diagrams for which external momentum is carried through the "spine" in Green functions, oriented from left to right. Integrating over the range $k' \ll k$ is fulfilled along all the lines n_q. Calculating the powers of each graph, one can easily see that for $\omega \lesssim k U_T$ all such diagrams have the same order as the value G_q^{-1}.

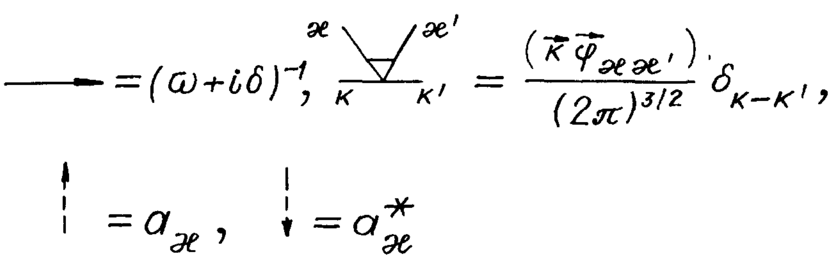
The sum of this diagram sequence has the form (0.5)

$$G_q = \langle \tilde{G}_q^0 \rangle_V, \quad \tilde{G}_q^0 = [\omega - \vec{k}\vec{V} + i\delta]^{-1}$$

where $\langle \dots \rangle_V$ means averaging over the velocity field ensemble $V(r,t)$ at an arbitrary point $\vec{r}$ with the help of the Wyld technique. Really, expanding the $\tilde{G}_q^0$ in $\vec{k}\vec{V}/\omega$ and expressing V via a according to (1.8), we obtain



where



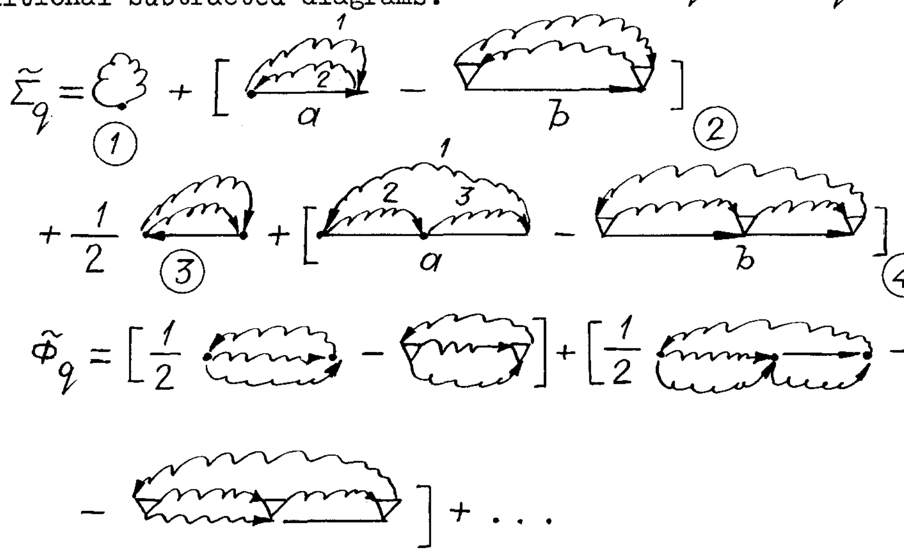
Substituting, then, a_q with the series over rangom force f_q and averaging over the Gaussian random force ensemble, we obtain all selected previously diagrams for $\tilde{G}^{o}_{q}$. Expression (0.5) for G_q corresponds to a simple physical situation of large and small scale interaction: from the Euler equations it follows that this interaction is mainly reduced to a simple transfer of small-scale vortices as a whole by large vortices with random velocity.

So the averaging should be fulfilled in two steps. The first step is the small-scale motion averaging for which the large-scale motion can be interpreted as a dynamic one . It is obvious that in the system of reference, moving with any large-scale vortex, the interaction between large- and small- scale motions is absent.

