A scale-invariant theory of fully developed hydrodynamic turbulence

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A systematic method of describing developed hydrodynamic turbulence in terms of the Navier-Stokes equation and diagrammatic perturbation-theory methods is proposed. A solution of the diagrammatic equations is obtained that corresponds to the well known Kolmogorov-Obukhov picture of stationary spatially uniform isotropic developed turbulence, in which the unequal-time velocity correlators are nonuniversal and their time dependence is determined by the energy-containing interval. To determine the equal-time correlators the kinematic effect of transport is eliminated at a certain point in space, and this necessarily leads to a diagram technique that is nondiagonal in the momenta. The convergence of all the integrals in the scale-invariant solutions, both in the ultraviolet and in the infrared region of the spectrum, is proved. This gives a diagrammatic proof of the hypothesis that the interaction is local, and provides a basis for a cascade energy-transfer mechanism. The asymptotic forms of the equal-time many-point velocity correlators are found.

INTRODUCTION

It is well known[1-3] that in the problem of fully developed hydrodynamic turbulence two substantially different interactions are present at the same time—the dynamical interaction of vortices with similar sizes $1/k$ ($\mathbf{k}$ is the momentum), which leads to the exchange of energy between them with frequency γ_k, and the kinematic effect of the transport of k-vortices as a whole by the almost uniform velocity field V_T of large-scale vortices of the energy-containing size L, this effect being characterized by the Doppler frequency $kV_T \gg \gamma_k$. The problem is to distinguish and study, in the formal apparatus of the theory (e.g., in the diagram technique (DT) of Wyld[4]) the relatively weak dynamical γ_k interaction determining the turbulence spectrum against the background of the transport effect masking it.[5] Unfortunately, numerous attempts to solve this problem, for all orders of perturbation theory, by the Lagrangian approach of Ref. 6 with the introduction of a separating scale,[7-9] by renormalization-group methods,[10-12] by an "internal" DT (Ref. 13), and by the introduction of "ballistic" modes[14] have not led to success.

In Sec. 1 of this paper we give a functional formulation of the Wyld DT for hydrodynamics in Eulerian variables, analyze the resulting divergences, and show that it is possible to sum the most divergent sequence of diagrams exactly and to determine the frequency dependence of the pair correlator (PC) and Green's function (GF). This frequency dependence is determined by statistical characteristics of the turbulence in the energy-containing interval of scales, i.e., by how the turbulence is excited, and is not universal. It may also be noted that the transport approximation in the formal scheme of the DT corresponds to the well-known approximation of "frozen turbulence" in the phenomenological theory of turbulence. In Sec. 2 different methods of eliminating the transport by means of a transformation to the comoving reference frame are discussed. If the velocity $\mathbf{V}$ of the latter is assumed to be statistically independent of the turbulent velocity $\mathbf{v}(t, \mathbf{r})$ (t is the time and $\mathbf{r}$ is the position), a relatively simple "internal" DT arises, in which, unfortunately, the divergences of the leading diagrams remain. If we set $\mathbf{V} = \mathbf{v}(t,\mathbf{r}_0)$, the transport in the neighborhood of the point $\mathbf{r}_0$ is eliminated completely. However, the DT in these variables—a quasi-Lagrangian DT—turns out to be nonlocal in $\mathbf{k}$-space. In Sec. 2 we have elucidated the basic properties of this DT. In the GF and PC it has been found extremely useful to go over from the purely momentum ($\mathbf{k}'$, $\mathbf{k}''$ representation to the mixed ($\mathbf{k}$,$\mathbf{r}$) representation [$\mathbf{k} = 1/2(\mathbf{k}' + \mathbf{k}'')$]. In this approximation it is easy to formulate a quasiclassical (QC) approximation, in which the GF and PC are assumed to be independent of $\mathbf{r}$ and are taken at the point $\mathbf{r}_0$ at which the transport is most effectively eliminated. We show that the QC approximation in the theory of turbulence gives qualitatively correct results, despite the fact that it has no formal parameter determining its applicability. In Sec. 2 we have show that in the QC approximation there are no divergences at all in the theory. In Sec. 3 we prove that this important result is preserved in the full theory. For this, in particular, we had to find the asymptotic forms of the GF and PC in ω (ω is the frequency) and r. Thus, we have executed a complete diagrammatic proof of the local-interaction hypothesis put forward by Kolmogorov and Obukhov in 1941. In particular, this implies that the turbulence spectrum for $\mathrm{Re} \to \infty$ should be described by a "5/3 law". However, our work not only proves this fact but also gives a systematic constructive apparatus for investigating other properties of turbulence as well. In particular, we have found the asymptotic forms of the single-time nth-order velocity correlators (n-VC) when one of the momenta or the sum of a group of momenta tends to zero.

1. THE TRANSPORT APPROXIMATION

1.1. The diagram technique. We shall start from the Navier-Stokes equations for an incompressible liquid[1]:

$$\partial \mathbf{v}/\partial t + (\mathbf{v}\nabla)\mathbf{v} + \nabla p - \nu\Delta\mathbf{v} = 0, \quad \operatorname{div}\mathbf{v} = 0. \tag{1.1a}$$

In the $(t,\mathbf{k})$ representation these can be written in the form

$$\frac{\partial v^\alpha(t,\mathbf{k})}{\partial t} = \frac{1}{2}\Delta_{\mathbf{k}}^{\alpha\beta}\int \gamma_{\mathbf{k}12}^{\beta\gamma\delta} v^\gamma(t,\mathbf{k}_1)\, v^\delta(t,\mathbf{k}_2)\, d\mathbf{k}_1\, d\mathbf{k}_2 - \nu k^2 v^\alpha(t,\mathbf{k}), \tag{1.1b}$$

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where $\Delta_{\mathbf{k}}^{\alpha\beta}$ is the transverse projector, and the Eulerian vertex is a homogeneous first-order function of $\mathbf{k}$:

$$\Delta_{\mathbf{k}}^{\alpha\beta}=\delta_{\alpha\beta}-k^{\alpha}k^{\beta}/k^{2},$$

$$\gamma_{\mathbf{k}12}^{\alpha\beta\gamma}=i\,(k^{\beta}\delta_{\alpha\gamma}+k^{\gamma}\delta_{\alpha\beta})\,\delta\,(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2). \qquad (1.2)$$

To describe the turbulence that arises in the flow around some body, e.g., a grid, we should have to solve these equations with given boundary conditions. Rather than consider this very complicated problem, we shall follow the idea of Kraichnan and Wyld[5,4] and consider an unbounded volume of liquid with zero average velocity, modeling the excitation of the turbulence by means of a random force $\mathbf{f}(t,\mathbf{k})$ introduced into the right-hand side of the equation. For convenience we shall assume that the statistics is Gaussian and that its correlator $D^{\alpha\beta}(\omega,\mathbf{k})$ is concentrated in a region of order L^{-1} in k and of order V_T/L in ω. In addition, for formal reasons we shall also add a vanishingly small regular external force $\mathbf{h}(t,\mathbf{k})$. Thus, the total external force is $\boldsymbol{\varphi}=\mathbf{f}+\mathbf{h}$. To calculate the velocity correlators $\mathbf{v}(\omega,\mathbf{k})$ it is sufficient to know the generating functional of the correlators of the velocity field:

$$Z(\boldsymbol{\psi},\mathbf{h})=\left\langle \exp\left(i\int \boldsymbol{\psi}^{*}(q)\mathbf{v}(q)d^{4}q\right)\right\rangle,\quad q=(\omega,\mathbf{k}). \qquad (1.3)$$

