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CONTRIBUTION TO THE NONLINEAR THEORY OF SOUND AND HYDRODYNAMIC TURBULENCE OF A COMPRESSIBLE LIQUID

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The interaction of sound with hydrodynamic turbulence has been studied in detail. The sound absorption decrement, the correlation time and length and the frequency diffusion coefficient for the acoustic wave packet are calculated. The spectral composition of the sound radiated by a unit, turbulent volume and the spectral energy density of sound in equilibrium with the turbulence are studied. The region of applicability of the kinetic equation for sound with a linear dispersion low is found.

INTRODUCTION

A great number of papers and monographs has been dedicated to description of turbulent liquid (of gas) flow. However, even in the simplest case, i.e. in the case of homogenious isotropic flow of an incompressible liquid the turbulence problem cannot be regarded as solved. The common concept of developed hydrodynamic turbulence (Kolmogorov theory) is phenomenological. It is based upon hypothesis of interaction locality and upon dimension analysis.

Kolmogorov theory is in good agrement with the experiment, therefore it is natural to accept it is a zero approximation while studying turbulence in compressible liquid, due to weak interaction between vortex and potential flows at small Mach numbers $M=v_T/c_s$ (v_t- characteristic velocity of vortex motion, - sound velocity). Taking compressibility into account leads to appearance of dimensionless parameters, and dimension analysis cease to be effective. Description should be made proceeding from dynamic equations of motion. For this purpose, the Euler equations for barotropic flow of a compressible liquid in Sec.1 are represented in Hamiltonian form (1.2), (1.3) with the aid of the Clebsch variables (1.5) [1]. Then, assuming the Mach number to be small, we constructed the non-linear canonical transformation (1.9)-(1.16) to the new variables in which the vortical motion of the liquid (a_k, a_k^*) and the potential motion (b_k, b_k^*) are separated in maximum fashion[2]. In these variables, the Hamiltonian of the problem takes the form

$$H=H_s+H_t+H_{st}\,. \tag{1}$$

Here H_s is the Hamiltonian for sound in a quiescent liquid 3 :

$$H_s=\int\omega_{\vec k}\, b^*_{\vec k}\, b_{\vec k}\, d^3k+1/2\int V_{1,23}\,(b_1^* b_2 b_3+c.c\,)\,\delta(\vec k_1-\vec k_2-\vec k_3)\,d^3k_1\, d^3k_2\, d^3k_3, \tag{2}$$

is the Hamiltonian for turbulent pulsations of the incompressible liquid[3]:

$$H_t = 1/4 \int T_{12,34}\, a_1^* a_2^* a_3 a_4 \delta(\vec{\kappa}_1+\vec{\kappa}_2-\vec{\kappa}_3-\vec{\kappa}_4)\, d^3\kappa_1\, d^3\kappa_2\, d^3\kappa_3\, d^3\kappa_4, \quad (3)$$

is the Hamiltonian of interaction of sound with turbulence calculated by us:

$$H_{st} = \int S_{12,34}\, a_1^* a_2\, b_3^* b_4\, \delta(\vec{\kappa}_1-\vec{\kappa}_2+\vec{\kappa}_3-\vec{\kappa}_4)\, d^3\kappa_1\, d^3\kappa_2\, d^3\kappa_3\, d^3\kappa_4 + \quad (4)$$

$$+ 1/4 \int W_{\kappa 12,34} (b_\kappa^* + b_{-\kappa}) a_1^* a_2^* a_3 a_4\, \delta(\vec{\kappa}+\vec{\kappa}_1+\vec{\kappa}_2-\vec{\kappa}_3-\vec{\kappa}_4)\, d^3\kappa\, d^3\kappa_1\, d^3\kappa_2\, d^3\kappa_3\, d^3\kappa_4 .$$

The first term describes the scattering of sound by turbulence, the second one - the processes of radiation and absorption of sound by the turbulence. For the statistical description we use the canonical diagram technique of Wyld[3]. In a number of cases to estimate the diagrams we must sum an infinite set of diagrams (i.e. transport must be taken into account) and use the Kolmogorov hypothesis that the unteraction of the vortices is local[2,4].
In Sec.3 below we obtain an expression for the sound damping decrement in a homogeneous and isotropic turbulent medium:

$$\Gamma_{dis} \approx v_T M^2/L . \quad (5)$$

Here L is the external (integrated) scale of the turbulence. We note that this effect is absent in the approximation of a specified turbulence. In Sec.4, we study the spectral composition of the sound $I_s(\omega)$ radiated by a unit turbulent volume:

$$I_s(\omega) \simeq \rho_o v_T^2 M^5 (v_T/\omega L)^{7/2}. \quad (6)$$

The integrated intensity of the radiated sound $\int I_s(\omega) d\omega$ was obtained earlier [6]. In acoustically opaque turbulence the sound density $E_s(\omega)$ is determined by the equilibrim between the processes of emission (6) and absorption (5) of the sound:

$$E_s(\omega) = \rho_o v_T L M^3 (v_T/\omega L)^{7/2}. \quad (7)$$

Here the total energy density of "equilibrium" sound $E_s = \int E_s(\omega) d\omega$ is smaller by a factor of M^3 than the energy density of the turbulence pulsations $\rho_o v_T^2$.

In Sec.5 we consider the scattering of sound in a turbulent medium. As is well known [7,8,] these processes are almost elastic and lead to the isotropization of the acoustic packet with respect to directions. For very narrow packets ($L \Delta\kappa_s < 1$), the process of scatering from vortices of energy-containing scale L with characteristic time

$$\tau_r^{-1} \approx \kappa_s v_T (\kappa_s L M), \quad (8)$$

are important. Here κ_s is the acoustic wave vector. Scattering takes place here at the small angle $\Delta\theta \approx (\kappa_s L)^{-1}$. But this does not mean that the evolution of broad packets $L\Delta\kappa_s \gg 1$ can be considered in the differential approximation. The fact is that the small-angle scattering does not materially change the shape of the broad packet, and the basic role is played by scattering through angles of the order of its width. As a result, the characteristic time of change of the packet increases and, in place of (8), we obtain

$$\tau^{-1}(\Delta\kappa_S) \approx \kappa_S v_T(\kappa_S L M)(\Delta\kappa_S L)^{-5/3}$$
$$l_o = L\,Re^{-3/4}. \qquad (9)$$

The total time of isotropization τ_{is} is determined by the evolution of the packet at the last stage, when $\Delta\kappa_S \simeq \kappa_S$ whence

$$\tau_{is}^{-1} \approx M\,(\kappa_S L)^{1/3} v_T / L. \qquad (10a)$$

This expression is valid if κ_S lies in the inertial interval of scales $l_o < \kappa_S^{-1} < L$ ($l_o = L\,Re^{-3/4}$ is the internal turbulent scale). At $\kappa_S L < 1$, the scattering processes are strongly suppressed because of the small intensity of vortices with $\kappa_T L < 1$; in the case $\kappa_S l_o >> 1$, the scattering takes place at a small angle, and the differential approximation is valid; the time of isotropization is determined by the scattering from vortices of scale l_o :

$$\tau_{is}^{-1} \approx v_T M / L\,Re^{1/4} \qquad (10b)$$

Within the isotropization time, the sound is not able to give up the turbulence energy because the time of sound absorption by the turbulence τ_{dis}, turns out to be very large:

$$\tau_{dis}^{-1} \approx v_T M^2 / L \qquad (11)$$

It is interesting that τ_{dis} does not depend on the sound wave vector; therefore the initial shape of the sound energy distribution function over the frequencies does not change in the absorption process. The frequency evolution of the acoustic packet of low intensity is therefore determined by the inelastic part of the scattering of sound by the vortices. As is shown in [2], under the conditions $l_o < \kappa_S^{-1} < L$, the characteristic time of frequency evolution τ_{diff} is

$$\tau_{diff}^{-1} \approx M^3 (\kappa_S L)^{1/3} v_T / L \qquad \kappa_S L M < 1, \text{ at} \qquad (12)$$

$$\tau_{diff}^{-1} \approx M^{5/2} (\kappa_S L)^{-1/6} v_T / L. \text{ at} \qquad \kappa_S L M > 1. \qquad (13)$$