In this system of reference $\tilde{G}_q=(\omega-\tilde{\Sigma}_q)^{-1}$, with $\tilde{\Sigma}_q \ll k U_T$ for $kL \gg 1$. Coming back to the laboratory system of reference replace ω by $\omega-\vec{k}\vec{V}(t)$, where $\vec{V}(t)$ is the velocity of large-scale vortices. Then we must average over large-scale motion. Therefore it is reasonable to seek the Green function in the form of (0.6). Similarly,

$$n_q=\langle \tilde{n}_{k,\omega-\vec{k}\vec{V}}\rangle_V, \quad \tilde{n}_q=|\tilde{G}_q|\tilde{\Phi}_q \tag{3.3}$$

Direct calculation shows that the series for $\tilde{\Sigma}_q$ and $\tilde{\Phi}_q$ contain the diagrams of the initial series for Σ_q and Φ_q and additional subtracted diagrams:



(3.4)

Convergence of these diagrams in small momentum range improves. For example, diagram (2a) diverges when $q_1 \simeq q_2 \ll q$ as $k_1^{-2/3}$; the difference of diagrams (2a) and (2b) converges as $k_1^{2/3}$.

The procedure described, however, gives no complete convergence in the small k range. For example, diagram (4a) with a "triple loop" diverges when integrating in the modules k_1, k_2, k_3 for $k_1 \simeq k_2 \simeq k_3 \ll k$ as k_1^{-1} after subtraction the logarithmic divergence remains, connected, as it has been already mentioned, with interference of the transfer effect and dynamic interaction. It is significant that after elimination of the transfer the divergence in the first diagrams (DIA) disappears.

4. KOLMOGOROV SPECTRUM IN THE IMPROVED DIA MODEL

The procedure of summation of the diagrams responsible for the transfer can also be performed in the Wyld diagram technique directly for Euler equations. In so doing the equations for the Green function $G_q^{\alpha\beta}$ and second moment $J_q^{\alpha\beta}$ can be obtained by the same way as (0.6) and (3.3). For isotropic case:

$$G_q^{\alpha\beta}=G_q\,\Delta_\kappa^{\alpha\beta},\quad J_q^{\alpha\beta}=J_q\,\Delta_\kappa^{\alpha\beta},$$

$$G_q=\langle \tilde G_{\kappa,\omega-\vec\kappa\vec V}\rangle_V,\quad J_q=\langle \tilde J_{\kappa,\omega-\vec\kappa\vec V}\rangle_V, \tag{4.1}$$

$$\tilde G_q^{-1}=\omega-\Sigma_q,\qquad \tilde J_q=|\tilde G_q|^2\Phi_q .$$

Expressions for Σ and $\tilde\Phi$ are, of course, different from (3.4):

$$\tilde\Sigma_q=\int \Gamma_\kappa^\alpha\Big|_{\kappa_1,-(\kappa+\kappa_1)}^{\beta\gamma}\Gamma_{\kappa_1}^\beta\Big|_{\kappa-(\kappa+\kappa_1)}^{\alpha\gamma}$$

$$\tilde G_1^*\,\tilde J_2\,(\delta_{q+q_1+q_2}-\delta_{q+q_1})\,dq_1\,dq_2,$$

$$\tilde\Phi_q=\int \Big|\Gamma_\kappa^\alpha\Big|_{\kappa_1,-(\kappa+\kappa_1)}^{\beta\ \gamma}\Big|^2\tilde J_1\,\tilde J_2 \tag{4.2}$$

$$\left(\frac{1}{2}\,\delta_{q+q+q_2}-\delta_{q+q_1}\right)dq_1\,dq_2$$

These equations take into account dynamic interaction in the DIA (in the second order over Γ), the transfer effect is taken accurately. As a result, the integrals in (4.2) contain no divergences. Let us find solution of these equations in scale-invariant form:

$$\tilde G_q=\frac{1}{\kappa^s}\,g\left(\frac{\omega}{\kappa^s}\right),\quad \tilde J_q=\frac{1}{\kappa^{s+\gamma}}\,f\left(\frac{\omega}{\kappa^s}\right).$$

The first relation between s and γ indices is the scaling relation:

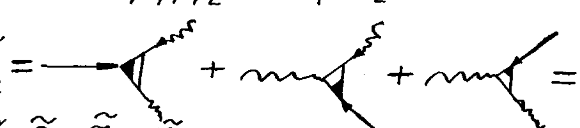
$$2s+\gamma=5.$$

To determine another relation between s and γ in our approximation write out the triple correlator expression:

$$J_{\kappa\kappa_1\kappa_2}^{\alpha\beta\gamma}=\Big\langle\int \tilde J_{q q_1 q_2}^{\alpha\beta\gamma}\,\delta_{\omega+\omega_1+\omega_2}\,d\omega\, d\omega_1\, d\omega_2\Big\rangle_V$$

where

$$\tilde J_{q q_1 q_2}^{\alpha\beta\gamma}=$$



$$=\sum_{\substack{\text{cycle}\\(q q_1 q_2)}}\Gamma_\kappa^\alpha\Big|_{\kappa_1\kappa_2}^{\beta\gamma}\tilde G_q\,\tilde J_{q_1}\,\tilde J_{q_2}$$

Substituting this expression in (2.3) we see the integral convergence. Thus, the triple correlator index is $\gamma_3 = -7 = -s - 2\gamma + 1$. From the two relations obtained we have Kolmogorov values of the indices $\gamma = 11/3$, $s = 2/3$. It indicates that this approximation is a reasonable one and can be used for investigation of the spectra close to the Kolmogorov one.