Here $\mathbf{v}(q)$ is a functional of the force $\mathbf{h}(q)$, and the averaging is performed over the Gaussian random field $\mathbf{f}(q)$. Using the functional methods of quantum field theory for Z, we can obtain an expression in the form of a double functional integral over the random velocity $\mathbf{v}(q)$ and the momentum $\mathbf{p}(q)$ conjugate to it,[2,11,12,15,16] with $\mathbf{v}^{*}(q)=\mathbf{v}(-q)$ and $\mathbf{p}^{*}(q)=\mathbf{p}(-q)$:

$$Z(\boldsymbol{\psi},\mathbf{h})=\int\prod_{i,q}dv^{i}(q)\,dp^{i}(q)\exp\Big[i\int d^{4}q\,(\mathscr{L}(q)+\boldsymbol{\psi}^{*}(q)\,\mathbf{v}(q)+\mathbf{h}^{*}(q)\,\mathbf{p}(q)\Big],\quad \mathbf{kv}(q)=\mathbf{kp}(q)=0, \qquad (1.4)$$

where the Lagrangian $\mathscr{L}$ has the form

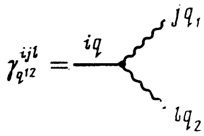
$$\mathscr{L}=p^{i*}(q)\left[(i\omega+\nu k^{2})v^{i}(q)+{}^{1}/_{2}iD^{ij}(q)p^{j}(q)+\int\gamma_{k12}^{ijl}v^{j}(\omega-\omega',\mathbf{k}_1)v^{l}(\omega',\mathbf{k}_2)d\omega'\,d\mathbf{k}_1\,d\mathbf{k}_2.\right. \qquad (1.5)$$

The expressions (1.4) and (1.5) lead to a DT with the usual Feynman rules for the two-component vector field $x_1^i v_i$, $x_2^i=p^i$, the matrix Green's function $\hat{g}(q)$ of which satisfies a Dyson equation, from which, for the velocity pair correlator $F^{ij}=g_{11}^{ij}$ and Green's function $G^{ij}=g_{12}^{ij}$ there follows the well-known system of Dyson-Wyld equations:

$$\hat{G}=\hat{G}^{0}+\hat{G}^{0}\hat{\Sigma}\hat{G}, \qquad (1.6)$$

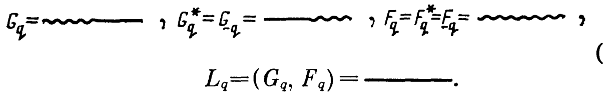
$$\hat{F}=\hat{G}(\hat{D}+\hat{\Phi})\hat{G}^{*}, \qquad (1.7)$$

in which the mass operators (MO) Σ and Φ can be expressed in terms of G and F using Feynman rules with a vertex of one type



$$\gamma_{q12}^{ijl}=\;\cdots\;=\gamma_{\mathbf{k}}^{ijl}\;\delta(q-q_1-q_2). \qquad (1.8)$$

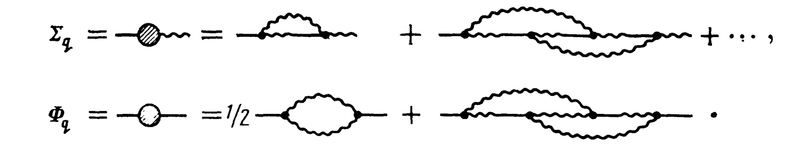
One solid line enters the vertex and two wavy lines leave it, if for G and F we adopt the "natural" graphical notation dictated by their definitions:



(1.9)

The common graphical notation for the line L_q will be used in cases when it is not necessary to distinguish pair correlators from Green's functions.

We give the first diagrams for Σ_q and Φ_q:



(1.10)

The topological properties of the diagrams are described in Ref. 17. We note also that in the isotropic case the incompressibility condition gives $L_q^{ij}=\Delta_{\mathbf{k}}^{ij}L_q$, and the vector equations (1.7) become scalar.

1.2. Kolmogorov scaling. In the inertial range of scales we can omit the external-force correlator D in Eqs. (1.6), (1.7). Then it is natural to seek their solution in the scale-invariant form[18,17]

$$G_q=k^{-\nu}g(\omega/k^{\nu}),\quad F_q=k^{-(\nu+\mu)}f(\omega/k^{\nu}). \qquad (1.11)$$

Here, for simplicity, we have omitted the dimensional constants needed to make f and g dimensionless functions of dimensionless arguments. If the integrals in the diagrammatic series (1.10) converge in the region of large and small momenta, it would be possible to extend them beyond the limits of the inertial interval to the whole of q-space. Then the indices ν and μ would satisfy the following scaling relation[18]:

$$2\nu+\mu=5 \qquad (1.12)$$

and all the diagrams in the series (1.10) would be of the same order of magnitude. The relation (1.12) is easily obtained if we take into account that each succeeding order of perturbation theory adds two vertices (1.8), two GF, one PC, and one integration over d^4q. The second relation for the indices ν and μ follows from the condition of constancy of the energy flux over the spectrum ε_k, i.e., is independent of the length scale the quantity $\varepsilon_k:\varepsilon_k\propto k^{0}$. It can be found from the size dependence of the 3-VC $F^{(3)}$:

$$\varepsilon\approx k^{3}\langle v(t,\mathbf{k})\dot{v}(t,\mathbf{k})\rangle\approx k^{7}F^{(3)}(t,\mathbf{k}),$$
$$F^{(3)}(t,k)\approx kG_q\omega^{2}F_q^{2}\approx k^{2\nu+1}G_qF_q^{2}. \qquad (1.13)$$

The length dependence of $F^{(3)}$ can be found from the first diagram if we assume that all the integrals in the leading diagrams converge and the relation (1.12) is fulfilled. Combining (1.12) and (1.13), we find the second relation for the scaling indices:

$$\nu+2\mu=8. \tag{1.14}$$

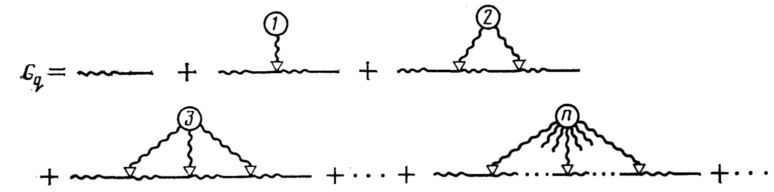
The solution of the relations (1.12) and (1.14) is the Kolmogorov solution $\nu = 2/3, \mu = 11/3$. This is not surprising, since these values of the indices follow from dimensional considerations with the assumption that L and λ do not appear in the expressions for F_q and γ_k because the interaction is local. In fact, the integrals in the diagrams (1.10) diverge in the region of small k when the scaling functions (1.11) are substituted into them, both for the Kolmogorov indices and for indices differing from the latter by no more than 2/3. Therefore, the solution of the form (1.11) with $\nu = 2/3$ and $\mu = 11/3$ is not actually realized, and the problem of the determination of the scaling indices remains open.

