Spin diagram technique for nonequilibrium processes in the theory of magnetism

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A new diagram technique developed for spin operators is as close as possible to that for Bose operators and borrows its standard graphical notation. The kinematic relationship between S^z and the transverse components $S^{\pm}$ is used in calculating $\langle S^z \rangle$ and the correlation functions of the longitudinal spin components, $K^{zz}_{\omega\mathbf{k}}$, K^{zzz}, etc. It thus becomes possible to carry out a more profound summation and to derive as a zeroth approximation an improved version of the self-consistent-field approximation which incorporates magnons and which leads to the correct temperature dependence of $\langle S^z \rangle$ and of the magnon spectrum $\omega_{\mathbf{k}}(T)$ at low temperatures. This new diagram technique is analogous to that of Keldysh for nonequilibrium processes. It is intended for describing the kinetics of magnons in states which are not necessarily near thermodynamic equilibrium. In particular, the expressions derived for $\langle S^z \rangle$ and $K^{zz}_{\mathbf{k}\omega}$ are functionals of the dispersion law at $T=0$, $\omega_{\mathbf{k}}(0)$ and of the magnon population numbers $n_{\mathbf{k}}$, which constitute a solution of the kinetic equation. The contribution of the dispersion part of the magnon spectrum to the expression for $K^{zz}_{\omega\mathbf{k}}$ gives K^{zz} a nonvanishing width along ω, i.e., gives rise to a damping of longitudinal spin correlations. The damping of nondispersive magnons is calculated. Their Green's function is shown to be Gaussian, rather than Lorentzian. The magnetic dipole interaction can be incorporated in this diagram technique. This technique can easily be generalized to the case of multiple-sublattice magnetic materials.

INTRODUCTION

There are three basic methods for describing spin waves in magnetic materials.[1–3] In the first method the spin operators are represented in terms of Bose operators in some way or other. Unfortunately, all versions of this method—those using the Holstein-Primakoff,[1–3] Dyson-Maleev,[1–3] or Bar'yakhtar-Yablonskiĭ[4] representations—suffer from a fundamental restriction on the effectiveness of the method at low temperatures. The second method, which is claimed valid for describing magnetic materials at all temperatures, involves constructing a chain of coupled equations for the spin Green's functions, and various techniques are used to close the chain.[2] This method, however, is not a systematic calculation method which would allow one to control the nature of the assumptions and to regularly calculate corrections to the first aproximation. These purposes are served in the third of these three methods, which is the diagram method proposed by Vaks, Larkin, and Pikin.[3] They constructed for a Heisenberg ferromagnet a temperature diagram technique which makes it possible to construct successive approximations in the dimensionless interaction volume R^{-3}. The Wick theorem for the spin operators was proved in the same studies; this proof made it possible for Izyumov and Kassan-Ogly to construct a temperature diagram technique directly in terms of spin operators.[5] Several interesting and important results have been derived by this diagram technique,[3,6] but it has not been adopted widely. The reasons are both the specific difficulties of these diagram techniques[3,5] and the unsuccessful graphical notation, which makes it difficult to perceive and establish analogies. At nearly all temperatures the magnons are well-defined collective excitations of the Bose type, damped only slightly, so that a diagram technique for magnons must be extremely close to the ordinary diagram technique.

Our purpose in this paper is to reformulate the diagram technique for spin operators to make it resemble as closely as possible the ordinary diagram technique and to use the standard notation of the latter. Entities of a new type appear in the spin diagram technique: end vertices, which are of a kinematic nature and unrelated to the interaction. Because of these end vertices, we need to distinguish the spin Green's functions G, which are defined in the standard way,[5] from the propagator g, which is standard in ordinary diagram techniques. This propagator is associated with a line on a diagram, and a dressing procedure making use of a Dyson equation is developed for it. The diagram technique which we have developed for the transverse spin operators S^+ and S^- is analogous to the well-known technique for nonequilibrium processes.[6,7] It thus becomes possible to describe the magnon kinetics in the most natural way, including the kinetics of magnons in states far from thermodynamic equilibrium. The longitudinal spin correlation functions enter our diagram technique as external parameters of the medium in which the magnons are propagating. In contrast with the temperature diagram technique which is standard in the theory of magnetism,[3,5] we cannot use perturbation theory to calculate the longitudinal correlation functions in a nonequilibrium diagram technique. To determine these correlation functions we should use kinematic relations which make it possible to find $\langle S^z \rangle$ and any arbitrary correlation functions of the spins as functionals of the magnon dispersion law $\omega^0_{\mathbf{k}}$ at $T=0$ and of the magnon population numbers $n_{\mathbf{k}}$, which are found from the kinetic equation and which are not necessarily equilibrium values.

We are thus proposing a systematic method for describ-

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ing kinetic effects in magnetic materials which starts from the spin Hamiltonian of the problem and which is not restricted to low temperatures. We derive several physical results. In particular, for the first time in a theory with a large interaction range R we determine the structure of the transverse spin Green's functions and vertex functions at $kR \gg 1$, and we calculate the damping time for short-wave magnons. We analyze the time-dependent structure of a binary longitudinal correlation Green's function. By making use of kinematic identities, we are able to carry out a summation which is more profound than in other versions of the diagram technique. As a result, an improved "spin-wave self-consistent-field approximation," incorporating magnons, arises here in the zeroth approximation. At equilibrium we thus immediately find the correct temperature dependence of the magnetization M and of the frequency $\omega_{\mathbf{k}}$ over the entire temperature range outside the critical region. Our method can easily be generalized to the case of complex spin Hamiltonians.

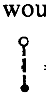
The diagram technique for the operators $S^{\pm}$ and S^z can be used not only in the theory of magnetism but also in any quantum-mechanical problem for which the state space is finite-dimensional or the direct product of finite-dimensional spaces. In such a system the diagram technique for $S^{\pm}$ and S^z is canonical in the sense that the diagram technique based on the operators a^+, a in an ordinary Bose system is canonical. The most profound algebraic property, which makes it possible to construct a simple diagram technique, is $[S^-[S^-[S^-, B]]] = 0$ for $B = S^{\pm}$, S^z and is analogous to the property $[a[a, B]] = 0$ for $B = a^+, a$ in the case of Bose operators. These identities make it possible to construct a simple version of the Wick theorem and to construct a simple diagram technique with few lines and vertices. The basic results of this study have been published as a preprint.[8]