5. GYROTROPIC TURBULENCE SPECTRA

The interest to the investigation of gyrotropic turbulence arose due to the turbulent "dinamo" problem 22,23. Besides scalar value $\mathcal{J}_\kappa$, the spectral tensor of velocity field of such turbulence

$$\mathcal{J}_{\alpha\beta}(\kappa) = \mathcal{J}_\kappa \Delta_\kappa^{\alpha\beta} + i\,\varepsilon_{\alpha\beta\gamma}\frac{\kappa_\gamma}{\kappa}A(\kappa) \tag{5.1}$$

contains a pseudoscalar $A(\kappa)$ one which is closely connected with topological structure of the flows. For homogeneous turbulence the value $A(\kappa)$ is expressed via the density of Hopf invariant (1.9):

$$\langle \vec{V}, \mathrm{rot}\ \vec{V} \rangle = \int \kappa A(\kappa)\,d\kappa\ .$$

Just as for the Kolmogorov spectrum corresponding to the constant flux P of energy, which conserves in the inertial interval, a helicity spectrum must exist corresponding to the constant helicity flux.

Determine this spectrum with the help of dimension analysis in the limits of weak gyrotropy, i.e., considering the second term in (5.1) as a perturbation to the Kolmogorov spectrum. In this case the characteristic nonlinear time is, obviously, of the order of ω_κ^{-1}. So, the helicity flux P_s can be estimated as

$$P_s \sim \omega_\kappa \kappa^4 A(\kappa)\ .$$

When P_s = const it follows that [24]:

$$A(\kappa) \backsim \kappa^{-11/3}\left(\frac{P}{\rho}\right)^{2/3}\frac{1}{\kappa\tilde{L}} = \mathcal{J}(\kappa)\frac{1}{\kappa\tilde{L}} \tag{5.2}$$

where $\tilde{L} = P/\rho P_s$ is a scale characterizing the total system helicity.

Illustrate now how this result can be obtained in the limits of the improved DIA. For this purpose linearize Dyson and Wyld equations for G and $\mathcal{J}$ on the background of Kolmogorov solution (0.1), assuming for the perturbances:

$$\delta G_q^{\alpha\beta} = i\,\varepsilon_{\alpha\beta\gamma}\frac{\kappa_\gamma}{\kappa}H_q\,,\quad H_q = \kappa^{-s-2/3}h\left(\frac{\omega}{\kappa^{2/3}}\right)$$

$$\delta \mathcal{J}_q^{\alpha\beta} = i\,\varepsilon_{\alpha\beta\gamma}\frac{\kappa_\gamma}{\kappa}A_q\,,\quad A_q = \kappa^{-t-2/3}a\left(\frac{\omega}{\kappa^{2/3}}\right)$$

Then from scaling relations of the linearized equations it follows that

$$t = s + 11/3$$

The second relation between the t and s indices is determined from stationary equations for $A(\kappa)$:

$$\kappa \frac{\partial A(\kappa)}{\partial t} = -\frac{1}{2} \operatorname{Re} \int R^{\alpha}_{\kappa} \Big|^{\beta\gamma}_{\kappa_1 \kappa_2} \mathcal{J}^{\alpha\beta\gamma}_{\kappa \kappa_1 \kappa_2} \delta_{\kappa+\kappa_1+\kappa_2} d\kappa_1 d\kappa_2 \, .$$

From this, using relations (1.10) for R, the index of $\operatorname{Re} \mathcal{J}^{\alpha\beta\gamma}_{\kappa\kappa_1\kappa_2} : \gamma = -8$ is determined similarly to (4.3). Expressing, then, $\operatorname{Re} \mathcal{J}^{\alpha\beta\gamma}_{\kappa\kappa_1\kappa_2}$ via G_q and $\mathcal{J}_q$ and restricting ourselves to the improved DIA we find that $s=1$, $t=11/3+1$ give the result which corresponds completely to dimensional estimations (5.2). As for the spectrum locality, the proof of this fact is analogous to that in Section 4 and is presented in [14].