1.3 The "bare backbone" representation and the Green's function in the transport approximation. For the investigation of the structure of the contributions of the region of small momenta to the GF the Dyson representation for the GF, in which the one-particle-irreducible graphs are separated out, is inconvenient. More adequate is the "bare backbone" representation, in which the Green's functions on the backbone remain bare while the Green's functions on the ribs of the diagrams, and all pair correlators, are taken to be dressed. It can be seen from the structure of this series that the sum of all possible many-point diagrams with a fixed number n of exits onto the backbone is the complete dressed n-VC. Consequently, in this series we can sum the subsequences with a fixed number $n + 1$ of bare Green's functions in the backbone:

$$G_q = \sum_{n=0}^{\infty} \int \Pi^{\alpha_1\ldots\alpha_n}_{q_1\ldots q_n}(q)\, F^{\alpha_1\ldots\alpha_n}_{q_1\ldots q_n}\, d^4q_1 \ldots d^4q_n, \tag{1.14a}$$

$$\Pi^{\alpha_1\ldots\alpha_n}_{q_1\ldots q_n}(q) = {}^1/_2\Delta_{\mathbf{k}}{}^{ij} G_q^{0,\, i\beta_1} \gamma_{\mathbf{k}}^{\beta_1\alpha_1\beta_2} G^{0,\, \beta_2\beta_2'}_{q+q_1} \ldots \gamma^{\beta_{n-1}\alpha_n\beta_n}_{\mathbf{k}+\mathbf{k}_1+\ldots+\mathbf{k}_{n-1}} G_q^{0,\, \beta_n j}.$$

Here Π is the backbone and F is the complete n-VC. Analysis of this expression shows that the main contribution to it is given by the integration in the region of small ω_j, k_j flowing into the backbone. This makes it possible to neglect q_j in comparison with q in the arguments of the GF of the backbone and to integrate over all q_j in the n-VC, which thus become single-point correlators. Graphically, this procedure can be represented in the form

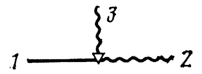


(1.15)

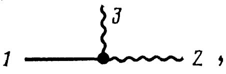
Here n is the complete (reducible) single-point n-VC

$$\langle v^{\alpha_1}(t,\mathbf{r}) \ldots v^{\alpha_n}(t,\mathbf{r})\rangle = \int F^{\alpha_1\ldots\alpha_n}_{q_1\ldots q_n}\, d^4q_1 \ldots d^4q_n, \tag{1.16}$$

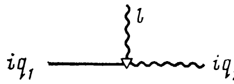
and



is the asymptotic form of the Eulerian vertex



when $q_3 \to 0$:



$$= ik_1{}^l \delta_{ij} \delta(q_1 - q_2). \tag{1.17}$$

Henceforth, we shall call this vertex "kinematic". It is not difficult to see that the n-term of the series (1.15) is $(G_q^0)^{n+1}\langle (kv)^n\rangle$, and, correspondingly, the sum of this geometric progression can be represented in the form designated in Ref. 13 as a Green's function in the transport approximation:

$$G_q = G_q^{\rm tr} = \langle (i\omega - i\mathbf{k}\mathbf{V} + \nu k^2)^{-1}\rangle \mathbf{V}. \tag{1.18}$$

It is a GF of noninteracting k-vortices, in which the frequency $\omega - \mathbf{k}\cdot\mathbf{V}$ includes a Doppler shift on account of the uniform transport of the vortices with velocity $\mathbf{V}$ and in which the averaging is performed over the ensemble of the turbulent velocity $\mathbf{v}(t,\mathbf{r})$ at a fixed point of space and at a single time. It follows from (1.18) that

$$G_q^{\rm tr} = (kV_T)^{-1} g(\omega/kV_T), \quad V_T{}^2 = \langle (\mathbf{v}(r,t))^2\rangle \approx L^{2(1-\nu)}. \tag{1.19}$$

Thus, it must be regarded as established that the scaling index ν for the Green's function is equal to 1 and not 2/3. In other words, the frequency in the Green's function is made dimensionless by dividing it by the Doppler transport frequency kV_T and not by the frequency $\gamma_{\mathbf{k}}$ of the dynamical interaction of the vortices.

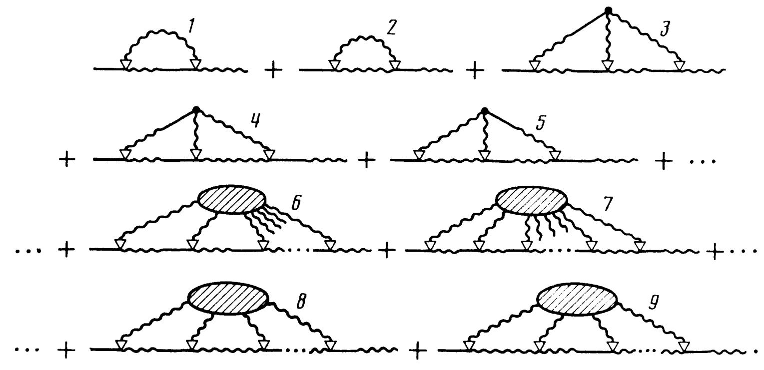
1.4. The pair correlator in the transport approximation. In the transport approximation it is more convenient to find the pair correlator not from the Wyld equation (1.7) but from the generalized kinetic equation (GKE), which is easily derived from the Wyld equation by multiplying by G_q and G_q^* and adding the results:

$$S_q = \mathrm{Re}\,[\hat{\Sigma}_q \hat{F}_q + \hat{\Phi}_q \hat{G}_q] = 0. \tag{1.20}$$

The term GKE arises from the fact that in the case of weak coupling, after the integration over ω, S_q goes over into the usual collison integral. We note also that S_q can be expressed in terms of the complete 3-VC, convolved with the bare vertex (1.2). In accordance with the results of Sec. 1.3, the PC must be sought in the form

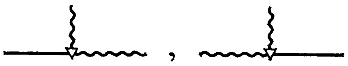
$$F_q = V_T k^{-4} (kL)^{-\eta} f(\omega/kV_T). \tag{1.21}$$

Here, in contrast to (1.12), the dimensional factors have been retained. Substituting (1.21) into the diagrammatic series (1.10), we can convince ourselves that for $\eta > 0$ the self-similar form (1.21) is reproduced if everywhere we take into account the leading IR divergences, cut off at $kL \cong 1$. Here, all the terms of the series for Φ_q are of the same order of magnitude and have the following structure: The large external momentum q flows along one of the split backbones of the diagram for Φ_q, and the whole of the remaining part of the diagram is attached to this backbone by wavy lines, along which free integration in the region $k' \ll k$, $\omega' \ll \omega$ is performed. Consequently, this part is one of the contributions to the one-point, equal-time VC. In graphical form, the GKE in the transport approximation has the form



(1.22)

The graphs 1, 3, and 6 with a wavy line F_q to the right correspond to the contribution $\Sigma_q F_q$, and the other graphs correspond to the contribution $\Phi_q G_q$. A small circle with wavy tails terminated by triangles denotes a one-point n-VC. These correlators are real. We now explain the mechanism by which S_q given by (1.20) vanishes identically. The graphs 1 and 2 differ in sign because the kinematic vertices



differ in sign, and, by virtue of the diagonal character of the kinematic vertex with respect to the four-momentum, the GF and PC can be interchanged. It is easy to see that the graphs 3, 4, and 5 also cancel out. The vanishing of the sum of graphs of the general form 6–9 (1.22) is guaranteed by the following algebraic identity:

$$G_q^{n}-(-1)^{n}G_q^{*n}=(G_q+G_q^{*})\sum_{m=0}^{n-1}(-1)^{m}G_q^{n-m-1}G_q^{*m}. \qquad (1.23)$$

Thus, the PC in the transport approximation remains arbitrary. Any F_q of the form (1.21) with $\eta > 0$ and an arbitrary structure function satisfies the diagrammatic equations. This result has a simple physical interpretation: In the transport approximation one does not take into account the dynamical interaction of the vortices, which determines the distribution of their energy over the scales. Consequently, this quantity remains arbitrary. However, if to complement the transport approximation we invoke the physical consideration that the frequency γ of the dynamical interaction is much smaller than kV_T, then the structure of F_q in ω can be determined uniquely:

$$F_q=V_T^2k^{-3}(kL)^{-\eta}\langle\delta(\omega-\mathbf{kV})\rangle_{\mathbf{V}}. \qquad (1.24)$$

It follows from this that the frequency dependence of the PC and GF is not universal, but is determined by statistical characteristics of the turbulent velocity in the energy-containing interval. The question of the structure of the single-time velocity correlators and of their universality in the transport approximation remains open.[13]