The time τ_{diff} was calculated earlier in the work of Krasil'nikov and Pavlov [8] under the conditions $\kappa_S L M < 1$, $\kappa_S l_o > 1$. With increase in the sound intensity, the necessity arises of taking into account the interaction of sound with sound (SS-interaction) and the problem is how this is to be done. It is known that the approximation of almost random phases (the kinetic equation) is inapplicable for the description of acoustic tirbulence in the case of a linear dispersion law $\omega_\kappa = c_S \kappa_S$. The fact is that all the waves propagating is one direction have the same velocity, and the interaction between them leads to a strong phase correlation. In the presence of dispersion, a spreading of the acoustic packet takes place and the kinetic equation is applicable if the time of SS - interaction

$$\tau_{SS}^{-1} \approx \kappa_S c_S E_S / \rho_o c_S^2 \qquad (14)$$

is longer than the time of randomization of the phases in the packet τ_d : $\tau_d^{-1} = \omega''(\Delta\kappa)^2$. Another reason exists for the randomization of the phases for sound in a turbulent medium - its scattering from the random vortex field. It is therefore natural to estimate the time of randomization from the time of scattering of the sound by the vortices, i.e., to assume $\tau_d^{-1}=\tau_r^{-1}=(\kappa_s L)(\kappa_s LM)$

$$\tau_d^{-1}=\tau_r^{-1}=(\kappa_s v_T)(\kappa_s LM)$$

(see (8)). Thus the kinetic equation is applicable if $\tau_{ss}>\tau_r$, i.e.,

$$E_s<\rho_o v_T^2(\kappa_s L) \tag{15}$$

In the region $\kappa_s LM<1$ this criterion is obtained in Sec.6 by the analysis of the diagram series for renormalization of the vertex that discribes the interaction of sound with sound. At $\kappa_s LM>1$, this analysis leads to another criterion for the applicability of the kinetic equation

$$E_s<\rho_o v_T^2 M^{-1}(\kappa_s LM)^{-1/4} \tag{16}$$

Let us clarify the reason for this difference. The parameter $\kappa_s LM$ has the meaning of a phase lag $\Delta\varphi$ over the distance L, which arises because of its interaction with the vortex velocity field of scale L. At $\kappa_s LM<1$, $\Delta\varphi<1$ and the time of destruction of the phase correlation τ_{cor} is determined by the random distribution of the phase over a large number of vortices; at $\kappa_s LM>1$, the phase shift over the path L is large and the destruction of the correlations takes place over a distance Λ that is smaller than L. We can therefore assume that the acoustic packet is transported as a whole in the almost homogeneous velocity field of the large-scale vortices; the time τ_{cor} is determined by the Doppler effect from these vortices and $\tau_{cor}\approx(\kappa_s v_T)^{-1}$.

The transport of the packet as a whole does not destroy the correlation of phases between the waves inside the packet and therefore the criterion of applicability of the kinetic equation is not determined by the time τ_{cor}. To obtain this criterion, as we have shown in Sec.6, it is necessary to compare the interaction length $\Lambda_{int}=c_s\tau_{ss}$ with the distance Λ_{cor} in which the phase correlation in the wave is destoyed. The correlation length Λ_{cor}, as is shown in Sec.5, is determined by the sound scattering from vortices of scale $\Lambda_{cor}<L$; determining it self-consistently, we obtain

$$\Lambda_{cor}\approx L(\kappa_s LM)^{-3/4}. \tag{17}$$

The relation $\Lambda_{int}>\Lambda_{cor}$ is equivalent to the criterion (16). It is seen from all that has been said above that the interaction of the sound with the hydrodynamic turbulence is characterized on the whole by a set of times which describe the sound attenuation, the scattering and correlation properties of the acoustic packets and so on. For convenience in comparing these, we have collected the corresponding expressions for the frequencies τ^{-1} in a table.

In the region of weak turbulence upon satisfaction of the criteria (16) and (17) the interaction of sound with sound appears first in the frequency evolution of the packet. It becomes decisive when

$$E_s>\rho_o v_T^2 M^2(\kappa_s L)^{-2/3} \tag{18}$$

If now $\tau_{ss}<\tau_{dis}$, i.e.,

$$\begin{aligned} &E_s>\rho_o v_T^2 M(\kappa_s L)^{-1}, \quad M<\kappa_s L<M\,Re^{1/2}, \\ &E_s=\rho_o v_T^2(\kappa_s L)M^{-1}Re^{-1}, \quad \kappa_s L>M\,Re^{1/2}, \end{aligned} \tag{19}$$

then the role of turbulence is reduced merely to the isotropization of the packet and is determined in other respects by the kinetic equation of the interaction of sound with sound.

If we consider the problem of the spectrum of acoustic turbulence, then the Zakharov-Sagdeev spectrum $E_S \propto K^{-3/2}$ will be produced in the region (16), (17) and (19) as the exact solution of the kinetic equation. If the sound intensity is large and the criterion (16), (17) is violated, a region of strong acoustic turbulence begins, the effect of the vortices can be neglected and the question about spectrum is open[1] because the alternative possibility (i.e. the Kadomtsev-Petviashvili spectrum $E_S \propto K^{-2}$) exists [10].

1. CANONICAL VARIABLES IN HYDRODYNAMICS. THE HAMILTONIAN

The equations of ideal hydrodynamics, which describe the barotropic flow of a compressible liquid, permit locally the introduction of canonical variables - the Clebsch variables $\rho, \Phi; \lambda, \mu$, [1] the Hamilton equations for which are of the form

$$\dot{\lambda} = \delta H / \delta \mu = -\mathrm{div}(\lambda \vec{v}), \quad \dot{\mu} = -\delta H / \delta \lambda = -(\vec{v}\,\nabla)\mu ;$$

$$\dot{\rho} = \delta H / \delta \Phi = -\mathrm{div}(\rho \vec{v}), \quad \dot{\Phi} = -\delta H / \delta \rho = -\vec{v}^2/2 + \lambda \rho^{-1}(\vec{v}\,\nabla)\mu - \delta\varepsilon/\delta\rho$$

Here

$$H = \int [\rho v^2/2 + \varepsilon(\rho)]\, d^3 r,$$

while

$$\vec{v} = \nabla \Phi + \lambda \nabla \mu / \rho .$$

The case $\lambda = 0$ or $\mu = \mathrm{const}$ corresponds to potential motions of the liquid, which are described by the pair of variables ρ and Φ in correspondence with Eqs. (1.2). In the case of an incompressible liquid, $\dot{\rho} = 0$ and Eqs. (1.2) reduce to the relation div $V = 0$, which allows us to express Φ in terms of λ and μ:

$$\Phi = -\Delta^{-1} \mathrm{div}(\lambda \nabla \mu / \rho) \tag{1.5}$$

Then, from (1.4),

$$\vec{v} = -\Delta^{-1} \mathrm{rot}\,[\nabla \lambda \times \nabla \mu] \tag{1.6}$$

and Eqs. (1.2) for λ and μ describe the nonpotential motions of the incompressible liquid.

In the general case, we cannot state that the pair ρ and Φ describe the potential motions and the pair λ and μ the vortical motion. Actually, by dividing V into two parts

$$\vec{v} = \vec{v}_l + v_t, \quad \mathrm{rot}\,\vec{v}_l = 0, \quad \mathrm{div}\,\vec{v}_t = 0, \tag{1.7}$$

it is easy to see that

$$v_l = \nabla \tilde{\Phi}, \quad \tilde{\Phi} = \Phi + \Delta^{-1} \mathrm{div}(\lambda \nabla \mu / \rho) \tag{1.8}$$

Therefore, the initial Clebsch variables are unsuitable for a description of the turbulence of the compressible liquid: the fields (ρ, Φ) and (λ, μ) turn out to be strongly coupled even in the case in which the velocity of the pulsations is not large, i.e., at $M \ll 1$. Formally, this is manifest by the fact that some matrix elements of the interaction Hamiltonian of these fields increase with increase in the sound velocity, $\propto C_S^{1/2}$.

Assuming the Mach number M to be small, we construct a canonical transformation that separates the potential and vortical motions of the liquid in the principal order in the new variables $q, p; Q, P$. We specify the transformation with the aid of the generating functional $F(q, Q; \Phi, \mu)$, which depends on the new coordinates and the old momenta[9]:

$$\rho(\vec{r},t)=\delta F/\delta\Phi(\vec{r},t), \qquad p(\vec{r},t)=\delta F/\delta q(\vec{r},t), \tag{1.9}$$
$$\lambda(\vec{r},t)=\delta F/\delta\mu(\vec{r},t), \qquad P(r,t)=\delta F/\delta Q(\vec{r},t).$$

Denoting by F_o the identity-transformation functional

$$F_o=\int(\Phi q+\mu Q)\,d^3r, \tag{1.10}$$

we represent F in the form $F=F_o+F_1$, where $F_1=F_1(q, Q; \Phi, \mu)$ is so chosen that $q=\rho$ and $p=\tilde{\Phi}$. The functional F_1 does not depend on $\tilde{\Phi}$, is bilinear in μ and Q, and is a series in powers of the variable part of the density $\rho_1=\rho-\rho_o$:

$$F_1=\rho_o^{-1}\int\left[Q\,(\nabla\mu)\,\Delta^{-1}\nabla\rho_1\right]d^3\vec{r}+\ldots \tag{1.11}$$