6. COMPRESSIBILITY INFLUENCE ON KOLMOGOROV TURBULENCE SPECTRUM

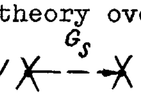
At first obtain qualitatively the spectrum correction $\delta \mathcal{J}_\kappa$ due to compressibility. According to the interaction locality hypothesis the spectrum distortion can be determined only by the velocity fluctuations $V(\kappa)$ with the same scale. Taking into account that $V(\kappa) \backsim V_T (\kappa L)^{-1/3}$ in the inertial interval one can easy obtain that

$$\frac{\delta\rho_T(\kappa)}{\rho_0} \simeq \frac{V^2(\kappa)}{c_s^2} \simeq M^2 (\kappa L)^{-2/3}$$

and the supposed relative addend to the spectrum must be of the same order:

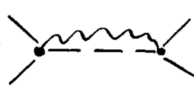
$$\frac{\delta \mathcal{J}_\kappa}{\mathcal{J}_\kappa} \simeq (\kappa L)^{-2/3} \tag{6.1}$$

From the formal viewpoint the compressibility results in additional interaction vertices for the canonical variables describing vortex motions. The simplest vertex $T^{(-1)}$ of the kind occurs in the second order of the perturbation theory over the vertices W in Hamiltonian H_{st} (1.12).

$T^{(1)} = W$ 

W

where G_s is the sound Green function $(\omega - \Omega_\kappa)^{-1}$. In hydrodynamical diagrams the $T^{(-1)}$ vertex leads to the δT correction to the vertex T :

$\delta T =$  $+ \dots$

Using explicit expression for W (1.13) obtain

$$\delta T_{12,34} = T_{12,34} \int \frac{\kappa}{c_s} \mathcal{J}_{q'} G_q \delta_{q_1 - q_2 + q' - q} dq' dq$$

and then devide δT into two parts:

$$\delta T^{(1)}_{12,34} = T_{12,34} \int \frac{\kappa}{c_s} \mathcal{J}_{q'} G_q \delta_{q_1 - q_2 - q} dq' dq \simeq T_{12,34} M^2 ,$$

$$\delta T^{(2)}_{12,34} = T_{12,34} \int \frac{K}{c_s} J_{q'} G_q \left(\delta_{q_1-q_2+q'-q}\, \delta_{q_1-q_2-q}\right) dq\, dq' \simeq$$
$$\simeq T_{12,34} M^2 (\kappa L)^{-2/3}.$$

These estimations were used the assumption that all κ_i are of the same order. Thus, the addition $\delta T^{(1)}$ has the same index as T and, therefore, does not result in the change of index δJ. As for the $\delta T^{(2)}$ it has another index. In the diagram equations $\delta T^{(2)}$ is compensated by terms proportional to δJ. Thus,

$$\frac{\delta J_\kappa}{J_\kappa} \sim \frac{\delta T^{(2)}}{T} \sim \frac{M^2}{(\kappa L)^{2/3}}$$

which coincides with qualitative estimation (6.1).

CONCLUSION

The presented approach for strong hydrodynamic turbulence investigation is based on the conformal transformation and the transfer exclusion procedure. It permits to determine other corrections to the Kolmogorov spectrum. For example, the spectrum addition due to viscous boundary λ of the inertial interval has been calculated [9]:

$$\frac{\delta J_\gamma}{J_\kappa} \sim (\kappa\lambda)^{2/3}.$$

Of extraordinary interest is the investigation of the turbulence spectrum isotropization, i.e., calculation of the spectrum correction caused by anisotropy of turbulence excitation. This addend is determined with the help of the dimension reosons [25]:

$$\frac{\delta J_a}{J_\kappa} \propto (\kappa L)^{-2/3}$$

We think that the correction is of nonlocal nature and, thus, cannot be determined by the conformal transformation. To determine it one should not only eliminate the divergencies but also know the interaction locality power.

It should be recalled that the results of this review are based on the improved DIA, for which the locality has been proved. To construct a complete theory of the turbulence spectra in the inertial interval, analysis of the locality problem for higher diagrams is required. For this purpose, in our opinion, one should refuse the Euler description and find such a method which, on one hand, does not contain the transfer effect and, on the other hand, is much simpler than the Lagrange turbulence description.

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(1) I.e. for the r scales, which are less than streamlined body scale L, but larger than that of a "viscous" λ.

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