2. ELIMINATION OF THE TRANSPORT AND FORMULATION OF THE THEORY IN QUASI-LAGRANGIAN VARIABLES

2.1. Procedure for elimination of the transport, and the internal diagram technique. In order to get rid of the transport effect masking the dynamical interaction of the vortices, we go over from the laboratory frame to the comoving frame moving with the random velocity $\mathbf{V}(t)$ of the vortices. This transformation is implemented in the $(t,\mathbf{k})$ representation by multiplying the Eulerian velocity $\mathbf{v}(t,\mathbf{k})$ by the Doppler factor due to the random velocity $\mathbf{V}(t)$:

$$\mathbf{v}(t,\mathbf{k})=\hat{U}_{\mathbf{k}}(t,t_0)\mathbf{u}(t,\mathbf{k}),\quad \mathbf{p}(t,\mathbf{k})=\hat{U}_{\mathbf{k}}(t,t_0)\boldsymbol{\pi}(t,\mathbf{k}),$$
$$\hat{U}_{\mathbf{k}}(t,t_0)=\exp\Big[i\int_{t_0}^{t}\mathbf{kV}(\tau)\,d\tau\Big]. \qquad (2.1)$$

This change of variables leads to the appearance in the Lagrangian of a "kinematic" interaction vertex (1.17) describing the transport effect. There also appears a renormalization of the random force, which becomes non-Gaussian but, as before, is concentrated in the energy-containing region. We can convince ourselves that the topological structure of the diagrams in the inertial interval is not changed by this. It is important to note that at $t=t_0$ the velocity $\mathbf{u}(t,\mathbf{k})$ and the momentum $\boldsymbol{\pi}(t,\mathbf{k})$ conjugate to it coincide with the Eulerian velocity and momentum, and this leads to the equality of the equal-time GF and PC at $t=t_0$ in the two reference frames. This makes it possible to use the DT based on $\mathbf{u}_q$ and $\boldsymbol{\pi}_q$ to find the equal-time statistical characteristics of the velocity field and to eliminate the divergences inherent in the Eulerian DT.

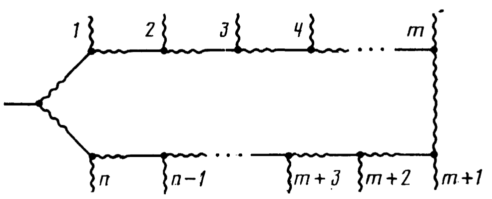
First we shall assume that the random velocity $\mathbf{V}(t)$ is statistically independent of $\mathbf{v}(t,\mathbf{k})$ and $\mathbf{p}(t,\mathbf{k})$, since the velocity correlators of $\mathbf{V}(t)$ are determined by the relation (1.16) and do not depend on the time. In this case, from the condition that the diagrammatic series for the Green's functions $\tilde{G}_q$ and pair correlators $\tilde{F}_q$ coincide, after averaging over the transport velocity $\mathbf{V}(t)$ we obtain the rules of the "internal static DT"[13]:

$$G_q=\langle\tilde{G}_{\omega-\mathbf{kV},\,\mathbf{k}}\rangle_{\mathbf{V}},\quad F_q=\langle\tilde{F}_{\omega-\mathbf{kV},\,\mathbf{k}}\rangle_{\mathbf{V}}. \qquad (2.2a)$$

An explicit expression for the GF and PC of the internal DT in terms of its MO in this case is given by the usual Dyson and Wyld equations (1.6) and (1.7). The graphical notation for the diagrams in the internal DT is not changed, but the rules for associating analytical expressions with them are changed and will contain counterterms that subtract the transport from all the irreducible velocity correlators (pair, three-point, etc.). For example,

$$F_q \to K_q = F_q - \delta^4(q) \int F_{q'}\, d^4q'. \tag{2.2b}$$

The above approach to the study of turbulence was proposed in a paper of one of the authors (V. S. L.)[13] and led to the cancellation of the IR divergences in second order in the vertices in the mass operators $\tilde{\Sigma}_q$ and $\tilde{\Phi}_q$, making it possible to formulate an "improved" direct-interaction approximation in terms of which the Kolmogorov-Obukhov solution is obtained.[19] Diagrams not containing GF outside the backbones and describing Gaussian fluctuations of the velocity field also converge in the IR region. However, the fourth-order diagrams with GF outside the backbone, which describe the 3-VC, already contain IR divergences on a spectrum that is close to the Kolmogorov-Obukhov spectrum. These divergences grow in higher orders, and the theory turns out to be nonnormalizable in the IR region. The internal DT also contains divergences in the UV region, which are associated with the fact that the combination of the vertex and GF on the Kolmogorov-Obukhov solution behaves like $k^{1/3}$. Therefore, a large ring containing 1-PC and n-FG



diverges like $k^{n/3}$. The theory (Eulerian and internal) contains Ward identities,[20] which reflect the invariance of the equal-time quantities (e.g., $F_{\mathbf{k}}$) under a Galilean transformation: $\omega \to \omega - \mathbf{k}\cdot\mathbf{V}$. These identities can be obtained by differentiating $F_{\omega - \mathbf{k}\cdot\mathbf{V},\mathbf{k}}$ with respect to $\mathbf{V}$, using here the Wyld and Dyson equations and then integrating over ω. The Ward identities guarantee the vanishing of the renormalization of the nth-order vertex for external momenta tending to zero, and lead to subtractions from the ring shown above, thereby improving the convergence. But complete cancellation of the UV divergences does not occur.[20]

2.2. The quasi-Lagrangian VT. Since the attempt to eliminate the transport in a statistically independent manner turns out to be unsound, we shall take as the transport field $\mathbf{V}(t)$ the velocity $\mathbf{u}$ at a certain spatial point $\mathbf{r}_0$: $\mathbf{V}(t) = \mathbf{u}(t,\mathbf{r}_0)$.

In the $\mathbf{r}$-representation this leads to the following replacement of the Eulerian velocity $\mathbf{v}(\mathbf{r},t)$ by the new velocity $\mathbf{u}_{\mathbf{r}_0 t_0}(\mathbf{r},t)$:

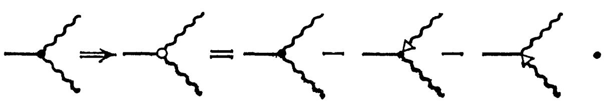
$$\mathbf{v}(\mathbf{r},t) = \mathbf{u}_{\mathbf{r}_0 t_0}\left(\mathbf{r} + \int_{t_0}^{t} \mathbf{u}_{\mathbf{r}_0 t_0}(\mathbf{r}_0,\tau)\, d\tau, t\right). \tag{2.3}$$

If the velocity of the liquid were constant, the quantity $\mathbf{u}_{\mathbf{r}_0 t_0}(\mathbf{r},t)$ would coincide with the Lagrangian velocity of a particle moving along a rectilinear trajectory. Since the true trajectories of the particles on the space-time scales of interest to us are almost rectilinear, it is natural to call the velocity $\mathbf{u}_{\mathbf{r}_0 t_0}(\mathbf{r},t)$ the Lagrangian velocity in the straight-line orbit approximation, or, more briefly, the "quasi-Lagrangian" velocity. We emphasize that the relation (2.3) is simply a change of variables, and does not contain any approximations. The equation for $\mathbf{u}_{\mathbf{r}_0 t_0}(\mathbf{r},t)$ follows from (1.1a) and (2.3) and differs from the Navier–Stokes equations (1.1a) by a term that subtracts the transport at the point $\mathbf{r}_0$:

$$\frac{\partial \mathbf{u}_{\mathbf{r}_0}(\mathbf{r},t)}{\partial t} + [(\mathbf{u}_{\mathbf{r}_0}(\mathbf{r},t) - \mathbf{u}_{\mathbf{r}_0}(\mathbf{r}_0,t))\nabla \mathbf{u}_{\mathbf{r}_0}(\mathbf{r},t)]$$
$$+\nabla p_{\mathbf{r}_0}(r,t) - \nu\Delta \mathbf{u}_{\mathbf{r}_0}(\mathbf{r},t) = 0, \quad \operatorname{div} \mathbf{u}_{\mathbf{r}_0}(\mathbf{r},t) = 0. \tag{2.4}$$