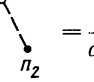
§1. THE DIAGRAM TECHNIQUE

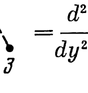
1. The rules of the diagram technique. Analysis of the Wick theorem for spin operators (reported in the preprint[8]) leads to rules for the diagram technique which can be reproduced most clearly by means of the following representations for the spin operators in terms of the Bose operators a_n^+, a_n and a random field φ_n:

$$S_n^- \equiv S_n^x - iS_n^y = a_n, \qquad S_n^+ \equiv S_n^x + iS_n^y = -a_n^+ a_n a - 2a_n^+\varphi_n,$$
$$S_n^z = a_n^+ a_n + \varphi_n. \tag{1.1}$$

Here n is the index of the lattice node with spin S_n. The field φ_n is determined by its irreducible correlation functions Φ_1, Φ_2, Φ_3, etc., which depend on the Hamiltonian of the problem. In the simplest case of noninteracting spins in an external magnetic field H_0 we would have

$\Phi_1^0(n) =$  $= bp_s(y),$

$\Phi_2^0(n_1 n_2) =$  $= \dfrac{d}{dy}(bp_s(y))\,\delta_{12},$ (1.2)

$\Phi_3^0(1, 2, 3) =$  $= \dfrac{d^2}{dy^2}(bp_s(y))\,\delta_{12}\delta_{13}, \ldots$

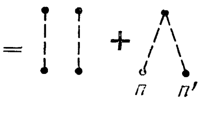
Here $bp_s(y)$ is the difference between the Brillouin function[1,5] $b_s(y)$ and the Planck function $n(y)$:

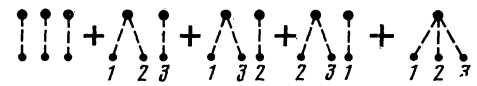
$$bp_s(y) = -S - (2S+1)\,[\exp{(2S+1)y} - 1]^{-1}, \qquad y = \mu_B H_0/T. \tag{1.3}$$

In § 2 we will calculate the irreducible correlation functions Φ_1, Φ_2, Φ_3,..., for a system of interacting spins. Working with these quantities in the standard way, we can find expectation values of the products φ_n at various nodes:

$$\langle\varphi_n\rangle = \Phi_1, \qquad \langle\varphi_n\varphi_{n'}\rangle = \Phi_1^2 + \Phi_2(n, n'),$$
$$\langle\varphi_1\varphi_2\varphi_3\rangle = \Phi_1^3 + \Phi_1[\Phi_2(1,2) + \Phi_2(1,3) + \Phi_2(2,3)]$$
$$+ \Phi_3(1, 2, 3), \ldots \tag{1.4}$$

With these expressions we can associate some clear graphical notation analogous to that used in the procedure for averaging the potentials of random impurities:

$\langle\varphi_n\rangle =$ , $\quad\langle\varphi_n\varphi_{n'}\rangle =$  (1.5)

$\langle\varphi_1\varphi_2\varphi_3\rangle =$ 

We should point out that in the case $\varphi = -S$ Eq. (1.1) is the same as the Dyson-Maleev representation,[1] which, as we know, describes the low-temperature properties of magnetic materials for an arbitrary spin. With $\varphi = (2S+1)b^+b - S$ (where b^+, b are Fermi operators) we find the Bar'yakhtar-Krivoruchko-Yablonskiĭ representation, which was recently proposed[9] for describing the equilibrium properties of magnetic materials by means of a temperature diagram technique. We wish to emphasize that representation (1.1) is like the Dyson-Maleev and Bar'yakhtar-Krivoruchko-Yablonskiĭ representations in that it should not be understood as an operator identity. All of these representations simply specify the rules for calculating Green's functions by a diagram technique.

It can be seen from (1.1) that the rules of our diagram technique for spin operators are the same as the rules of the diagram technique for a Bose field in the presence of a random impurity field φ_n. In contrast with the impurity problem, on the other hand, the correlation functions of the field φ are not given; as will be explained in § 2, neither a perturbation theory which starts from the seed correlation functions (1.2) nor the Bar'yakhtar-Krivoruchko-Yablonskiĭ representation can be used to calculate these correlation fucntions in a nonequilibrium diagram technique. Instead, we use known kinematic relations[1,8] to derive an expression for the correlation functions of the field $\varphi_n(t)$, (1.4), "dressed" by the interaction, in terms of the "dressed" Green's functions of a Bose system, a_n^+ and a_n.

2. Transverse Green's functions and free propagator. In a nonequilibrium diagram technique, the ordering of the product of operators is carried out along a special temporal contour c, which begins at $t = -\infty$, goes along the upper side of the time-axis cut out to the longest time, and then returns to $-\infty$ along the lower side of the cut.[6,7] We assign an index 1 to operators on the upper side and an index 2 to those on the lower side. As a result we find four binary Green's functions: G^{11}, G^{12}, G^{21}, and G^{22}. We accordingly introduce a 2×2 matrix of transverse Green's functions:

$$G_{-+}^{ij}(nt,\ n't')=\langle T_c(SS_n^{-i}(t)S_{n'}^{+j}(t'))\rangle, \tag{1.6}$$

where T_c is the ordering along the contour c, and the S-matrix is

$$S=T_c\exp\left\{i\int_{-\infty}^{\infty}\mathscr{H}_{int}(t)\,dt\right\}. \tag{1.7}$$

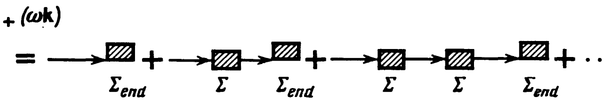
For noninteracting spins in an external field H we would have

$$G_{-+}^{ij}(\omega\mathbf{k})=-2\langle S^z\rangle_0 g_0^{ij}(\omega\mathbf{k}), \tag{1.8}$$

where $\langle S^z\rangle = b_s(\omega_0/T)$, $\omega_0=\mu_B H$, and g_0^{ij} is the Bose free propagator,[8] given by

$$g_0^{ij}(\omega,\mathbf{k})=\begin{bmatrix}\dfrac{1+n_0}{\omega-\omega_0-i\delta}-\dfrac{n_0}{\omega-\omega_0+i\delta}, & 2\pi i(1+n_0)\delta(\omega-\omega_0)\\ 2\pi i n_0\delta(\omega-\omega_0), & \dfrac{n_0}{\omega-\omega_0-i\delta}-\dfrac{1+n_0}{\omega-\omega_0+i\delta}\end{bmatrix}. \tag{1.9}$$

Examining the transverse Green's function G_{-+}, we single out in the perturbation-theory series a sequence of diagrams which are intersected along a common solid line:



(1.10)