As an expansion parameter, we use

$$\xi=\kappa_T\rho_1/\kappa_S\rho_o\approx\sqrt{E_S/\rho_o c_S^2}\;\lambda_S/L, \tag{1.12}$$

where κ_S and κ_T are the characteristic wave vectors of sound and turbulence, E_S is the energy of the acoustic motions. Substituting (1.10) and (1.11) in (1.9), and solving the resultant relation by the iteration method (in terms of the paramenter $\xi<<1$), we get

$$q=\rho, \quad P=\Phi+\Delta^{-1}\mathrm{div}\,(\lambda\rho^{-1}\nabla\mu)=\tilde{\Phi};$$
$$\lambda=Q+\rho_o^{-1}(\nabla(\Delta^{-1}\nabla\rho_1))Q+O(\xi^2),$$
$$\mu=P+\rho_o^{-1}((\nabla P)(\Delta^{-1}\nabla\rho_1))+O(\xi^2)$$

In the new variables

$$\vec{v}_l=\nabla P, \quad \rho=q, \quad \vec{v}_t=-(\rho_o\Delta)^{-1}[\nabla\times[\nabla Q\times\nabla P]]- \tag{1.15}$$
$$-\Delta^{-1}\nabla[\nabla\times[\nabla\times[(\Delta^{-1}\nabla\rho_1)\times[\nabla P\times\nabla Q]]]]+O(\xi^2).$$

Thus, the potential motions are described only by the pair q and p; the principal contribution to the vortex motion is made by the "turbulence" variables Q and P. The last term in the expersion for $\vec{V}_t$ describes the effect of compressibility on the vortex motion. We transform in the $\vec{\kappa}$-representation in standard fashion [2] from the variables $q_\kappa, p_\kappa; Q_\kappa, P_\kappa$ to the complex conjugate variables $b_\kappa, b_\kappa^*; a_\kappa, a_\kappa^*$

$$\rho_{\vec{\kappa}}=(\rho_o\kappa/2c_s)^{1/2}(b_{\vec{\kappa}}+b_{-\vec{\kappa}}^*), \quad \tilde{\Phi}_{\vec{\kappa}}=i(c_s/2\rho_o\kappa)^{1/2}(b_{\vec{\kappa}}-b_\kappa^*),$$
$$Q_{\vec{\kappa}}=(a_{\vec{\kappa}}+a_{-\vec{\kappa}}^*)/2^{1/2}, \quad P_{\vec{\kappa}}=(a_{\vec{\kappa}}-a_{\vec{\kappa}}^*)/i\,2^{1/2}.$$

In these variables, the equations of hydrodynamics take the form

$$i\dot{a}_{\vec{k}}=\delta H/\delta a_{\vec{k}}^*, \quad i\dot{b}_{\vec{k}}=\delta H/\delta b_{\vec{k}}^*$$

with the Hamiltonian H obtained from the substitution of (1.16) and (1.17) in (1.3):

$$H=H_s+H_t+H_{st}$$

The acoustic Hamiltonian H_s has the usual form (2), in which

$$\omega_{\vec{\kappa}} = \kappa c_s, \quad V_{\vec{\kappa}_1|\vec{\kappa}_2\vec{\kappa}_3} = 2^{-1}(2\pi)^{-3/2}(c_s/2\rho_0)^{1/2}(\kappa_1\kappa_2\kappa_3)^{1/2} \times$$
$$\times \left[(\vec{n}_1\vec{n}_2 + \vec{n}_1\vec{n}_3 + \vec{n}_2\vec{n}_3) + (\gamma - 2) \right]$$

$\vec{n} = \vec{\kappa}/\kappa$, γ is the adiabatic exponent: $\gamma = c_p/c_v$. The Hamiltonian H_t is identical with the Hamiltonian (3) for the turbulence of an incompressible liquid, with

$$T_{\vec{\kappa}_1\vec{\kappa}_2|\vec{\kappa}_3\vec{\kappa}_4} = \rho_0(\vec{\varphi}_{\vec{\kappa}_1\vec{\kappa}_3}\vec{\varphi}_{\vec{\kappa}_2\vec{\kappa}_4} + \vec{\varphi}_{\vec{\kappa}_1\vec{\kappa}_4}\vec{\varphi}_{\vec{\kappa}_2\vec{\kappa}_3}), \tag{1.19}$$

$$\vec{\varphi}_{\vec{\kappa}\vec{\kappa}'} = (2\rho_0)^{-1}(2\pi)^{-3/2}\left[\vec{\kappa} + \vec{\kappa}' - (\vec{\kappa} - \vec{\kappa}')(\kappa^2 - \kappa'^2)/|\vec{\kappa} - \vec{\kappa}'|^{-2}\right]$$

In the incompressible liquid,

$$v_t(\vec{\kappa}) = \int \vec{\varphi}_{\vec{\kappa}_1\vec{\kappa}_2} a^*_{\vec{\kappa}_1} a_{\vec{\kappa}_2} \delta(\vec{\kappa} + \vec{\kappa}_1 - \vec{\kappa}_2) d^3\vec{\kappa}_1 d^3\vec{\kappa}_2 \tag{1.20}$$

In the interaction Hamiltonian H_{st} (4), the matrix elements are of the form

$$\delta_{\vec{\kappa}_1\vec{\kappa}_2|\vec{\kappa}_3\kappa_4} = (2\pi)^{-3/2}(\kappa_3\kappa_4)^{1/2}\vec{\varphi}_{\vec{\kappa}_1\vec{\kappa}_2}(\vec{n}_3 + \vec{n}_4)/2, \quad W_{\vec{q}_s\vec{\kappa}_1\vec{\kappa}_2|\vec{\kappa}_3\vec{\kappa}_4} = \tag{1.21}$$
$$= (2\pi)^{-3/2}(2\rho_0 q/c_s)^{1/2}\left[(\vec{\varphi}_{\vec{\kappa}_1\vec{\kappa}_3}\cdot\vec{n}_{\vec{q}})(\vec{\varphi}_{\vec{\kappa}_2\vec{\kappa}_4}\cdot\vec{n}_{\vec{q}}) + (\vec{\varphi}_{\vec{\kappa}_1\vec{\kappa}_4}\vec{n}_{\vec{q}})(\vec{\varphi}_{\vec{\kappa}_2\vec{\kappa}_3}\vec{n}_{\vec{q}})\right]$$

In Eq. (4), terms of the form $S^{(n)}a_1a_2^*b^{n+2}$ and $W^{(n)}a^2a^{*2}b^{n+1}$ are not written down since they are unimportant for what follows. These terms differ from Saa^*bb^* and $Wa^{*2}a^2b$ by the small factor $\xi = [(\kappa_s/\rho_0 c_s)^{1/2}b]^n$

2. STATISTICAL DESCRIPTION OF NONLINEAR TURBULENT AND ACOUSTIC FIELDS

1. Diagram Technique. For the statistical description of nonlinear fields a_κ and b_κ, we use the canonical diagram technique of Wyld, which is similar to that analyzed in Ref.3. In contrast with Ref. 3, we consider a system of two coupled fields. We introduce the graphic notation for the Green's function (which has the meaning of a linear response to the force F, f):

$$\delta^4(q-q')G_q = \langle \delta b_q/\delta F_{q'}\rangle, \quad \delta^4(q-q')g_q = \langle \delta a_q/\delta f_{q'}\rangle \tag{2.1}$$
$$G_q \sim \dashrightarrow, \quad g_q \sim \longrightarrow .$$

and the pair correlators

$$\delta^4(q-q')N_q = \langle b_q b^*_{q'}\rangle, \delta^4(q-q')n_q = \langle a_q a^*_{q'}\rangle, q = (\kappa,\omega), \delta^4(q) = \delta(\omega)\delta^3(\kappa) \tag{2.2}$$
$$N_q \sim \rightsquigarrow, \quad n_q \sim$$

These quantities satisfy the Dyson equations

$$G_q = (\omega - \omega_\kappa - \Sigma_q)^{-1}, \quad g_q = (\omega - \sigma_q)^{-1}, \tag{2.3}$$

$$N_q = |G_q|^2\Phi_q, \quad n_q = |g_q|^2\varphi_q, \tag{2.4}$$

where $\Sigma_q, \sigma_q; \Phi_q, \varphi_q$ are the sums of the corresponding irreducible diagrams.