The time t_0 does not appear explicitly in this equation, and therefore the correlators of the quasi-Lagrangian velocity $\mathbf{u}_{\mathbf{r}_0}(\mathbf{r},t)$, like the correlators of the Eulerian velocity $\mathbf{v}(\mathbf{r},t)$, are invariant under displacement in time. Therefore the index t_0 in the velocity $\mathbf{u}_{\mathbf{r}_0 t_0}(\mathbf{r},t)$ is omitted in Eq. (2.4) and below. In the $(t,\mathbf{k})$ representation the equation of motion for the quasi-Lagrangian velocity $\mathbf{u}_{\mathbf{r}_0}(t,\mathbf{k})$ reproduces the original Navier-Stokes equations (1.1b), but with another quantity $\tilde{\gamma}$ which we call "dynamical":

$$\gamma^{ijl}_{q^{12}} \to \tilde{\gamma}^{ijl}_{q^{12}} = i\{k_2{}^j \delta_{il}[\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) - e^{i\mathbf{k}_1\mathbf{r}_0}\delta(\mathbf{k}-\mathbf{k}_2)]$$
$$+ k_1{}^l \delta_{ij}[\delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2) - e^{i\mathbf{k}_2\mathbf{r}_0}\delta(\mathbf{k}-\mathbf{k}_1)]\}\,\delta(\omega-\omega_1-\omega_2),$$



(2.5)

The fundamental difference between the quasi-Lagrangian theory with the vertex (2.5) and the theory with the Eulerian vertex (1.2) is that the momentum is no longer conserved at the vertex, and, consequently, the GF and PC become nondiagonal in the momentum:

$$L^{ij}(\omega,\mathbf{k})\delta^4(q-q') \to L_{r_0}{}^{ij}(\omega,\mathbf{k},\mathbf{k}')\delta(\omega-\omega'). \tag{2.6a}$$

In the case of isotropic turbulence the incompressibility condition makes it possible to express L^{ij} in terms of two scalar functions L_1 and L_2 of ω, k^2, k'^2, and $(\mathbf{k}\cdot\mathbf{k}')$:

$$L^{ij}(\omega,\mathbf{k},\mathbf{k}') = \Delta_{\mathbf{k}}^{ii'}\Delta_{\mathbf{k}'}^{jj'}(L_1\delta_{i'j'} + L_2 k'^{i'}k^{j'}/kk'). \tag{2.6b}$$

In the Eulerian diagram technique, $L(q',q'')$ was proportional to $\delta^4(q'-q'')$. The theory is nondiagonal in the difference $\mathbf{s} = \mathbf{k}' - \mathbf{k}''$ as a consequence of the explicit spatial nonuniformity of the theory, since in the original variables we have singled out a point $\mathbf{r}_0$ determining the velocity $\mathbf{u}_{\mathbf{r}_0}(t,\mathbf{r}_0)$ with which the comoving reference frame moves. It is convenient to go over to the mixed $(\mathbf{k},\mathbf{r})$ representation:

$$L_{r_0}{}^{ij}(\omega,\mathbf{k},\mathbf{r}) = \int \exp(i\mathbf{k}\mathbf{r}) L_{r_0}{}^{ij}(\omega,\mathbf{k},\mathbf{s})\, d\mathbf{s}, \tag{2.7}$$

$$L_{r_0}{}^{ij}(\omega,\mathbf{k},\mathbf{s}) = L_{r_0}{}^{ij}(\omega,\mathbf{k}',\mathbf{k}''), \quad \mathbf{k} = {}^1/_2(\mathbf{k}'+\mathbf{k}''), \quad \mathbf{s} = \mathbf{k}'-\mathbf{k}''.$$

This method is dictated by the quasiclassical (QC) approximation,[21] which is adequate for the problem of the motion of a liquid with two substantially different scales L and k^{-1}. The quantities $G_q(\mathbf{r})$ and $F_q(\mathbf{r})$ have the physical meaning discussed above: They are the pair correlator and Green's function near the point $\mathbf{r}$. Both this circumstance, and the fact that the formulas (2.7) also have a strictly defined meaning outside the framework of the QC situation, will be important for us in the following.

We shall show that the GF and the pair correlator $L_{\mathbf{r}_0}(q,\mathbf{r})$ depend in fact on the difference $\mathbf{r}-\mathbf{r}_0$ and not on $\mathbf{r}$ and $\mathbf{r}_0$ separately. This is a consequence of the spatial uniformity of the problem of fully developed hydrodynamic turbulence. In fact, if in Eqs. (1.1b) with the vertex (2.5) we replace the velocity $u_{\mathbf{r}_0}(q)$ by $\tilde{\mathbf{u}}_{\mathbf{r}_0}(q)\exp(i\mathbf{k}\cdot\mathbf{r}_0)$, the dependence on $\mathbf{r}_0$ vanishes in the vertex (2.5); this proves that the entire dependence on $\mathbf{r}_0$ is contained in $\exp(i\mathbf{k}\cdot\mathbf{r}_0)$, and this, in combination with formula (2.7) for $L(q,\mathbf{s})$, quickly proves our assertion. In the following we can set $\mathbf{r}_0=0$, and the transport is then eliminated at the coordinate origin at $t=t_0$. We note that the quasi-Lagrangian variables (2.1) and quasi-Lagrangian equations (2.4), (2.5) were proposed by one of us (V. S. L.) in 1980. The necessity of using these variables for a scale-invariant formulation of the theory of turbulence was noted in the review Ref. 19. For the Hamiltonian formulation of the theory of turbulence in Clebsch variables, calculations similar to (2.5) (but with $\mathbf{r}_0=0$) were proposed independently in Ref. 14. However, in this and subsequent papers starting from Clebsch variables, a detailed analysis of the resulting theory was not carried out. In particular, the authors of Ref. 14 did not draw attention to the fact that a theory with subtractions of the type (2.5) becomes nondiagonal in the momenta, and that its objects $L_{q'q''}$ depend on two three-dimensional variables $\mathbf{k}'$, $\mathbf{k}''$ and ω', and not on ω and $\mathbf{k}$. In a more recent paper,[22] the nondiagonal form of $L_{q'q''}$ in the momenta was taken into account, but the structure of these functions in $q'-q''$ was not investigated; without such an investigation the conclusions concerning the convergence of the integrals in the theory remain hypotheses.

The rules of the quasi-Lagrangian DT are more complicated than those of the Eulerian DT because of the nondiagonal form of the GF and PC in the momenta $\mathbf{k}',\mathbf{k}''$: Momentum $\mathbf{k}+\frac{1}{2}\mathbf{s}$ flows into each line, while $\mathbf{k}-\frac{1}{2}\mathbf{s}$ flows out. Consequently, each line "loses" the momentum $\mathbf{s}$, over which the integration is performed. Momentum $\mathbf{k}+\frac{1}{2}\mathbf{s}$ flows into each vertex, and momenta $\mathbf{k}_1-\frac{1}{2}\mathbf{s}_1$ and $\mathbf{k}_2-\frac{1}{2}\mathbf{s}_2$ flow out. In view of the complication of the quasi-Lagrangian theory involving the nondiagonal form of the GF and PC in the momentum, we shall formulate and analyze first the QC approximation, which pretends only to a qualitatively correct description of the properties of turbulence.