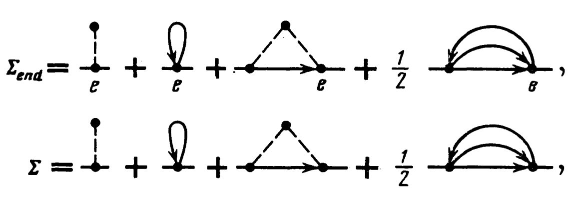
This sequence differs from the ordinary sequence in the theory of Bose and Fermi particles in the presence of an end mass operator $\Sigma_{\rm end}$ at the end of the diagram for G_{-+}. This end operator arises from the nonlinear dependence of S^+ and S^z on a^+, a, and φ. It is natural to single out $\Sigma_{\rm end}$ from the Green's function G_{-+}, determining the propagtor g by means of the relation

$$G_{-+}(\mathbf{k}\omega)=g(\mathbf{k}\omega)\Sigma_{end}(\mathbf{k}\omega). \tag{1.11}$$

We then can write the customary expression for the propagator g, which is a matrix propagator in terms of Keldysh indices:

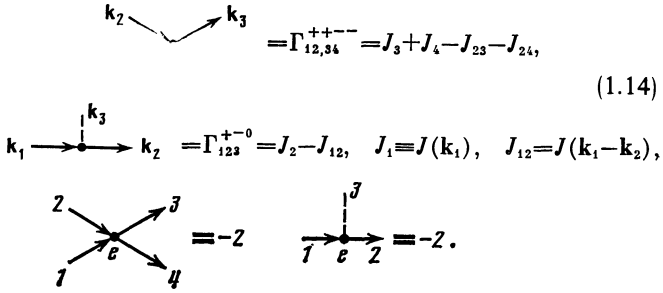
$$g^{ij}(\omega\mathbf{k})=[g^0(\omega\mathbf{k})^{-1}-\Sigma(\omega\mathbf{k})]_{ij}^{-1}, \tag{1.12}$$

where g^0 is the propagator (1.10), and Σ^{ij} is the ordinary Keldysh mass operator. In the lowest order of our perturbation theory (the approximation of a self-consistent field), $\Sigma_{\rm end}$ and Σ in the Heisenberg model are



(1.13)

where



$$=\Gamma_{12,34}^{++--}=J_3+J_4-J_{23}-J_{24},$$

(1.14)

$$=\Gamma_{123}^{+-0}=J_2-J_{12},\quad J_1\equiv J(\mathbf{k}_1),\quad J_{12}=J(\mathbf{k}_1-\mathbf{k}_2),$$

Here $J(\mathbf{k})$ is the Fourier transform of the exchange integral $J(\mathbf{n}-\mathbf{n}')$, and e represents an end vertex: a terminal of kinematic origin (it originates from the nonlinear dependence of the operators S^+ and S^z on a^+, a, and φ), which is unrelated to the interaction Hamiltonian.

To find the spectrum of magnons and their distribution function at a finite temperature, we perform a standard unitary transformation over Keldysh indices:

$$\tilde{g}=\mathrm{R}g R^{-1},\quad R=R^{-1}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}. \tag{1.15}$$

In this representation we have

$$g^r=g^{0r}(1+\Sigma^r g^r),\quad g^a(\mathbf{k}\omega)=(g^r(\mathbf{k}\omega))^*,$$
$$g^d=g^r(\Sigma^{0d}+\Sigma^d)g^a,\quad \Sigma^{0d}=2i\delta(1+2n_0), \tag{1.16}$$
$$g^r=g^{11}-g^{21},\quad g^a=g^{11}-g^{12},\quad g^d=g^{12}-g^{21};$$
$$\Sigma^r=\Sigma^{11}+\Sigma^{21},\quad \Sigma^a=(\Sigma^r)^*,\quad \Sigma^d=-(\Sigma^{12}+\Sigma^{21}),$$
$$g^{11}+g^{22}=g^{12}+g^{21},\quad \Sigma^{11}+\Sigma^{22}+\Sigma^{12}+\Sigma^{21}=0. \tag{1.17}$$

Equations (1.17) thus have the form of the standard Dyson equations in the Feynman diagram technique, and the magnon spectrum $\omega_{\mathbf{k}}$ and the magnon damping $\gamma_{\mathbf{k}}$ are determined by the real and imaginary parts, respectively, of the operator $\Sigma^a(\mathbf{k}\cdot\omega_{\mathbf{k}})$ or by the coefficient of $2n_{\mathbf{k}}$ in collision integral (1.19) below. In lowest order in the interaction we find from (1.14) and (1.15)

$$\omega_{\mathbf{k}}=\langle S^z\rangle(J_{\mathbf{k}}-J_0)+\lambda\int(J_{\mathbf{k}'}-J_{\mathbf{k}-\mathbf{k}'})n_{\mathbf{k}'}d\mathbf{k}',\quad \lambda=v_0(2\pi)^{-3}. \tag{1.18}$$

Here $\langle S^z\rangle$ is the expectation value of S^z, v_0 is the volume of the unit cell, and $n_{\mathbf{k}}$ is the magnon distribution function in a state of the magnetic material which is not necessarily an equilibrium state. The function $n_{\mathbf{k}}$ is determined from Eq. (1.18), which becomes a kinetic equation in the case of a weak interaction. To see this, we take the magnon interaction into account, $\Sigma^d\neq 0$; if we turn on the interaction in an adiabatic way, such that $\delta\to 0$, we can ignore $\Sigma^{0d}\sim\delta$ in comparison with Σ^d in Eq. (1.16). These equations then lose all memory of the original distribution function n_0:

$$\frac{dn_{\mathbf{k}}}{dt}=\mathrm{St}(n)=\int\frac{d\omega}{(2\pi)^4}[\Sigma^a g^d-\Sigma^d g^a]. \tag{1.19}$$

In particular, for the four-magnon processes described by the last diagram for Σ in (1.13) we find

$$\mathrm{St}^{(4)}=\int d1d2d3\,|A_{\mathbf{k}1,23}|^2 N_{\mathbf{k}1,23}\delta(\mathbf{k}+\mathbf{1}-\mathbf{2}-\mathbf{3})$$
$$\delta(\omega_{\mathbf{k}}+\omega_1-\omega_2-\omega_3), \tag{1.20}$$
$$N_{\mathbf{k}1,23}=n_{\mathbf{k}}n_1(n_2+1)(n_3+1)-(n_{\mathbf{k}}+1)(n_1+1)n_2n_3;$$
$$|A_{\mathbf{k}1,23}|^2=\pi\lambda^2\Gamma_{\mathbf{k}123}^{++--}\Gamma_{1\mathbf{k}23}^{--++}. \tag{1.21}$$

At thermodynamic equilibrium we should have $\mathrm{St}=0$ and $N_{\mathbf{kl},23}=0$, and

$$n_{\mathbf{k}}=(\exp\beta\omega_{\mathbf{k}}-1)^{-1},\quad \beta=1/T, \tag{1.22}$$

would be a Bose distribution function with a dispersion law $\omega_{\mathbf{k}}$. In this approximation the propagator g differs from g^0 of (1.9) in the replacements $\omega_0\to\omega_{\mathbf{k}}$ and $n_0\to n_{\mathbf{k}}$. After calculating the dependence of $\langle S^z\rangle$ on T in § 2 we will see that Eq. (1.18) gives the correct temperature dependence $\omega_{\mathbf{k}}(T)$ except in the critical region. In particular, at $T\ll T_c$ we find $\omega_{\mathbf{k}}(T)-\omega_{\mathbf{k}}(0)\sim T^{5/2}$ from (1.18).