We write down the Dyson equation for $N_{\kappa\omega}$ in the form

$$L_{\vec{k}\omega} \equiv \Sigma''_{\vec{k}\omega} N_{\vec{k}\omega} - \Phi_{\vec{k}\omega} G''_{\vec{k}\omega} = 0 \tag{2.5}$$

In the case in which the interaction is weak, only diagrams of second order in the vertices need be retained in the series for Σ''_q and Φ_q, and Eq. (2.5) can be integrated with respect to ω. Then the condition $L_k = \int d\omega L_{k\omega} = 0$ will coincide with the stationary kinetic equations for the waves:

$$0 = -\Gamma_{\vec{k}} N_{\vec{k}} + \pi \Phi_{\vec{k}} \tag{2.6}$$

where

$$\Gamma_{\vec{k}} = -\Sigma''_{\vec{k},\omega(k)}, \quad \Phi_{\vec{k}} = \Phi_{\vec{k},\omega(k)}$$

It is seen from Eq. (2.6) that physically vertex Γ_k - the "damping decrement" - has the meaning of the departure term and Φ_k the meaning of the apporoach term in the kinetic equation. The kinetic-equation approximation in a number of cases is insufficient and it is necessary to substitute the already partially summed quantities Γ_k and Φ_k in Eq. (2.6). We shall use a similar equation for the hydrodynamic pulsations of the velocity:

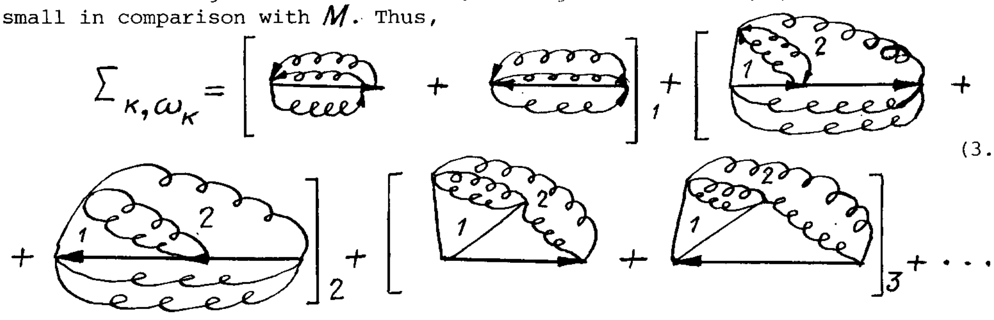
$$l_{\vec{k}\omega} = n_{\vec{k}\omega} \delta''_{\vec{k}\omega} - \varphi_{\vec{k}\omega} g''_{\vec{k}\omega} . \tag{2.7}$$

In the equations $L_{k\omega} = 0$, $l_{k\omega} = 0$, a cancellation of the longwave divergences responsible for transport takes place. The reason for this is that the transport of the elementary excitation (vortices, sound waves) by a homogeneous velocity field does not lead to a redistribution of the energy in k space.

3. DAMPING OF SOUND IN A TURBULENT MEDIUM

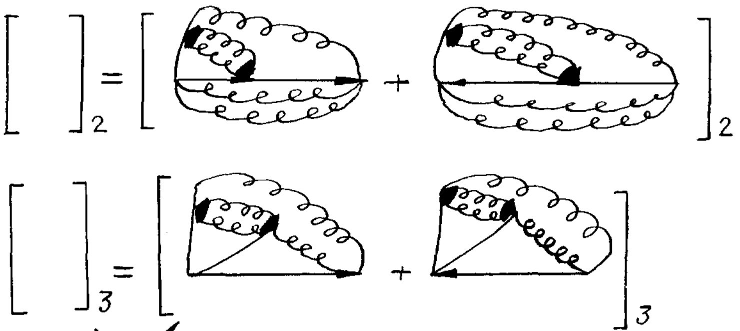
In a turbulent medium, the contribution to the damping decrement Γ_k arises both as a result of the direct absorption of the sound energy by the turbulent pulsations and from processes of sound scattering. We study the first of these mechanisms.

1. Sound absorption. We calculate the contribution to $\Sigma_{k,\omega(k)}$ made by sound absorption in processes described by the W vertices in (4). Here it suffices to take into account diagrams of the order W^2; diagrams containing W^4 and $S^n W^2$ are small in comparison with M. Thus,



(3.1)

In the calculation of $\mathrm{im}\,\Sigma_{k\omega}$ a strong cancellation takes place in the diagrams that differ in the arrow directions. As a result, the sums of each of the two diagrams in the brackets in (3.1) turn out to be of the order of $(kv_T)(kL)^{2/3} M^{3/2}$. Moreover, the sum of the entire diagram series (3.1) is still less than this quantity in terms of the parameter $(M/kL)^{1/3}$. Actually, the principal contribution to the intergration over those internal lines n_q which are not joined to an input or output $[n_q, n_q$ in diagrams (3.1)$]$ is made by the energy-containing region: $k_1, k_2 \backsim L^{-1}$; $\omega_1, \omega_2 \backsim v_T L^{-1}$. In this region, we can take in place of the diagram $[\ ]_2$ and $[\ ]_3$



(3.2)

Here

$$\backsim T_{12,34}\,\delta(\vec{k}_2-\vec{k}_4)\,\delta(\omega_2-\omega_3)$$

and differs from usual vertex

$$\backsim T_{12|34}\,\delta(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\,\delta(\omega_1+\omega_2-\omega_3-\omega_4).$$

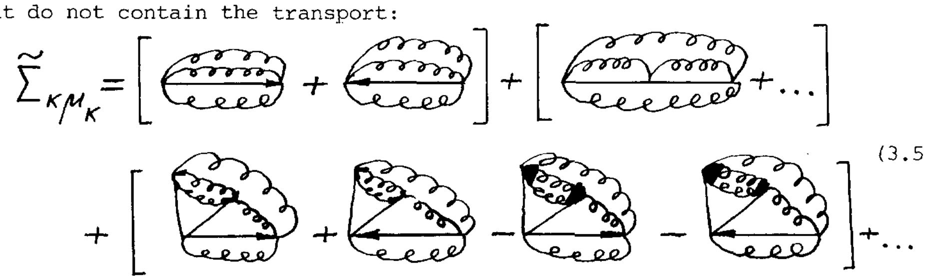
The diagrams (3.2) describe the transport of the absorbing volume by vortices of scale $k_i \simeq k_T$. It can be summed by the method discribed in [4]. (See also the review by V. L'vov, E.Kuznetsov in this ussue of Phys. Rep.) Then the Green's function $G_{k\omega}, g_{k\omega}$ takes the form

$$G_{\vec{k}\omega} = \langle [\omega - \vec{k}\vec{v} - \tilde{\Sigma}_{k,\omega-\vec{k}\vec{v}}]^{-1} \rangle_{\vec{v}}, \quad g_{\vec{k}\omega} = \langle [\omega - \vec{k}\vec{v} - \delta_{k,\omega-\vec{k}\vec{v}}]^{-1} \rangle_{\vec{v}} \tag{3.3}$$

in which $\langle\ \rangle_{\vec{v}}$ denotes averaging over the ensemble of the turbulent velocity field $\vec{V}$ at the arbitrary point $\vec{r}, t$, and we obtain for $\tilde{\Sigma}_{k\omega}$ a diagram series which must be calculated with the Kolmogorov functions

$$\tilde{n}_{k\omega} = \rho k^{2} f(\omega L/v_T (kL)^{2/3}), \quad \tilde{g}_{k\omega} = \omega^{-1} g(\omega L/v_T (kL')^{2/3}) \tag{3.4}$$

that do not contain the transport:



(3.5)

The analytic expression for the first pair of diagrams has the form

$$\Sigma''_q = \int |W_{\vec{k},\vec{k}_1,\vec{k}_2|\vec{k}_3\vec{k}_4}|^2 n_{\vec{k}_1\omega_1} n_{\vec{k}_2\omega_2} n_{\vec{k}_3\omega_3} g''_{\vec{k}_4\omega_4} [\delta(\vec{k}+\vec{k}_1-\vec{k}_2+\vec{k}_3-\vec{k}_4)\delta(kc_s+\omega_1-\omega_2+\omega_3-\omega_4) - \delta(-\vec{k}+\vec{k}_1-\vec{k}_2+\vec{k}_3-\vec{k}_4)\delta(-kc_s+\omega_1-\omega_2+\omega_3-\omega_4)] \times$$
$$\times d^3\vec{k}_1 d\omega_1 d^3\vec{k}_2 d\omega_2 d^3\vec{k}_3 d\omega_3 d^3\vec{k}_4 d\omega_4 .$$

The basic contribution to this expression is made by integration over the region of scales $k_i \simeq k_T$, where the characteristic Kolmogorov frequency $v_T/L(k_T L)^{2/3}$ is of order kc_s. We assume that k_T falls in the inertial $L^{-1} < k_T < L^{-1}Re^{3/4}$, or

$$ML^{-1} < k_s < ML^{-1}Re^{1/2} \tag{3.6}$$

With account of this circumstance, it is easy to estimate

$$\Gamma_K = -\operatorname{Im}\Sigma_{K,\omega_K} \approx v_T M^2/L. \tag{3.7}$$

We obtain this result qualitatively by considering the change in energy of turbulent pulsations with a characteristic frequency of motion:

$$\omega_T \approx (\kappa_T L)^{2/3} v_T/L \approx \omega_3 = \kappa c_S. \tag{3.8}$$

In the field of a sound wave of intensity E_S, the density of the liquid oscillates with amplitude $\delta\rho_S = \rho_0 \sqrt{E_S/\rho_0 c_S^2}$ which produces changes in the velocity v by an amount $\delta v_T \approx v_T \delta\rho_S/\rho_0$ within the time of single period ω_S^{-1}. The corresponding change in the energy of the vortex motion has the form

$$\delta E_T \approx E_T\left[\delta\rho_0/\rho_0 + \delta\rho_S^2/\rho_0^2 + \ldots\right]$$

whence

$$dE_S/dt = d\overline{\delta E_T}/dt \approx \omega_S \overline{\delta E_T} \approx \omega_S E_T(\delta\rho_S/\rho_0)^2 \approx \omega_S E_S E_T/\rho_0 c_S^2$$

Then, substituting $E_T \approx \rho v_T^2 (\kappa_T L)^{-2/3}$ and expressing κ_T in terms of κ_S we get, with the help of the relation $\omega_T \approx \kappa_S c_S$,

$$dE_S/dt \approx -v_T M^2 E_S/L,$$

which corresponds to the estimate (3.7).