2.3. The quasiclassical approximation. Since in the QC approximation the GF and PC have the form of a sharp peak in the variable $\mathbf{s}$, and, with $\mathbf{r}_0=0$ in (2.7), every diagram is averaged over $\mathbf{s}$, we can neglect the dependence on the given $\mathbf{s}_i$ in all parts of the diagram except $L(q_i,\mathbf{s}_i)$, and then integrate over all the variables $\mathbf{s}_i$ and obtain closed Dyson and Wyld equations in terms of local GF and PC L^l_q defined by the simple relation

$$L^l(q)=\int L(q,\mathbf{s})\,d\mathbf{s}. \tag{2.8}$$

In this case the Dyson and Wyld equations take an integral form:

$$G^l_q=G^0_q+\int G^0(\omega,\mathbf{k}+{}^1/{}_2\mathbf{s})\Sigma^l_\omega(\mathbf{k},\mathbf{s})G^l(\omega,\mathbf{k}-{}^1/{}_2\mathbf{s})\,d\mathbf{s}, \tag{2.9}$$

$$F^l_q=\int d\mathbf{s}\,G^l(\omega,\mathbf{k}+{}^1/{}_2\mathbf{s})\Phi^l_\omega(\mathbf{k},\mathbf{s})G^{*l}(\omega,\mathbf{k}-{}^1/{}_2\mathbf{s}),$$

$$L^{ij,l}_q=\Delta^{ij}_{\mathbf{k}}L^l_q.$$

Here $\Sigma^l_\omega(\mathbf{k},s)$ and $\Phi^l_\omega(\mathbf{k},s)$ are mass operators with inflowing momentum $\mathbf{k}+\frac{1}{2}\mathbf{s}$ and outflowing momentum $\mathbf{k}-\frac{1}{2}\mathbf{s}$. These MO are functionals of L^l_q. The diagrammatic series for the MO Σ^l and Φ^l have the same form as in the Eulerian DT, but the dynamical vertex (2.5) is the difference of the complete Eulerian vertex and the kinematic vertex. Consequently, the momentum in the MO Σ^l and Φ^l is not conserved, and the general δ-function expressing the law of conservation of momentum in the local theory cannot be separated out in the local mass operators; this is the reason for the presence of the additional integration in (2.9) in comparison with the usual DT.

The solution of the diagrammatic equations (2.9) for the local functions G^l_q and F^l_q is naturally sought in a scale-invariant form analogous to (1.11), by assuming the scaling relation (1.12) to be fulfilled. We shall prove that in the diagrammatic equations (2.9) both IR and UV divergences are absent. Our proof is based on the explicit asymptotic form of the dynamical vertex $\tilde{\gamma}^{\alpha\beta\gamma}_{\mathbf{k}12}$ (2.5) for the case when the momenta $\mathbf{k}$, $\mathbf{k}_1$, and $\mathbf{k}_2$ differ in order of magnitude. We shall give the asymptotic forms:

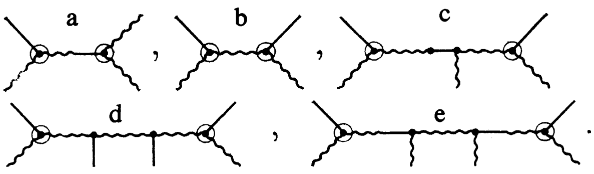
$$\begin{aligned}
&\text{A},\quad \mathbf{k}\to 0:\quad \tilde{\gamma}^{\alpha\beta\gamma}_{\mathbf{k}12}\to k^\beta\delta_{\alpha\gamma}[\delta(\mathbf{k}_1+\mathbf{k}_2)-\delta(\mathbf{k}_2)]\\
&\qquad\qquad +k^\gamma\delta_{\alpha\beta}[\delta(\mathbf{k}_1+\mathbf{k}_2)-\delta(\mathbf{k}_1)],\\
&\text{B},\quad \mathbf{k}_1\to 0:\quad \tilde{\gamma}^{\alpha\beta\gamma}_{\mathbf{k}12}\to \mathbf{k}_1^\gamma\delta_{\alpha\beta}[\delta(\mathbf{k}-\mathbf{k}_2)-\delta(\mathbf{k})]\\
&\qquad\qquad -k^\beta k_1^\delta\delta_{\alpha\gamma}\delta_\delta'(\mathbf{k}-\mathbf{k}_2)\\
&\text{C},\quad \mathbf{k}_1,\mathbf{k}_2\to 0:\quad \tilde{\gamma}^{\alpha\beta\gamma}_{\mathbf{k}12}\to -(k_1^\delta k_2^\beta\delta_{\alpha\gamma}+k_2^\delta k_1^\gamma\delta_{\alpha\beta})\delta_\delta'(\mathbf{k}),\\
&\text{D},\quad \mathbf{k},\mathbf{k}_1\to 0:\quad \tilde{\gamma}^{\alpha\beta\gamma}_{\mathbf{k}12}\to k_1^\gamma\delta_{\alpha\beta}[\delta(\mathbf{k}_2)-\delta(\mathbf{k}-\mathbf{k}_1)].
\end{aligned} \tag{2.9a}$$

Here $\delta'_\alpha(\mathbf{k})=\partial\delta(\mathbf{k})/\partial k^\alpha$. The asymptotic forms for $\mathbf{k}_2\to 0$ and $\mathbf{k}$, $\mathbf{k}_2\to 0$ are easily found by interchanging the indices 2 and 1. The asymptotic forms (2.9) show that the vertex $\tilde{\gamma}$ (2.5) possesses a whole series of remarkable properties. Namely, when one of the momenta $\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$ is much smaller than other two, the vertex is proportional to this momentum (relations A and B). When one of the momenta $\mathbf{k}$, $\mathbf{k}_1$, $\mathbf{k}_2$ is much larger than the other two, the vertex either, for $k\gg k_1$, k_2, tends rapidly to zero (relation C), or, for $k_2\gg k$, k_1, tends to a constant that is independent of $\mathbf{k}_2$ (relation D). There are enough of these properties to prove the convergence of the diagrams in the quasiclassical approximation. First, we shall perform the integration over the internal frequencies ω_j for an arbitrary diagram. Here we shall assume that the three-momenta of the line are independent, since the three-momentum at the vertex $\tilde{\gamma}$ (2.5) is not conserved and it is not worthwhile to distinguish in the graphs separate fragments corresponding to particular paths of momentum flow. As a result, in each pair correlator the scaling factor $k^{-\mu}$ appears. In addition, a function Y of many variables will appear as a common factor of the diagram:

$$Y\left(\frac{\omega}{k^\nu},\frac{\mathbf{k}}{k},\frac{\mathbf{k}_1}{k},\ldots,\frac{\mathbf{k}_{3n-1}}{k}\right)=\int g^0\left(\frac{\tilde{\omega}_{n+1}}{k^\nu_{n+1}}\right)\dot{g}\left(\frac{\tilde{\omega}_{n+2}}{k^\nu_{n+2}}\right)\ldots g\left(\frac{\tilde{\omega}_{3n+1}}{k^\nu_{3n-1}}\right)f\left(\frac{\omega_1}{k_1^\nu}\right)f\left(\frac{\omega_2}{k_2^\nu}\right)\ldots f\left(\frac{\omega_n}{k_n^\nu}\right)k_1^{-\nu}k_2^{-\nu}\ldots k_n^{-\nu}\,d\omega_1\ldots d\omega_n. \tag{2.10}$$