§ 2. CALCULATION OF THE MAGNETIZATION AND OF THE LONGITUDINAL CORRELATION FUNCTIONS IN A DIAGRAM TECHNIQUE FOR NONEQUILIBRIUM PROCESSES

When we attempt to calculate $\langle S^z\rangle$ by perturbation theory, we find a diagram series in which some of the terms are regular while others are inversely proportional to the adiabatic parameter δ. The sequence of lowest order in δ^{-1} is summed in such a manner that only the total propagators g and the dressed correlation functions Φ_n of the field φ remain in it. As a result, we again find kinetic equations which determine $n_{\mathbf{k}}$ and which leave the correlation functions Φ_n arbitrary. The reason for this result is that our diagram technique is based exclusively on the spin Hamiltonian and the spin commutation relations. It embodies information on the magnitude of the spin only in the initial conditions (the bare correlation functions Φ_n^0), which, as we know, are forgotten in the nonequilibrium diagram technique. To determine the dressed correlation functions Φ_n we thus need to make use of kinematic relations which, on the one hand, are comparable with the equations of motion and, on the other, fix the value of the spin. For this purpose we introduce the projection operators π^m, which project onto states with definite values of the spin projection:

$$\pi^m=\prod_{\substack{p=-S,\\ p\neq m}}^{S}(S^z-p)(m-p)^{-1},$$

$$\pi^m\pi^{m'}=\delta_{mm'}\pi^m,\qquad \sum_{m=-S}^{S}\pi^m=1, \tag{2.1}$$

$$(S^z)^p=\sum_{m=-S}^{S}m^p\pi^m,\qquad (S^z)^p\pi^m=m^p\pi^m.$$

We see that the problem of calculating $(S^z)^p$ reduces to one of calculating expectation values of π^m. Multiplying the relation $\hat{S}^2=S(S+1)$ by π^m, we find

$$\gamma_{S,m-1}^2\pi^m=\pi^m S^+S^-,\qquad \gamma_{S,m}^2=(S-m)(S+m+1). \tag{2.2}$$

We now introduce the operator $z^m=\pi^m S^+$, which has simple commutation relations with the active operator S^-:

$$[S^-,Z^m]=\gamma_{S,m-1}^2(\pi^{m-1}-\pi^m). \tag{2.3}$$

Taking the average of Eq. (2.2) in lowest order in the interaction, and using (2.3), we find

$$\langle\pi^m\rangle\mathfrak{N}[\langle\pi^{m-1}\rangle-\langle\pi^m\rangle],\qquad \mathfrak{N}=\lambda\int n_{\mathbf{k}}\,d\mathbf{k}. \tag{2.4}$$

Solving (2.4), and using the normalization condition in (2.1), we find

$$\langle\pi^m\rangle=\exp[y(S-m)]Z^{-1}(y),\qquad y=\ln[(1+\mathfrak{N})/\mathfrak{N}], \tag{2.5}$$
$$Z(y)=[\exp(2S+1)y-1][\exp y-1]^{-1}.$$

Hence

$$\langle S^z\rangle=-S+\mathfrak{N}-\frac{(2S+1)\mathfrak{N}^{2S+1}}{(\mathfrak{N}+1)^{2S+1}-\mathfrak{N}^{2S+1}}=b_s(y). \tag{2.6}$$

This expression has been derived previously by a method involving the splitting of correlation functions for the case of thermodynamic equilibrium.[1,2,10] It follows from our derivation, however, that this expression holds in the first approximation in the reciprocal of the interaction range even if we do not assume thermodynamic equilibrium. We wish to emphasize that this expression does not contain a temperature, and it demonstrates the important circumstance that $\langle S^z\rangle$ and (as we will see below) all the higher-order correlation functions of S^z are functionals of the magnon population numbers $n_{\mathbf{k}}$, which are found from the solution of the kinetic equation and thus do not have to be positive. To demonstrate this point, we note that expression (1.22) for $\mathrm{St}^{(4)}$ shows that if $n_{\mathbf{k}}$ is a solution of the kinetic equation then $\tilde{n}_{\mathbf{k}}=-1-n_{\mathbf{k}}$ is also a solution of this equation, after $\langle S^z\rangle$ is replaced by $-\langle S^z\rangle$. This property is in agreement with the circumstance that in (2.6) the quantity $\langle S^z\rangle$ becomes $-\langle S^z\rangle$ upon the replacement $\mathfrak{N}\to-1-\mathfrak{N}$. The kinetic equation and Eq. (2.6) thus make it possible to describe the behavior of $\langle S^z\rangle$ over the permissible range from $-S$ to S. At equilibrium, Eq. (2.6) can be assigned the graphic physical meaning of the equation of a self-consistent field if we note that it determines the expectation value of a noninteracting spin in the effective magnetic field determined by (2.4):

$$\langle S^z\rangle=b_s(\beta H_{eff}),\qquad [\exp\beta H_{eff}-1]^{-1}=\lambda\int[\exp\beta\omega_{\mathbf{k}}-1]^{-1}\,d\mathbf{k}. \tag{2.7}$$

Working from expression (1.18) for $\omega_{\mathbf{k}}$, we easily see that this equation gives us the well-known Bloch law $\Delta\langle S^z\rangle\propto T^{3/2}$ in the limit $T\to 0$ and the Curie-Weiss law $\langle S^z\rangle\propto(T_c-T)^{1/2}$ in the limit $T\to T_c$. From (2.1) and (2.5) we find the standard expressions[5] for $\langle(S^z)\rangle^2$, etc.:

$$\langle(S^z)^2\rangle=b_s^2(y)+b_s'(y),\ldots \tag{2.8}$$

To calculate the longitudinal correlation function of spins at different nodes and at different times,