We compare the sound absorption by turbulence (3.7) with its damping due to viscosity and heat conduction $\Gamma_0 \approx \nu\kappa_S^2$ [12]. At small κ_S, the damping (3.7) predominates and is comparable with Γ_0 at $\kappa_S = \kappa_0$, where

$$\kappa_0^2 \approx v_T M^2/\nu L = (M^2/L^2)(v_T L/\nu) = M^2 \mathrm{Re}/L^2. \tag{3.9}$$

Thus, the damping due to turbulence predominates over the whole range (3.6).

4. STUDY OF THE SOUND OF TURBULENCE

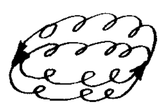
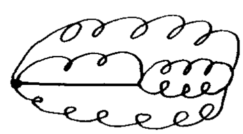
1. Study of "transparent turbulence". We use the kinetic equations for sound (2.6), to which we add the phenomenological component $\dot{N}_K^{\circ}$, which describes the losses due to radiation of sound from the turbulent volume:

$$\dot{N}_{\vec{K}}^{\circ}/2 = -\Gamma_K N_K + \pi\Phi_{\vec{K},\omega_{\vec{K}}}. \tag{4.1}$$

If the volume occupied by the turbulence is so small that it is acoustically transparent, then the losses due to damping Γ_K can be neglected. Then we get for the energy flux density (in a unit interval of frequency per unit time per unit volume)

$$I_{\vec{K}}(\omega) = 4\pi\kappa^3 N_{\vec{K}} = 4\pi^2\kappa^3\Phi_{\vec{K},\omega(\vec{K})}. \tag{4.2}$$

The principal sequence of diagrams for Φ is proportional to W^2 and does not contain sound lines:

$\Phi_{K\omega_K} =$  $+$  $+ \cdots$ (4.3)

It is not difficult to understand that this sequence is summed to the fourth correlator of the turbulent velocity:

$$\Phi_{\vec{K},\omega_{\vec{K}}}=(2\pi)^{-3}2\rho_0\kappa c_s^{-1}\int I^{zzzz}_{1,2,3,4}\delta(\vec{K}+\vec{K}_1+\vec{K}_3)\delta(\vec{K}+\vec{K}_2+\vec{K}_4)\delta(\omega_{\vec{K}}+\omega_T+$$
$$+\omega_3)\delta(\omega_{\vec{K}}+\omega_2+\omega_4)d^3\vec{K}_1\,d\omega_1\,d^3\vec{K}_2\,d\omega_2\,d^3\vec{K}_3\,d\omega_3\,d^3\vec{K}_4\,d\omega_4\,. \tag{4.3}$$

The z axis is oriented along $\vec{K}$ and

$$\langle v^z_{\vec{K}_1\omega_1}v^z_{\vec{K}_2\omega_2}v^z_{\vec{K}_3\omega_3}v^z_{\vec{K}_4\omega_4}\rangle=I^{zzzz}_{1,2,3,4}\delta^4(q_1+q_2+q_3+q_4)\,. \tag{4.4}$$

Estimate of this expression on the Kolmogorov spectrum of isotropic turbulence gives

$$\Phi_{\kappa,\omega(\kappa)}\approx\rho_0 v_T^2 L^3 M^2(M/\kappa L)^{3/2}. \tag{4.5}$$

In correspondence with (4.2), this expression determines the spectral content of the sound radiated acoustically by the transparent turbulence:

$$I_s(\omega)=\rho_0 v_T^2 M^5(M/\kappa L)^{7/2}. \tag{4.6}$$

We note that the principal contribution to the emission of sound at a frequency is made by vortices of scale $1/\kappa_T$, for which the characteristic angular frequency is $\omega_T\approx v_T L^{-1}(\kappa_T L)^{2/3}$ is of the order of ω_s. The most intensively radiated is sound of vortices of the energy content scale L at frequency v_T/L.

We obtain the estimate (4.6) qualitatively. In the decay of the vortex of scale $1/\kappa_T$ in time $1/\omega_T$, density pulsations develop which produce changes in the volume of the vortex V at a rate

$$\delta\rho_T\approx\rho_0 v_T^2(\kappa_T)/c_s^2\approx\rho_0 M^2(\kappa_T L)^{2/3}$$
$$\dot{V}=dV/dt\approx V\omega_T\,\delta\rho_T/\rho_0\approx(v_T/L)(M^2\kappa_T^{-3})$$

The pulsating volume causes radiation of sound with intensity

$$I\approx\rho_0\omega_s^2\,\dot{V}^2/c_s^2$$

(see Ref.12). Substituting there the estimate for $\dot{V}$, multiplying the result by the number of vortices per unit volume, κ_T^{-3}, and expressing κ_T in terms of κ_s according to the formula $\omega_T\approx\kappa_s c_s$, we obtain the formula (4.6) for $I(\omega)=dI/d\omega\approx I/\omega$.

The total flow is determined by the integral of the expression (4.6) with respect to ω:

$$I\approx\rho_0 v_T^3 M^5/L. \tag{4.7}$$

The estimate (4.7) for the integrated intensity was obtained previously by other methods [6].

2. The sound spectrum in a non-transparent medium. It is determined by Eq.(4.1), in which it is necessary to neglect losses due to radiation:

$$N_{\vec{K}}=\pi\Phi_{\vec{K},\omega_{\vec{K}}}/\Gamma_{\vec{K}}\,. \tag{4.8}$$

This spectrum is determined by the equilibrium between the process of sound radiation (4.5) and the reverse process of its absorption (3.7). Scattering processes lead only to isotropization and, of course, do not affect the spectral composition of $I_s(\omega)$ (4.6). Thus,

$$N_\kappa\approx\rho_0 v_T L^4(M/\kappa L)^{13/2} \tag{4.9}$$

This expression is valid in the interval (4.4) for κ_S at which κ_T falls in the inertial interval. The sound energy density per unit frequency interval is of the form

$$E_S(\omega_\kappa) = 4\pi\kappa_S^3 N_\kappa \approx \rho_0 v_T L M^3 (M/\kappa L)^{7/2} \tag{4.10}$$

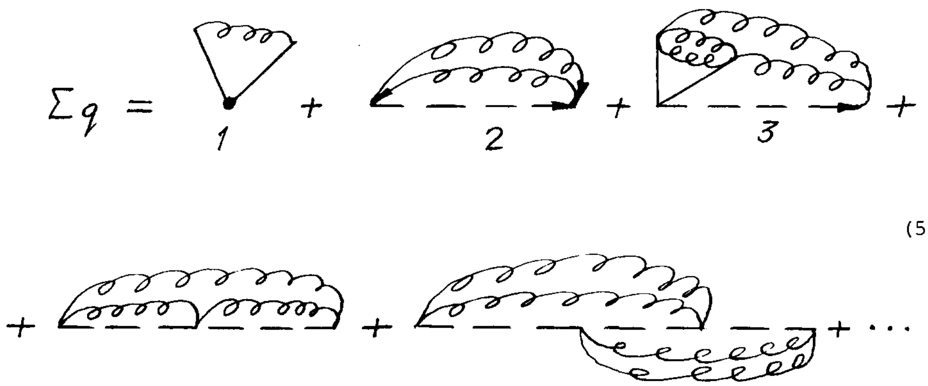
The total "equilibrium" sound energy density

$$E_S = \int E_S(\omega)\, d\omega \approx \rho_0 v_T^2 M^3 \tag{4.11}$$

is smaller by a factor of M^3 than the energy density of the turbulent pulsations.

5. SCATERRING OF SOUND IN A TURBULENT MEDIUM

1. Scattering of a plane wave. The diagrams $\Sigma_{\kappa\omega}$, which describe the scattering of sound by turbulent pulsations, have the following structure:



(5.1)

Diagrams with a single vertex S (for example,1) describe the Doppler shift of the frequency $\kappa\langle v_T\rangle$ and are equal to zero in a system of coordinates where the liquid is at rest on the average.