Here ω and $\mathbf{k}$ are the external variables of the diagram, $\mathbf{k}_1,...,\mathbf{k}_{3n-1}$ are the momenta corresponding to all the lines of a diagram with n pair correlators, and $\tilde{\omega}_{n+1},...,\tilde{\omega}_{3n-1}$ are linear combinations of the external frequency ω and the integration frequencies, as determined by the laws of conservation of the frequencies at the vertices. In the diagrams for the Wyld equation the expression for Y is modified slightly—the zero subscript on the first GF disappears, and one of the GF is replaced by a PC. The statement that the function Y is regular is very important. This function is finite in the whole range of variation of its arguments and can be expanded in a series in them. This follows from its explicit form (2.10). In the proof we shall assume that the structure function $f(\xi)$ is finite everywhere, and falls off sufficiently rapidly as $\xi\to\infty$; $\operatorname{Re}g(\xi)$ behaves analogously, while $\operatorname{Im}g(\xi)$ is equal to zero at $\xi=0$ and behaves like ξ^{-1} as $\xi\to\infty$. Consequently, the integral (2.10) exists for all values of the parameters appearing in it, and in analyzing the convergence of the integrals we can ignore the presence of the function Y in a diagram, assuming it to be a constant. Now our diagrams contain only three-dimensional integrations over the variable $\mathbf{s}$ and the $3n-1$ variables $\mathbf{k}$ inside the MO. After the integration the GF and PC can be assumed to be power functions $G_q\propto k^{-\nu}$, $F_q\propto k^{-\mu}$, and the analysis of the divergences is performed fairly simply. It is necessary to consider the IR and UV situations, when some fragment of the diagram contains a momentum much smaller or much greater than the external momentum $\mathbf{k}$. We must also distinguish two situations: A large (small) momentum flows inside the diagram or emerges at the ends of the diagram. We shall consider the IR situation inside some diagram. Topologically, three cases are possible: 1) A small momentum starts at one vertex and ends at another; 2) a small momentum flows around a ring; 3) a small momentum branches and encompasses a certain region.

In the first case the following variants of the flow of the small momentum are possible:



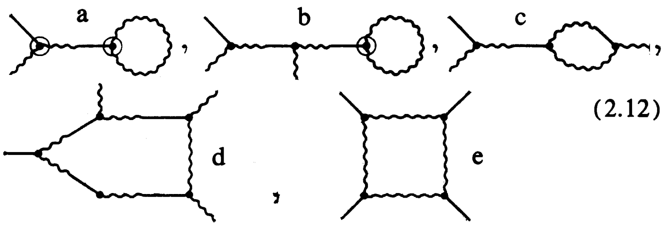
(2.11)

Here the vertices at which the small momentum starts or ends are surrounded by a small circle. Using the asymptotic representation (2.9a) of the vertex, we can easily calculate the IR convergence indices σ of the diagram with allowance for the scaling relation (1.12):

$$\sigma_a=5-\nu,\quad \sigma_b=2\nu,\quad \sigma_c=1+\nu,\quad \sigma_e=2,$$

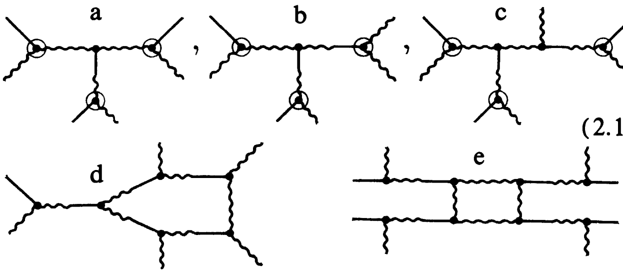
while the graph d simply does not agree with the asymptotic form (2.9), proportional to $\delta'_\alpha(\mathbf{k})$, of the vertex $\tilde{\gamma}$. Since $1>\nu>0$, all the $\sigma_a,...,\sigma_e$ are greater than zero and the fragments of the graphs (2.11) converge on the Kolmogorov spectrum.

In the second case we have the following variants of the flow of the small momentum:



(2.12)

The IR convergence indices of these graphs are as follows: $\sigma_a=\nu$, $\sigma_b=1$, and $\sigma_c=1$, while the graphs d and e do not agree with the asymptotic form (2.9a) of the vertex. We note also that the diagrams a and b are equal to zero because the pair correlators are transverse. Thus, the smallest margin of convergence in this group of graphs, equal to 1, is possessed by the graph c. In the third case we have the following variants of the flow of the small momentum:



(2.13)

The graphs (2.13) differ from the graphs (2.11) and (2.12) in that the small momentum branches. This implies that in the calculation of the convergence index of the diagram the phase volume must be taken to be not k^3 but k^{3n+3}, where n is the number of branchings of the small momentum. Taking this into account, we obtain the following IR convergence indices:

$$\sigma_a=3\nu-1,\quad \sigma_b=4,\quad \sigma_c=2\nu,\quad \sigma_d=4+(1-\nu)(n-1),$$

(n is the number of GF in the ring), and $\sigma_e=3+m\nu$ (m is the number of PC in the ring). On the Kolmogorov spectrum all these indices are positive. The analysis of the simplest types of paths of the small momentum in the diagrams proves the convergence of all these diagrams. Here an increase of the length of the path of the small momentum only improves the convergence. The situation when the paths of the small momenta $\mathbf{k}$ cover a certain region inside the diagram also gives rise to no dangers. The scaling relation guarantees that the increased complexity of the diagram does not lead to divergences.

We now consider the UV situation inside the diagrams. Topologically, the paths of large momentum are the same as the small-momentum paths considered above. In the first case, the graph (2.11a) is forbidden, since a large momentum cannot terminate at a solid-line entrance to a vertex [see the asymptotic form C in (2.9a)]. The UV convergence indices are as follows:

$$\sigma_a=2\nu-2,\quad \sigma_c=-2+\nu,\quad \sigma_d=6(\nu-2),\quad \sigma_e=-2.$$

Since all the UV indices are negative, the fragments of the graphs converge on the Kolmogorov spectrum. In the second case, the UV indices are as follows:

$$\sigma_a=-1+\nu,\quad \sigma_b=-1,\quad \sigma_c=-1,\quad \sigma_d=-2-(n-2)\nu,$$

(n is the number of GF in the ring), and $\sigma_e=3$

$-m(5-2\nu)$ (m is the number of PC in the ring). In this case the UV indices are also all negative. In the third case, we have the UV indices

$$\sigma_a=-3+3\nu, \quad \sigma_c=-3+2\nu, \quad \sigma_d=-1-\nu(n-1),$$

(n is the number of GF in the ring), and $\sigma_e = -m(1-\nu)$ (m is the number of PC in the ring). Thus, in all the cases considered, the UV convergence indices are negative and the diagrams converge. When the large momentum encompasses a considerable internal part of the diagram the scaling relation is operative and this considerable fragment also converges.

The fact that a small or a large momentum can emerge at the end of a graph gives rise to some concern. For example, the graphs

(2.14)

diverge logarithmically in the IR region. This is the only example of a divergence. Happily, the Ward identity considered in Ref. 20 makes it possible to prove that the divergent parts of these graphs cancel each other. UV divergences associated with the ends of graphs do not arise, because the large integration momentum **s** always encompasses at least one PC. Consequently, we have proved that IR and UV divergences are absent in the quasiclassical approximation for fully developed hydrodynamic turbulence. The interaction in this approximation is local, and, by virtue of the scaling relation (1.12) and energy conservation (1.14), the Kolmogorov picture is realized. In Sec. 3 we shall show that it is also preserved in the exact theory.

3. STRUCTURE FUNCTIONS OF THE QUASI-LAGRANGIAN THEORY AND THE LOCAL NATURE OF THE INTERACTION

The QC approximation considered in Sec. 2 gives a qualitatively correct picture of fully developed turbulence, but, naturally, cannot pretend to give a quantitative description of it, since the structure functions of the GF and PC $L(q,\mathbf{s})$ are not localized in **s** in the region $s \ll k$. In reality, we must expect that the GF and PC are localized in **s** in the region $\mathbf{s}\sim\mathbf{k}$, since (in the case of convergence of the integrals) there is no other parameter with the dimensions of momentum in the theory. In this section we shall find the asymptotic forms of the structure functions of the GF and PC in the variables **s** and ω, we shall show that the integrals of the structure functions over **s** converge, i.e., that the PC and GF for $\mathbf{r}=0$ exist ($L_q^l \neq \infty$; see (2.8)), and shall prove that the interaction is local in **k**-space, i.e., the integrals over **k** converge. Constructively, the proof of these statements is performed more conveniently in the reverse order. First we assume that the integral in (2.9) exists and prove that the interaction is local, and then we find the asymptotic forms of $L(q,\mathbf{s})$ in s and ω and convince ourselves that the integral of (2.9) over **k** converges.