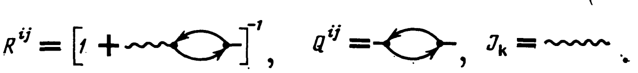
$$K^{ij}(1,2)=\langle S_1^{zi}S_2^{zj}\rangle$$

(i,j are the Keldysh indices), we need to determine the relationship between K^{ij} and the binary correlation function Φ^{ij}. Using representation (1.1) for $\langle S^z\rangle$, and summing the lowest-order sequence of "chain" diagrams, which is known[3,5] to be the lowest-order sequence in terms of R^{-3}, where

$$R^{-3}=\lambda\int J_{\mathbf{k}}^2\,d\mathbf{k}/J_0^2, \tag{2.9}$$

we find

$K_q^{ij}=$  $=[R_q(Q_q+Q_qJ_{\mathbf{k}}Q_q+\Phi_q)R_q]^{i,j}, \qquad q=\mathbf{k},\ \omega,$ (2.10)



We now consider the Green's function

$$K_{zm}^{ij}(1,2)=\langle T_c(S_1^{zi}\pi_2^{mj})\rangle_{\mathscr{H}}=\gamma_{S,m-1}^{-2}\langle T_c(S_1^{zi}Z_2^{mj}S_2^{-j})\rangle_{\mathscr{H}},$$

$$\sum_{m=-S}^{S} K_{zm}^{ij}=0. \tag{2.11}$$

Using (2.3), and summing the sequence of chain diagrams in the lowest order in R^{-3}, we find the system of equations

$$K_{zm}^{ij}(q)=\Re[K_{z,m-1}^{ij}-K_{z,m}^{ij}]+T^{ij}(q)[\langle\pi^{m-1}\rangle-\langle\pi^m\rangle],$$
$$T^{ij}(q)=[R_q(Q_q+Q_qJ_{\mathbf{k}}Q_q-\Phi_qJ_{\mathbf{k}}Q_q)R_q]^{ij}. \tag{2.12}$$

Solving these recurrence relations, and using expressions (2.5) and (2.11), we find

$$K_{zm}^{ij}(q)=4\exp[y(S-m)]\,\mathrm{sh}^2(y/2)Z^{-1}(y)[m-b_s(y)]T_q^{ij}.$$

Here y is given as function of $\Re$ and Z in (2.5). Summing over m, we find a second equation relating K_q^{ij} and Φ_q^{ij}:

$$K_q^{ij}=c(\Re)T_q^{ij}, \quad c(\Re)=4\,\mathrm{sh}^2(y/2)b_s'(y). \tag{2.13}$$

The system of linear equations (2.10), (2.13) can be solved easily:

$$K_q^{ij}=c(\Re)[Q_q(1+c(\Re)J_{\mathbf{k}}Q_q)^{-1}]^{ij},$$
$$\Phi_q^{ij}=[1-c^{-1}(\Re)]^2K_q^{ij}+[1-c^{-1}(\Re)]Q_q^{ij}. \tag{2.14}$$

Calculating the inverse matrix in terms of Keldysh indices, we find

$$K_q^{r,a}=c(\Re)Q_q^{r,a}[1+c(\Re)J_{\mathbf{k}}Q_q^{r,a}]^{-1},$$
$$K_q^d=c(\Re)Q_q^d|1+c(\Re)J_{\mathbf{k}}Q_q^r|^{-2}, \tag{2.15}$$

where

$$Q_q^{r,a}=\lambda\int[n_{\mathbf{k}'-\mathbf{k}}-n_{\mathbf{k}'}][\omega+\omega_{\mathbf{k}'-\mathbf{k}}-\omega_{\mathbf{k}'}\mp i\delta]^{-1}dk',$$
$$Q_q^d=2\pi\lambda\int[n_{\mathbf{k}'}+n_{\mathbf{k}'-\mathbf{k}}+2n_{\mathbf{k}'}n_{\mathbf{k}'-\mathbf{k}}]\delta(\omega+\omega_{\mathbf{k}'-\mathbf{k}}-\omega_{\mathbf{k}'})d\mathbf{k}'. \tag{2.16}$$

We wish to emphasize that the expressions for K^a, K^r, and K^d hold even if we do not assume thermodynamic equilibrium. These quantities are functionals of the magnon population numbers $n_{\mathbf{k}}$, the magnon dispersion law $\omega_{\mathbf{k}}$, and the vertex functions of the theory (in this case, $J_{\mathbf{k}}$ alone). We note that at equilibrium we would have

$$K^{ij}(nt,nt)=\int K_q^{ij}d_q^{\ 4}=b_s'(y) \tag{2.17}$$

in agreement with expression (2.8) for $\langle(S^z)^2\rangle$. In general, $n_{\mathbf{k}}$ is the solution of a kinetic equation and not unambiguously related to $\omega_{\mathbf{k}}$. There is thus no point in pursuing the general analysis of the expressions for K_q^{ij}. At equilibrium, however, these quantities do have several simple properties. In the first place, K_q^d and K_q^r are related by a fluctuation-dissipation theorem:

$$K_q^d=2(2n_\omega+1)\,\mathrm{Im}\,K_q^r, \quad n_\omega=[\exp\beta\omega-1]^{-1}. \tag{2.18}$$

In the low-temperature limit we have $c(\Re)\to 1$ and $\Phi_q^{ij}\to 0$, and the expressions for the correlation function K_q^{ij} in (2.15) simplify to the point that they become an expression which can be derived by means of a Dyson-Maleev representation without taking into account projection operators.

The single-time longitudinal Green's function $K(\mathbf{k})$ can be expressed in terms of both the temperature Green's function $K_{n\mathbf{k}}^T$ and K_q^d:

$$K(\mathbf{k})=T\sum_n K_{n\mathbf{k}}^T=(4\pi)^{-1}\int K_q^d\,d\omega$$
$$=b_s'(y)[1-\beta J_{\mathbf{k}}b_s'(y)]^{-1}. \tag{2.19}$$

This result is in agreement with the results derived previously, and in the limit $T\to T_c$ it gives us the well-known Ornstein-Zernike correlation function.[5]

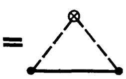
The single-time longitudinal Green's function at small $\mathbf{k}$ (at large distances) is determined by the dispersive part of the spectrum and is a small quantity of order R^{-3}:

$$K(\mathbf{k})\approx\lambda\int n_{\mathbf{k}'}n_{\mathbf{k}'-\mathbf{k}}\,d\mathbf{k}'\approx v_0T^2/8\alpha^2k, \tag{2.20}$$

where $\alpha=\partial\omega_{\mathbf{k}}/\partial k^2$ in the limit $k\to 0$ for a cubic magnetic material. It follows from (2.20) that we would have $K(r)\propto r^{-2}$ in the limit $r\to\infty$. In a corresponding way, we can use a diagram technique for the projection operators π^m to derive expressions for the longitudinal Green's functions of higher order.