In the region $\kappa_S L < 1$, diagrams of the series (5.1) make no contribution to $\Sigma_{\kappa\omega}$ because of the laws of energy-momentum conservation, under the assumption that there are no turbulent pulsations with scale $\kappa L < 1$.

At $1 < \kappa_S L < M^{-1}$, it is necessary to take into account only diagrams of the order of S^2: numbers 2, 3, etc., from series (5.1). It is not difficult to see that the sum of this series has the form

$$\Sigma^{(2)} = (1/32\pi^3) \iint I^{\alpha\beta}_{\kappa''\omega''} G_{\kappa'\omega'} (n^\alpha + n^{\alpha'})(n^\beta + n^{\beta'}) \times \tag{5.2}$$

$$\times\, \delta(\vec{\kappa} - \vec{\kappa}' - \vec{\kappa}'')\, \delta(\omega - \omega' - \omega'')\, d^3\kappa'\, d^3\kappa''\, d\omega'\, d\omega'', \; n^\alpha = \kappa^\alpha/|\vec{\kappa}|.$$

Here $I^{\alpha\beta}$ is the pair correlator of the velocity:

$$\langle v_{\kappa\omega}{}^{\alpha}\, v_{\kappa'\omega'}{}^{\beta} \rangle = I_{\vec{\kappa}\omega}^{\alpha\beta}\, \delta(\omega' + \omega)\, \delta(\vec{\kappa} + \vec{\kappa}'). \tag{5.3}$$

In the calculation of the correction to the sound dispersion law $\Delta\omega_\kappa = \mathrm{Re}\,\Sigma_{\kappa,\omega(\kappa)}$, it is necessary to substitute the complete Green's function in (5.2). As a result,

$$\Delta\omega_k = k v_T M^2 \ln\left[(kL)^2 M\right]. \tag{5.4}$$

We now estimate $\operatorname{Im}\Sigma_{k,\omega(k)}$ on the mass shell, substituting the bare Green's function in (5.2). The principal contribution to the integration over k'' is made by the energy-containing region $k'' \approx L^{-1}$. As a result,

$$\Gamma_k = -\operatorname{Im}\Sigma_{\vec{k},\omega_{\vec{k}}} \approx k v_T (kLM). \tag{5.5}$$

In the scattering processes, the frequency and, consequently, the energy of the sound is conserved (the inelasticity is small in the parameter M), broadening of the Green's function and of the pair correlator in ω is due to loss of phase coherence. Strictly speaking, the time $\tau = \Gamma^{-1}$ is the time of phase correlation and is not at all connected with the sound energy dissipation in the turbulent medium.

We now obtain the estimate of (5.5) qualitatively, considering the sound scattering from vortices of scale k_T^{-1}. In a single scattering act, the phase of the wave is altered, because of the Doppler frequency shift, by an amount

$$\Delta\varphi \approx M k_S / k_T (k_T L)^{1/3}$$

where $v_T/(k_T L)^{1/3}$ is the circumferential velocity of the vortex. In a time τ, $N = k_T c_S \tau$ acts of scattering take place and the phase change increases by $\sqrt{N}$. The time of loss of the correlations in scattering from vortices of scale k_T^{-1} is determined from the condition $\Delta\varphi\sqrt{N} = 1$:

$$\Gamma_{kS}(k_T) = \tau^{-1}_{k_S}(k_T) \approx k_S^2 v L M/(k_T L)^{5/3}. \tag{5.6}$$

The principal contribution to $\Gamma_k(k_T)$ is made by vortical motion with characteristic scale $k_T \sim L^{-1}$, which corresponds to the estimate (5.5).

In the shorter-wave region, when $k_S LM > 1$, the physical picture of the scattering of the wave is changed, since the Doppler phase shift by a single vortex of energy-containing scale exceeds π. Formally, this is expressed by the fact that the diagrams (5.1) with two, three and more vertices S turn out to be of the same order of magnitude. The principal contribution to $\Sigma_{k\omega}$ is made by those diagrams for which the external momentum is carried along the "backbone" of the sound Green's function, oriented from left to right, while integration is carried out over the remaining turbulent lines n_q, g_q in the energy-containing region $kL \approx 1$. This sequence is summed by the method described in Ref.4. As a result,

$$G_{\vec{k}\omega} = \left\langle \left[\omega - \omega_k - \vec{k}\vec{V} + i\delta\right]^{-1} \right\rangle_{\vec{V}}. \tag{5.7}$$

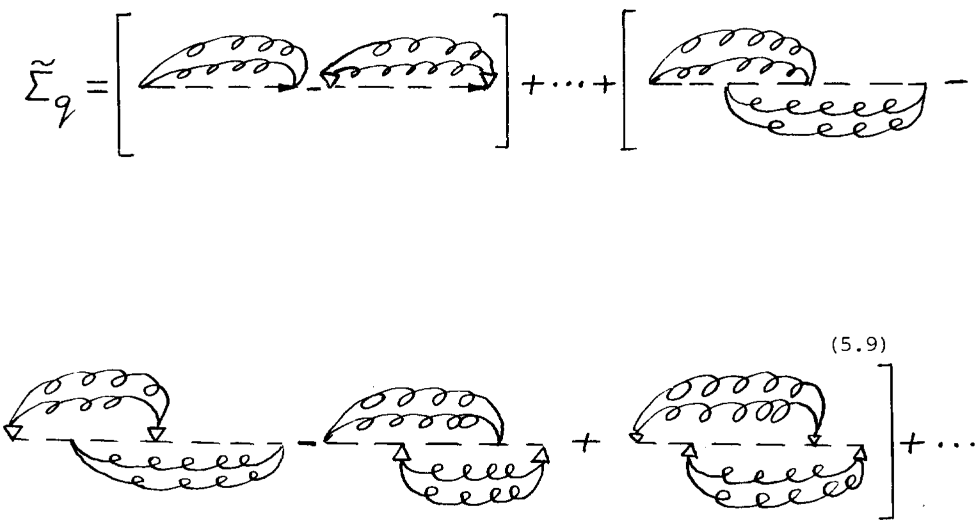
This Green's function describes the transport of the field of the sound wave as a whole with random velocity.

The width in ω determines the lifetime of the phase correlations $\tau_{cor}^{-1} \approx k_S v_T$. However, $\Lambda_{cor} \neq c_S \tau_{cor}$, because the sound wave vector is conserved in the approximation (5.7) corresponding to transport of the spatially homogeneous velocity field. Therefore, the correlation function $K(k_T O)$ is proportional to $\exp(i k_S R)$ and the spatial correlation of the phases is not destroyed.

2. Spatial correlation length. At $k_S LM \ll 1$, the distance at which the phase of the wave falls off by π,

$$\Lambda \approx c_S \tau_{cor} \tag{5.8}$$

is greater than the energy-conserving scale L and is the correlation length Λ_{cor}. At $kLM > 1$, we cannot assume the quantity (5.8) to be the correlation length, as has already been noted above, since the almost homogeneous transport of the scale L , making the principal contribution to τ_{cor}, does not lead to a destruction of the spatial correlations. We refine the approximation (5.7) seeking $G_{\kappa\omega}$ in the form (3.2). The diagram series for $\tilde{\Sigma}$ has the form



(5.9)

where the vertex-triangle is $S_{12,34}\delta(\kappa_2-\kappa_4)\,\delta(\omega_2-\omega_4)$ and differs from the usual vertex-point, which has the form $S_{12,34}\,\delta(q_1-q_2-q_3-q_4)$. Subtraction leads to cancellation of terms in Σ from the region of integration over $\kappa_T < \varkappa$ which is determined from the condition

$$\varkappa c_s \approx \Sigma''_{\vec{\kappa}\,\omega(\kappa)}. \tag{5.10}$$

With account of this, estimate of Σ from the first bracket gives

$$\tilde{\Sigma}_{\kappa,\omega(\kappa)} \approx (\kappa_s v_T)^2/(\varkappa L)^{2/3}\,\tilde{\Sigma}_{\kappa,\omega(\kappa)}. \tag{5.11}$$

All the remaining diagrams here are of the same order of magnitude. Thus,

$$\tilde{\Sigma}_{\kappa,\omega(\kappa)} \approx (\kappa v_T)(\kappa L M)^{-1/4}. \tag{5.12}$$

This contribution to $\tilde{\Sigma}$ arises because of the spatial inhomogeneity of the velocity field and therefore leads to a destruction of the spatial correlations:

$$\Lambda_{cor}(\kappa) \approx c_s\,\Sigma''^{-1}_{\kappa,\omega(\kappa)} \approx L\,(\kappa L M)^{-3/4}. \tag{5.13}$$

We note that $\Lambda_{cor} \approx \varkappa^{-1}$, where $\varkappa$ is determined from (5.10). Thus, Λ_{cor} is of the order of the size of the vortices which make the principal contribution in the destruction of the spatial correlations.

We now obtain an estimate for Λ_{cor} qualitatively. It is obvious that vortices with size $\kappa_T\Lambda_{cor} < 1$ do not destroy the phase correlations at a distance Λ_{cor}. Their effect reduces to the uniform transport of the sound field in the volume Λ^3_{cor}, while the shortwave vortices with $\kappa_T\Lambda_{cor} > 1$ weakly disrupt the correlations see (5.6) . Therefore, we can understand why the basic contribution to Λ_{cor} is made by vortices with size $\kappa_T\Lambda_{cor} \approx 1$.

From the qualitative estimate of (5.6), we obtain the following for the correlation length:

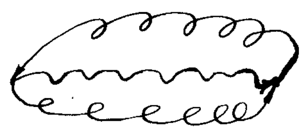
$$\Lambda_{cor}(\kappa_1) = c_s \tau_s(\kappa_T) = L(\kappa_T L)^{5/3}/M^2(\kappa_S L)^2$$

Substituting Λ_{cor}^{-1} as κ_T and solving the resultant equation relative to Λ_{cor}, we obtain the desired estimate (5.13).

3. Evolution of acoustic packet in direction. With the help of the nonstationary kinetic equation

$$dN_\kappa / 2dt = -\Gamma_\kappa N_\kappa + \pi \Phi_{\kappa,\omega(\kappa)} \tag{5.14}$$

we obtain the evolution of the acoustic packet in the linear approximation from the intensity of the sound. At $\kappa_S L < 1$, there is no sound scattering at all; in the range $M < \kappa_s LM < 1$, as has already been noted, we can limit ourselves to diagrams that are quadratic in the vertices S. Using the expression (5.2) for Γ_κ and summing the diagram series for $\Phi_{\kappa\omega}$ in analogous fashion,

$\Phi_{\kappa\omega} =$  $+ \ldots,$

we obtain

$$dN_{\vec{\kappa}}/dt = (4\pi)^{-2} \int I^{\alpha\beta}_{\vec{\kappa}''\omega''} (n^\alpha + n^{\alpha'})(n^\beta + n^{\beta'})(N_{\vec{\kappa}} - N_{\vec{\kappa}'}) \times$$
$$\times \delta(\vec{\kappa} - \vec{\kappa}' - \vec{\kappa}'')\delta(\omega_\kappa - \omega_{\kappa'} + \omega'')\, d\kappa' d\kappa'' d\omega''. \tag{5.15}$$

This equation was discussed previously in the work of Krasilnikov and Pavlov[8]. For isotropic turbulence in the approximation $M \ll 1$, this can be simplified:

$$dN_{\vec{\kappa}}/dt = (4\pi)^{-2} \int I_{\kappa''\omega''}(4 - \kappa''^2/\kappa\kappa')\,(N_{\vec{\kappa}} - N_{\vec{\kappa}'}) \times$$
$$\times \delta(\vec{\kappa} - \vec{\kappa}' - \vec{\kappa}'')\delta(\omega_\kappa - \omega_{\kappa'} + \omega_{\kappa''})\, d\vec{\kappa}' d\vec{\kappa}'' d\omega''. \tag{5.16}$$

At the begining of this section, in the calculation of the damping decrement of a plane wave in scattering processes, it was shown that the principal contribution to Γ_κ is made by large-scale vortices, which lead to scattering at small angles of the order of $(\kappa L)^{-1}$. Therefore, we can show that the diffusion approximation is valid over the angles [8], i.e., the process of scattering at large angles is the result of small angle scattering in stages. However, analysis of Eq. (5.16) shows that the diffusion approximation over the angles is valid if the sound wavelength is greater than the internal scale of the turbulence. The fact is that the contribution of the scattering at small angles $\Delta\theta$ and the evolution of the acoustic packet of width $\Delta\kappa_S$ is strongly suppressed (as a result of the scattering, the wave does not emerge from the packet) if $\Delta\theta < \Delta\kappa_S/\kappa_S$. Therefore, scattering from vortices with $\kappa_1 L > 1$ begins to play a role in the inertial interval. As a result, the evolution is essentially determined by the scattering at angles $\Delta\theta$ of the order of the width of the acoustic packet: $\Delta\theta \approx \Delta\kappa_S/\kappa_S$. Formally, all this means that a strong cancellation takes place in the integral (5.16) in the region $\kappa_T'' < \Delta\kappa_S$ and the fundamental contribution is made by $\kappa_T'' \simeq \Delta\kappa_S$. With account of this, the damping decrement $\Gamma_{\kappa_S}(\kappa_S)$ of a packet of width $\Delta\kappa_S$ turns out to be of the order of

$$\Gamma_{\kappa S}(\Delta\kappa_S) \approx \kappa_S^2 v L M (\Delta\kappa_S L)^{-5/3}. \tag{5.17}$$

This result is easily obtained from qualitative considerations by considering scattering from vortices of scale $\kappa_T \approx \Delta\kappa_S$. Taking it into account that the peripheral speed in these vortices is $v_{\kappa_T} = v_T(\kappa_T L)^{-1/2}$, it is not difficult to obtain an estimate for the scattering angle from a single vortex: $\Delta\theta(\kappa_S) \approx M(\kappa_T L)^{-1/2}$. By virtue of the random character of the scattering some $\kappa_T^2/(\kappa_T \Delta\theta(\kappa_S))$ acts are necessary for scattering at an angle of the order of κ_T/κ_S. Then the length of the path of the sound relative to the scattering processes from the vortices κ_T has the form

$$\Lambda_{is}(\kappa_T) \approx \kappa_T/\kappa_S^2(\Delta\theta(\kappa_S))^2 \approx \kappa_T(\kappa_T L)^{2/3}/\kappa^2 M^2 \tag{5.18}$$

Its corresponding damping decrement, $\Gamma_{\kappa_S}(\Delta\kappa_S) = c_S \Lambda^{-1}(\kappa_T)$, is identical with the estimate (5.17). It is clear from this consideration that we must take L^{-1} as κ_T if $\Delta\kappa_S < L^{-1}$. Then (5.17) is identical with the damping decrement of a plane wave (5.5). At $\Delta\kappa_S > L^{-1}$, $\kappa_T \simeq \Delta\kappa_S$. For broad angle packets $\Delta\kappa_S \simeq \kappa_S$ and

$$\Gamma(\kappa_S) \approx \kappa_S v_T M(\kappa_S L)^{-2/3}. \tag{5.19}$$

This estimate determines the isotropization time of the packet. Its corresponding path length determines the distance Λ_{is} over which the direction of propagation changes by an angle of order π:

$$\Lambda_{is} \approx L/M^2(\kappa_S L)^{1/3} \tag{5.20}$$

if $l_0 \Delta\kappa > 1$ (l_0 is the internal scale), then, in place of (5.20) we have

$$\Lambda_{is} \approx L M^2 (L/l_0)^{1/3} \approx L M^{-2} \mathrm{Re}^{-1/4}.$$

Here and in (5.20), we have assumed that the isotropization length is less than the viscous damping length.

4. Criterion for the transparency of the turbulent layer. In propagating through a turbulent layer, sound is chiefly absorbed by two mechanisms; in the region $M < \kappa_S L < M\,\mathrm{Re}^{1/2}$, as a consequence of direct absorption of the sound by the turbulence, with decrement $\Gamma_\kappa = v_T L^{-1} M^2$; in the region $\kappa_S L > M\,\mathrm{Re}^{1/2}$, because of viscosity and thermal conductivity of the medium. If $\kappa_S L < 1$, then the sound is propagated in a straight line and a layer of thickness

$$\Lambda > \Lambda_{tr} = c_s/\Gamma_{is} = L M^3 (\text{at } \kappa_S L < M\,\mathrm{Re}^{1/2})$$

$$\Lambda_{tr} = L/M(\kappa_S L)^2 (\text{at } \kappa_S L > M\,\mathrm{Re}^{1/2})$$

turns out to be opaque because of the sound absorption.

At $L^{-1} < \kappa < l_0^{-1}$ it is necessary to take into account processes of elastic scattering of the sound, which lead to a random walk of the phonons in the turbulent medium. After traveling a path $\Lambda_{tr} \gg \Lambda_{is}$, the phonon moves away from the initial point to a distance of the order of

$$\Lambda_{is}(\Lambda_{dis}/\Lambda_{is})^{1/2} \approx (\Lambda_{is}\Lambda_{dis})^{1/2}.$$

Thus the turbulence will be opaque if

$$\Lambda > \Lambda_{tr} = (\Lambda_{is}\,\Lambda_{dis})^{1/2} = L\,/\,M^{-5/2}(\kappa L)^{1/6}\ (\text{at } 1<\kappa L<M Re^{1/2})$$
$$\Lambda_{tr} = L\,/\,M^{3/2}(\kappa L)^{7/6}\cdot\ (\text{at } \kappa L > M Re^{1/2})$$

6. INTERACTION OF SOUND WITH SOUND IN A TURBULENT MEDIUM

In the study of the evolution of the sound field, we have up to now neglected the interaction of sound with sound (SS-interaction) in comparison with the interaction with hydrodynamic turbulence. The characteristic inverse time of the SS-interaction Γ_{SS} is easily estimated from the kinetic equation for the sound:

$$\Gamma_{SS} \approx N(\kappa_S)\,\kappa_S^5\,\rho_0^{-1} \approx \kappa_S\, c_S E_S/\rho_0 c_S^2 \qquad (6.1)$$

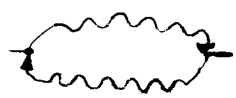
at low sound intensities, Γ_{SS} is small.