§ 3. DAMPING OF MAGNONS AND OF THE LONGITUDINAL GREEN'S FUNCTION

1. Damping of dispersion magnons. Expression (1.21) for the magnon damping is valid only at low temperatures. At $T>3T_c/S$, we need to supplement the diagrams (1.14) in the lowest order in R^{-3} with a sequence of "chain" diagrams with "springs." We easily see from the results of § 2 that this sequence reduces to the longitudinal Green's function

$\Sigma_q^{ij}=$  .

Substituting in the explicit expressions for K_q^{ij} from (2.15) and g_q^{ij}, and integrating over ω, we see that the collision integral (1.20) is of the standard form for four-magnon processes, (1.21), with an effective square matrix element

$$|A_{\mathbf{k}1,23}|^2=\frac{\pi\lambda^2(J_2-J_{\mathbf{k}-2})(J_{\mathbf{k}}-J_{\mathbf{k}-2})}{2|1+c(\Re)J_{\mathbf{k}-2}Q_{q-2}^r|^2}. \tag{3.1}$$

At equilibrium under the conditions $T>3T_c/S$ and $P\gg 1$, the nondispersive region makes the primary contribution to the expression for the damping which follows. As a result, this expression can be simplified to the familiar form[5]

$$\gamma_{\mathbf{k}}=\pi\lambda b_s'(y)\int\frac{(J_{\mathbf{k}'}-J_{\mathbf{k}-\mathbf{k}'})(J_{\mathbf{k}}-J_{\mathbf{k}-\mathbf{k}'})}{1-\beta J_{\mathbf{k}'-\mathbf{k}}b_s'(y)}\delta(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})d\mathbf{k}'. \tag{3.2}$$

2. Classification of diagrams for nondispersive magnons. In this subsection we show that the diagram for Σ_q which we are considering here is the most important diagram only at small momenta, $akR\ll 1(a^3=v_0)$, i.e., only in the dispersive part of the spectrum. For this purpose we will calculate $\gamma_{\mathbf{k}}$ at large k, for which the magnon dispersion

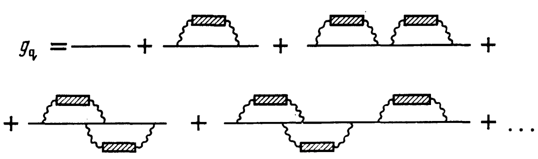
$\Delta\omega_{\mathbf{k}} = \max \omega_{\mathbf{k}} - \omega_{\mathbf{k}}$ is smaller than $\gamma_{\mathbf{k}}$. In this region the Green's functions cannot be assumed bare; correspondingly, the function $\delta(\omega_{\mathbf{k}} - \omega_{\mathbf{k}'})$ in (3.2) should be replaced by

$$\gamma_{\mathbf{k}}/\pi[(\omega_{\mathbf{k}}-\omega_{\mathbf{k}'})^2+\gamma_{\mathbf{k}'}^2].$$

The damping is thus determined in a self-consistent way, and instead of (3.2) we find

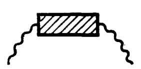
$$\gamma_{\mathbf{k}}^2=\pi b_s'(y) \int J_{\mathbf{k}'}^2[1-\beta J_{\mathbf{k}'} b_s'(y)]^{-1} d\mathbf{k}'. \qquad (3.3)$$

We see that with increasing $\mathbf{k}$ the damping $\gamma_{\mathbf{k}}$ ceases to depend on $\mathbf{k}$ and is given in order of magnitude by $J_0R^{-3/2}$. In the nondispersive region, where $\Delta\omega_{\mathbf{k}} < \gamma$, the Green's function can be estimated to be γ^{-1}; this estimate has some far-reaching consequences, since we cannot restrict the discussion to the diagrams of lowest order in Σ_q. The diagrams which are cut along one line and which lead to a Dyson equation for the propagator $g_{\mathbf{q}}$ are no longer the principal diagrams, and we must include in our perturbation-theory series the sequence of diagrams from overlapping dressed chains, i.e.,

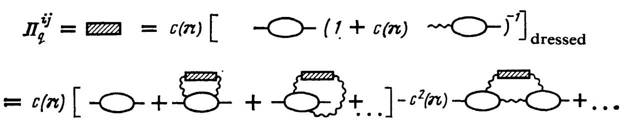


(3.4)

Here the symbol



denotes expression (2.14) in which each of the loops Q_q in the chain is dressed by a sequence (3.4):



(3.5)

Diagram (3.4) for the propagator $g_{\mathbf{q}}$ and diagram (3.5) for the polarization operator $\Pi_{\mathbf{q}}$ are the diagrams of lowest order in the parameter R^{-3} and are of the same order of magnitude. When we take into account the additional polarization operator with two springs we arrive at the factor

$$Z=J_{\mathbf{k}}^2\Pi_{\mathbf{q}} g_{\mathbf{q}-\mathbf{q}'} g_{\mathbf{q}-\mathbf{q}''}.$$

Integrating this factor over $\mathbf{k}$, and noting that the propagators $g_{q-q'}$ and $g_{q-q''}$ are in a nondispersive region, we find $Z \approx J_0^2/\gamma^2R^3$; under the condition $\gamma \approx J_0R^{-3/2}$ we have $Z \approx 1$. It can be shown that diagrams with three or more springs in the polarization operator and also diagrams which do not reduce to a summation of polarization operators are small, on the order of the parameter R^{-3}.