For isotropic acoustic turbulence, the approximation used above (which is linear in the sound amplitude) is valid if Γ_{SS} is less than the sound damping decrement due to the turbulence $\Gamma_{dis\,S}$ (5), i.e.,

$$N_\kappa \kappa^S/\rho_0 < v M^2/L \qquad E_S < \rho_0 v_T^2 M/\kappa L \qquad (6.2)$$

where E_S is the sound energy density.

In the opposite case, the SS-scattering is decisive. Limiting ourselves to the first diagrams for Σ_K and Φ_K:

$\Sigma_K =$  $\Phi_K =$ 

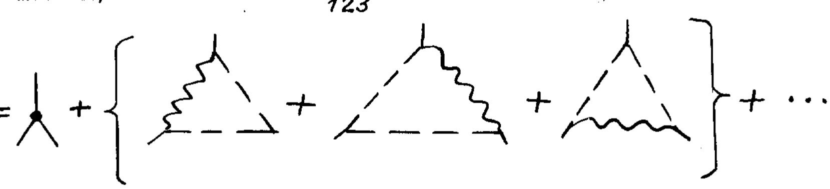
we obtain the kinetic-equation approximation for the sound [13]. This approximation is valid if the subsequent diagrams, which renormalize the vertex of the interaction, are small. For sound with dispersion, propagating in a nonturbulent medium [3], we have

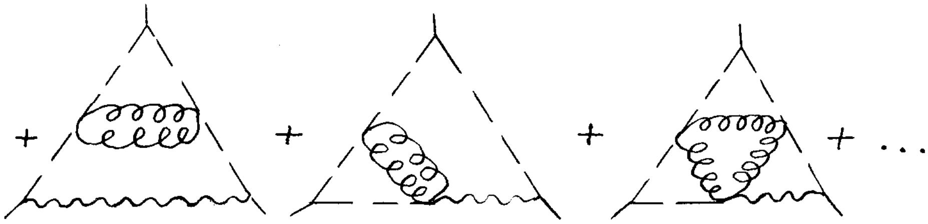
$\Gamma_{123}^{SS} =$ 

For an acoustic packet of width $\Delta\kappa$, calculation of the diagrams yields

$$\Gamma_{123} = V_{123}\,\left(1 + \Gamma_{SS}/\omega''(\Delta\kappa)^2 + \ldots\right)$$

The well-known criterion for the applicability of the kinetic equation for sound then follows: $\Gamma_{SS} < \omega''(\Delta\kappa)^2$. It can be shown that the length of the SS-interaction, $\Lambda_{SS} = c_S\,\Gamma_{SS}^{-1}$ should be large in comparison with the correlation length for the phase: $\Lambda_{cor} \approx c_S/\omega''(\Delta\kappa)^2$. The phase mismatch of the waves in the packet arises from the fact that waves with different κ propagate with different group velocities $\Delta v_{gr} = \omega''(\Delta\kappa)^2$.
In turbulent medium, the series for Γ_{123}^{SS} has the following form:

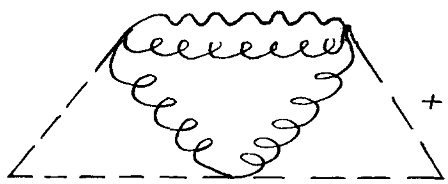
$\Gamma_{123}^{SS} =$ 



At $\kappa L M < 1$, the diagrams wich contain the turbulent lines have an additional smallness parameter: $(\kappa_s L M)^2$. Calculation of the diagram in the curly brackets with substitution of the solid lines $N_{\kappa\omega}$ and $G_{\kappa\omega}$, which take into account the effect of the turbulence, yields the following criterion for the applicability of the kinetic equation:

$$\Gamma_{ss} \ll \Gamma \approx \kappa_s v \dot{\kappa}_s L M. \qquad (6.3)$$

In the range $\kappa_s L M \gg 1$, the situation becomes more complicated and, as usual, it is necessary to take into account the whole series of diagrams in powers of the hydrodynamic velocity. It is also impossible to establish the fact that the role of hydrodynamic turbulence reduces to the renormalization of the functions $N_{\kappa\omega}$, $G_{\kappa\omega}$ and the vertex of the SS-interaction. For example, diagrams of the type

$\Sigma_{\kappa\omega}$ =


are important. Nevertheless, the whole series of the "kinetic" equation can be summed by the method described in Ref.[4], through transition to the "randomly moving reference system", which eliminates from the Green's function the Doppler shift from vortices with scale greater than Λ_{cor}. As a result, we obtain the following criterion of applicability of the kinetic equation for sound in a turbulent medium in the range $\kappa_s L M > 1$:

$$\Gamma_{ss} < \Sigma \approx \frac{c_s}{L} (\kappa L M)^{3/4}. \qquad (6.4)$$

The inequalities (6.3) and (6.4) have a simple meaning - the length of the interaction should be large in comparison with the length of the phase correlation. In this case, the phases have time to become stochastic and the kinetic equation is valid. We emphasize that the width of the acoustic packet does not enter into the criteria (6.3) and (6.4). The randomization of the phase in a turbulent medium at a distance Λ_{cor} takes place even for a single wave.

It is of interest to express the criteria (6.3) and (6.4) in terms of the energy density of the acoustic fiel:

$$E_s < (\rho_0 v_T^2) \kappa L, \quad M < \kappa L M < 1, \qquad (6.5)$$

$$E_S < (\rho_0 v_T^2)(\kappa L M)^{-1/4} M^{-1}, \quad \kappa L M > 1. \quad (6.5)$$

In conclusion, we discuss the problem of the spectrum of the acoustic turbulence excited by an external source. If the sound intensity satisfies the criterion (6.2), then no energy redistribution over the spectrum takes place, since the sound attenuation due to turbulence exceeds the inelastic scattering of sound (6.1). In this case, the acoustic spectrum outside the pumping region will be "equilibrium": $E(\omega_s) \sim \omega_s^{-7/6}$. If the sound intensity exceeds the threshold (6.2), but all the criteria (6.5) are satisfied, then the interaction of the sound with sound can be studied in the approximation of the kinetic equation, and the sound interaction with the turbulence leads to isotropization of the acoustic spectrum with a time $\tau^{-1} \approx \kappa_S v M (\kappa_S L)^{-2/3}$, see (5.19). In this case, the isotropic spectrum of Zakharov-Sagdeev is established $E_\kappa \sim \kappa^{-3/2}$ as the exact solution of the kinetic equation. At higher intensities, the criterion (6.5) is violated and we fall into the region of strong acoustic turbulence.

Footnote:

1) Another viewpoint is expressed in Ref.9, the authors of which assume that the Zakharov-Sagdeev spectrum has a much wider region of existence than the region of applicability of the kinetic equation.

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Table 1. Characteristic frequencies describing the interaction of sound with hydrodynamic turbulence

Frequency of the process	Region of applicability		
	$L^{-1}<\kappa_S<(LM)^{-1}$	$(LM)^{-1}<\kappa_S<l_0$	$\kappa_S > l_0$
τ_d^{-1}	$\kappa_S v_T(\kappa_S LM)$	$\kappa_S v_T$	
c_S/Λ_{cor}	$\kappa_S v_T(\kappa_S LM)$	$\kappa_S v_T(\kappa_S LM)^{1/4}$	
τ_{is}^{-1}	$v_T L^{-1}(\kappa_S L)^{1/3}$		$v_T L^{-1} M Re^{-1/4}$
τ_{diff}^{-1}	$v_T L^{-1} M^3(\kappa_S L)^{1/3}$	$v_T \kappa^{-1} M^{8/3}(\kappa_S LM)^{-1/6}$	$v_T L^{-1} M^2(Re^{-1}\kappa_S LM)^{1/2}$

Note. For comparison, we give the sound damping decrement: $\tau_{dis} \approx v_T L^{-1} M^2$ at $ML^{-1} < \kappa_S < ML^{-1} Re^{1/2}$,$\tau_{dis} \approx \nu/\kappa_S^2$ at $\kappa_S > ML^{-1} Re^{1/2}$.