3. Summation of the main sequence of diagrams. It follows from the topological structure of series (3.4) that this series describes a propagator g_q for some particle in a random Gaussian field ψ whose binary correlation function is determined self-consistently from (3.5). In the nondispersive region this propagator does not depend on $\mathbf{k}$, and the problem becomes effectively one-dimensional. A propagator of this type is conveniently sought in the t representation:

$$\begin{bmatrix} i\partial_{t_1}-\psi_1(t_1) & 0 \\ 0 & -i\partial_{t_1}+\psi_2(t_1) \end{bmatrix} g(t_1, t_2, \psi)=\delta(t_1-t_2). \qquad (3.6)$$

Here $\psi_i(t)$ with $i = 1,2$ are the Keldysh components of the random field $\psi(t)$. This equation can be solved easily:

$$g^{j_1j_2}(t_1, t_2, \psi)=\exp[i\Psi_{j_1}(t_1t_2)]g_0^{j_1j_2}(t_1-t_2), \qquad (3.7)$$

where the phase $\Psi^j(t_1\, t_2)$ and the free propagator $g_0(t_1t_2)$ are given by

$$\Psi^j(t_1t_2)=\int_{t_2}^{t} \psi_j(t')\,dt',\quad g_0(t)=\begin{bmatrix} \theta(t)+\mathfrak{N} & 1+\mathfrak{N} \\ \mathfrak{N} & \theta(-t)+\mathfrak{N} \end{bmatrix} e^{i\omega_0 t}. \qquad (3.8)$$

Here $\mathfrak{N} = [\exp(\beta\omega_0) - 1]^{-1}$. The propagator $g(t_1-t_2)$, which is the sum (3.4), can be written as a Gaussian functional integral:

$$g^{j_1j_2}(t_1-t_2)=Z^{-1}\int \prod_{tj} d\psi_j(t)\, g^{j_1j_2}(t_1, t_2, \psi) R(\psi),$$

$$Z=\int \prod_{tj} d\psi_j(t) R(\psi), \qquad (3.9)$$

$$R(\psi)=\exp\left[-\frac{1}{2}\sum_{j_1j_2} \psi_{j_1}(\tau_1) R^{-1}_{j_1j_2} \psi(\tau_2)\, d\tau_1\, d\tau_2\right].$$

Here the operator $R^{-1}_{j_1j_2}(\tau_1-\tau_2)$ is the inverse of the correlation function $R_{j_1j_2}(\tau_1-\tau_2)$ of the random Gaussian field $\psi_j(t)$. Expanding $g(\psi)$ in a series in ψ, and taking an average, we easily see that we find our original series, (3.4), for $g^{ij}(t)$. On the other hand, the functional integral in (3.9) can be evaluated easily; the result is

$$g^{ij}(t_1-t_2)=\exp\left[-\frac{1}{2}\int R^{ij}(\tau_1-\tau_2)\, d\tau_1\, d\tau_2\right] g_0^{ij}(t_1-t_2). \qquad (3.10)$$

The correlation function R^{ij} must be determined from nonlinear integral equation (3.5). The solution of this equation simplifies dramatically because the interaction-dressed loop $\tilde{Q}_q^{ij}$ reduces to the bare loop Q_q^{ij}. It can be seen from (3.5) that $\tilde{Q}_q^{ij}$ is found by averaging the bare loop Q_q^{ij} over the random field ψ. This averaging can be written with the help of a functional integral:

$$\tilde{Q}^{ij}(t_1t_2)=Z^{-1}\int \prod_{tj'} d\psi_{j'}(t)\, g^{ij}(t_1t_2\psi)\, g^{ij}(t_2t_1\psi) R(\psi). \qquad (3.11)$$

In the case $i=j$, the product $g^{ii}(12)g^{ii}(21)$ is obviously independent of ψ, so that we have $\tilde{Q} = Q$. A characteristic sum arises in the exponential function in the integral over ψ for $\tilde{Q}^{12}$ and $\tilde{Q}^{21}$:

$$\sum \theta_i\theta_j R^{ij}(1, 2)$$

($\theta_i = \pm 1$ for $i = 1,2$). This characteristic sum is zero, so that the quantity $Q^{ij}(t_1-t_2)$ is not renormalized. This assertion follows from a general relation for an arbitrary Green's function in a diagram technique for nonequilibrium processes:

$$\sum \theta_{i_1}\cdot \ldots \cdot\theta_{i_n} R_{i_1\ldots i_n}(\mathbf{x}_1, \ldots, \mathbf{x}_n)=0,$$

$$R_{i_1\ldots i_n}=\langle \psi_{1i_1}(\mathbf{x}_1), \ldots, \psi_{ni_n}(\mathbf{x}_n)\rangle. \qquad (3.12)$$

This relation follows from the unitarity of the S-matrix in a classical external field $A_p(\mathbf{x}_p)$ with the Hamiltonian

$$\mathscr{H}_{int}=\sum_p A_p(\mathbf{x})\psi_p(\mathbf{x}).$$

Writing the unitarity condition $\mathbf{S}^{-1}\mathbf{S}=1$, and carrying out a functional differentiation of this identity with respect to the classical variables $A_1(\mathbf{x}_1),\ldots,A_p(\mathbf{x}_p)$, we find (3.12). It can also be shown that not only the binary loop but also any closed loop of magnon lines with n vertices (and also any product of loops of any order) will remain bare. We might note that this general property also holds when the fluctuations of the field ψ are not Gaussian. The only requirements are that the magnon field be nondispersive and that the diagrams with springs be dominant; this property can be described by a non-Gaussian random field $\psi(t)$.

The fact that the loops in the nondispersive part of the spectrum are not dressed can be assigned a clear physical meaning: In the absence of dispersion, a magnon damping arises from random phase shifts in the field $\psi(t)$. For a particle which describes a closed loop, however, this phase shift is zero. Consequently, no damping of any sort need to be taken into account in the nondispersive region for magnons within a closed loop. We must emphasize that this assertion is quite general, applying to the damping of non-dispersive magnons of arbitrary nature: four- and three-magnon damping, damping due to an interaction with phonons, etc.

4. Structure of the Green's function and damping of nondispersive magnons. It follows from (3.5) for Π^{ij} and (2.16) for the Green's function K_q^d that the random-field correlation function $R^{ij}(t)$ does not depend on the time or on the Keldysh indices. At equilibrium we would have

$$R^{ij}(t)=P=\pi\lambda b_s'(y)\int J_\mathbf{k}^2[1-\beta J_\mathbf{k} b_s'(y)]^{-1}d\mathbf{k}\approx J_0^2R^{-3}. \quad (3.13)$$

Substituting (3.13) into (3.10), we find

$$g^{ij}(t)=g_0^{ij}(t)\exp(-Pt^2/2),$$

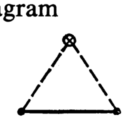
from which we find in turn, after a switch to the ω representation,

$$g^r(\omega)=P^{-1/2}\exp\left[-\frac{(\omega-\omega_0)^2}{2P}\right]\left[i\left(\frac{\pi}{2}\right)^{1/2}+\eta(\omega)\right], \quad (3.14)$$

$$\eta(\omega)=\int_0^y\exp\left(\frac{x^2}{2}\right)dx,\quad y=\frac{\omega-\omega_0}{P^{1/2}},$$

$$g^d(\omega)=2(2\mathfrak{N}+1)\,\mathrm{Im}\,g^r(\omega).$$

This expression for $g^r(\omega)$ agrees with that derived previously by one of the present authors[11] by the Wild diagram technique in a study of hydrodynamic turbulence. We wish to emphasize that Im g^r is a Gaussian function, not a Lorentzian function, as it would be if we considered only the one diagram



in the series for Σ_q. However, expression (3.13) for the halfwidth of this distribution, $P^{1/2}$, agrees within a number with the simple estimate (3.3) for γ based on the first diagram: $P\approx\gamma^2$.

5. Structure of unequal-time longitudinal spin correlation functions K^r, K^a, and K^d. It follows from the results of this section that the expressions for $K^r(\mathbf{k}\omega)$, $K^a(\mathbf{k}\omega)$ and $K^d(\mathbf{k}\omega)$ are given correctly in lowest order in $R^{-3/2}$ by expressions (2.16). Noting that $\mathbf{K}^a=(\mathbf{K}^r)^*$, while at equilibrium K^d is related to K^r by the fluctuation-dissipation theorem (2.18), we will analyze the frequency dependence of the retarded longitudinal spin correlation function $\mathbf{K}^r(\mathbf{k}\omega)$ in (2.16) in this subsection. We begin with the low-temperature case, $T<3T_c/S$, in which nondispersive magnons with an energy $\omega_\mathbf{k}=3T_c/S$ are not excited. In this case the integral (2.16) for Q^r is dominated by the long-wave part of the magnon spectrum, where we have

$$\omega_\mathbf{k}=\alpha k^2,\ \alpha\approx\omega_0(akR)^2,\ \omega_0=\langle S^z\rangle J_0=SJ_0.$$

As a result, at $\omega_k\ll T$ we find

$$Q^r(\omega\mathbf{k})=\frac{Ta^3}{8\alpha^2k}\left[\theta(\omega_\mathbf{k}+\omega)-\theta(\omega_\mathbf{k}-\omega)+\frac{i}{\pi}\ln\left|\frac{\omega_\mathbf{k}+\omega}{\omega_\mathbf{k}-\omega}\right|\right]. \quad (3.15)$$

The reason for the logarithmic singularity in this expression is that the pole form of the response is not spread out sufficiently upon the integration because of the pronounced localization of the soft-magnon distribution function at small $\mathbf{k}$. It can be seen from (2.15) and (3.15) that the characteristic width along ω of the longitudianl spin correlation functions $\mathbf{K}(\omega\mathbf{k})$ is determined by the magnon frequency $\omega_\mathbf{k}$.

The situation is different at high temperatures, specifically, at $T>3T_c/S$. Assuming that the exchange integral has a power-law behavior at $akR>1$,

$$J_\mathbf{k}=\delta J_0(akR)^{1-n},\ \delta\approx1,$$

we find

$$\mathrm{Im}\,Q^r(\omega\mathbf{k})=\frac{3\pi\mathfrak{N}(\mathfrak{N}+1)}{2T(n+3)}f\left(\frac{\omega}{\tilde{\Gamma}_\mathbf{k}}\right)$$

$$f(x)=x,\ x\leqslant1;\ f(x)=x^{-3/n},\ x\geqslant1. \quad (3.16)$$

Here $\tilde{\Gamma}_\mathbf{k}=kV_0=\delta(n-1)ak\omega_0R^{1-n}$, where V_0 is the group velocity at the boundary of the Brillouin zones.

The function Re$Q^r(\omega\mathbf{k})$ can be reconstructed from (3.16) with the help of the dispersion relations

$$\mathrm{Re}\,Q^r(\omega\mathbf{k})=\frac{\mathfrak{N}(\mathfrak{N}+1)}{T}\left[1-\frac{9\omega^2}{(3+n)(3+2n)\tilde{\Gamma}_\mathbf{k}^2}\right],\quad \omega\ll\tilde{\Gamma}_\mathbf{k};$$

$$\mathrm{Re}\,Q^r(\omega\mathbf{k})=\frac{3\pi\mathfrak{N}(\mathfrak{N}+1)}{T(3+n)}\begin{cases}2\tilde{\Gamma}_\mathbf{k}^2/3\omega^2, & n=3,\ \omega\ll\tilde{\Gamma}_\mathbf{k};\\ \left(\dfrac{\tilde{\Gamma}_\mathbf{k}^2}{\omega^2}\right)^{3/n}\mathrm{ctg}\dfrac{3\pi}{2n}, & n>3,\ \omega\gg\tilde{\Gamma}_\mathbf{k}\end{cases} \quad (3.17)$$

Using (3.19) and (3.17), we find an expression for $K^r(\omega\mathbf{k})$ at $\omega<\tilde{\Gamma}_\mathbf{k}$:

$$K^r(\omega\mathbf{k})=\frac{b_s'}{T}\left[1+i\frac{\omega}{\Gamma_\mathbf{k}}-\frac{J_\mathbf{k}b_s'(y)}{T}\right]^{-1}. \quad (3.18)$$

Here $\Gamma_\mathbf{k}=2(3+n)\tilde{\Gamma}_\mathbf{k}/3\pi$. We wish to emphasize that the results derived above are related in a fundamental way to the nonvanishing dispersion of magnons with $akR>1$. On the other hand, the proof that the contributions of the dressed propagators and of the vertex in the chains cancel out exactly was obtained in the absence of dispersion. We thus cannot rule out the possibility that a slight dispersion may disrupt this cancellation, to an extent which depends on the strength of the dispersion. In this case our results on the frequency dependence of the longitudinal correlation functions will be valid in order of magnitude.

Expression (3.18) simplifies near the point of the phase transition:

$$K^r(\omega\mathbf{k})=\frac{S(S+1)}{3T_c}\left[i\frac{\omega}{\Gamma_\mathbf{k}}+2\tau+\frac{3(akR)^2}{S(S+1)}\right]^{-1}, \qquad (3.19)$$

where $\tau=(T_c-T)/T_c$, and $\Gamma_\mathbf{k}\approx J_0(ak)\tau^{1/2}R^{i-n}$. In the ω plane here there is a pole which describes spin diffusion. This expression corresponds to the well-known expression for $K^r(\omega\mathbf{k})$ in terms of $G(\mathbf{k})$ [see, for example, Eq. (3.23) in Chapter 7 in Ref. 12], which was derived in the approximation of a self-consistent field. The hypothesis $\Gamma_\mathbf{k}\sim k^2$, however, is not justified.

It is a pleasure to thank S. V. Maleev for a discussion of the structure of longitudinal spin correlation functions.

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