3.1. Proof that the interaction is local. On the class of scale-invariant solutions of the hydrodynamic equations, the following representation is valid for the GF and PC:

$$G^{ij}(q,\mathbf{s})=s^{-3}k^{-\nu}g^{ij}\left(\frac{\omega}{k^\nu},\frac{\mathbf{k}}{k},\frac{\mathbf{s}}{k}\right),$$
$$F^{ij}(q,\mathbf{s})=s^{-3}k^{-\nu-\mu}f^{ij}\left(\frac{\omega}{k^\nu},\frac{\mathbf{k}}{k},\frac{\mathbf{s}}{k}\right), \qquad (3.1)$$

where the indices μ and ν satisfy the scaling relations (1.12). The structure functions should be everywhere finite and should tend to zero when $\omega/k^\nu\to\infty$, $s/k\to 0,\ \infty$. It is now obvious that the arguments proving the convergence of the diagrams in the QC approximation also remain valid in the complete theory. The integration of the MO over ω leads to a regular function

$$Y\left(\frac{\omega}{k^\nu},\frac{\mathbf{k}}{k},\frac{\mathbf{s}}{k},\frac{\mathbf{k}_1}{k},\frac{\mathbf{s}_1}{k},\frac{\mathbf{k}_2}{k},\ldots,\frac{\mathbf{k}_n}{k},\frac{\mathbf{s}_n}{k}\right), \qquad (3.2)$$

analogous to (2.10). Since each $\mathbf{s}_i$ is of the order of its $\mathbf{k}_i$, we can assume that the integration over $\mathbf{s}_i$ has been carried out, and then assume that $\mathbf{s}_i\cong\mathbf{k}_i$. Now the function Y that has arisen can be assumed to be a constant, and all the arguments of Sec. 2 concerning the convergence of the integrals over **k** remain valid. The problem of the "emergence" of the momenta at the ends of the graphs does not arise, since in the calculation of the diagrams for $G(q,\mathbf{s})$ and $F(q,\mathbf{s})$ it is not necessary to integrate over the momentum **s** corresponding to the ends of the diagrams.

The attempt to trace constructively the fact of the cancellation of the divergent contributions to the integrals in all ranges of variation of the integration variables for diagrams of high-order encounters calculational difficulties associated with the large number of terms in the corresponding analytical expressions. Even if we disregard the tensor structure (2.6b) of the GF and PC, each diagram with n vertices will contain 4^n terms, since the quasi-Lagrangian vertex (2.5) has four terms. The simplest graph ($n=2$) will contain 16 terms, and the next graphs ($n=4$) already have 256 terms. Therefore, we shall confine ourselves to illustrating the absence of IR divergences in the most dangerous region, when all the momenta k_i of the PC are small. In the internal DT, the integrals in this region diverge for $n\geqslant 4$. We note that the quasi-Lagrangian DT can be reformulated in terms of the Eulerian vertices (1.8) if the GF and PC G_ω $(\mathbf{k}',\mathbf{k}'')$ and F_ω $(\mathbf{k}',\mathbf{k}'')$ are replaced by the "subtracted" GF and PC $\widetilde{G}_\omega$ $(\mathbf{k}',\mathbf{k}'')$ and $\widetilde{F}_\omega$ $(\mathbf{k}',\mathbf{k}'')$:

$$\widetilde{G}_\omega(\mathbf{k}',\mathbf{k}'')=G_\omega(\mathbf{k}',\mathbf{k}'')-\delta(\mathbf{k}')\int G_\omega(\mathbf{k}_1,\mathbf{k}')\,d\mathbf{k}_1,$$
$$\widetilde{F}_\omega(\mathbf{k}',\mathbf{k}'')=F_\omega(\mathbf{k}',\mathbf{k}'')-\delta(\mathbf{k}')\int F_\omega(\mathbf{k}_1,\mathbf{k}'')\,d\mathbf{k}_1 \qquad (3.3)$$
$$-\delta(\mathbf{k}'')\int F_\omega(\mathbf{k}',\mathbf{k}_2)\,d\mathbf{k}_2+\delta(\mathbf{k}')\,\delta(\mathbf{k}'')\int F_\omega(\mathbf{k}_1,\mathbf{k}_2)\,d\mathbf{k}_1\,d\mathbf{k}_2.$$

In other words, the quasi-Lagrangian DT is also obtained if the vertices are taken to be Eulerian (conserving the four-momentum q), and the GF and PC are substituted into the internal lines in the subtracted form (3.3). In particular, in the quasi-Lagrangian DT the "bare backbone" representation (1.14a), or the representation equivalent to it for the MO Σ_q, is replaced by the subtracted representation

$$\Sigma_\omega(\mathbf{k}',\mathbf{k}'')=\sum_{n=1}^{\infty}\widetilde{\Pi}^{\alpha_1\ldots\alpha_n}_{\omega_1\mathbf{k}_1'\ldots\omega_n\mathbf{k}_n'}(\omega,\mathbf{k}',\mathbf{k}'')$$
$$[\delta(\mathbf{k}_1'-\mathbf{k}_1)-\delta(\mathbf{k}_1')]\ldots$$
$$\ldots[\delta(\mathbf{k}_n'-\mathbf{k}_n)-\delta(\mathbf{k}_n')]\,\widetilde{F}^{\alpha_1\ldots\alpha_n}_{iq_1\ldots q_n}\,d^4q_1\,d\mathbf{k}_1'\ldots d^4q_n\,d\mathbf{k}_n'. \qquad (3.4)$$

Here $\widetilde{\Pi}$ is the quasi-Lagrangian backbone constructed from

Eulerian vertices and subtracted GF (3.3), and $\widetilde{F}_i$ is the irreducible quasi-Lagrangian velocity correlator $\widetilde{F}_i \propto \delta(\omega_1 + \ldots + \omega_n)$, which, however, does not contain $\delta(\mathbf{k}_1 + \ldots + \mathbf{k}_n)$. It follows quickly from (3.4) that the IR divergences leading to the transport effect are absent in the quasi-Lagrangian DT, since, when the integration momenta $\mathbf{k}_1, \ldots, \mathbf{k}_n$ are small in comparison with the external momenta $\mathbf{k}', \mathbf{k}''$, the dangerous contribution to $\Sigma_\omega(\mathbf{k}',\mathbf{k}'')$ reduces to

$$\Sigma_\omega(\mathbf{k}', \mathbf{k}'') \backsim \sum_{n=1}^{\infty} \frac{\partial^n \widetilde{\Pi}^{\alpha_1 \ldots \alpha_n}_{\omega_1 \mathbf{k}_1 \ldots \omega_n \mathbf{k}_n}(\omega, \mathbf{k}', \mathbf{k}'')}{\partial k_1^{\beta_1} \partial k_2^{\beta_2} \ldots \partial k_n^{\beta_n}} \Bigg|_{k_1 = \ldots = k_n = 0} \times k_1^{\beta_1} \ldots k_n^{\beta_n} F^{\alpha_1 \ldots \alpha_n}_{q_1 \ldots q_n} d^4 q_1 \ldots d^4 q_n, \tag{3.5}$$

and on the scale-invariant solution (3.1) with the Kolmogorov index values $\nu = 2/3$ and $\mu = 11/3$ the integrals (3.5) converge.

3.2. Asymptotic forms of the structure functions in s and r. Under the assumption that all the integrals converge, we shall find the asymptotic form of the structure functions in s. For $s \gg k$ we can set $\mathbf{k} = 0$ in all the diagrams, since the arguments in the lines are the combinations $\mathbf{k} \pm \frac{1}{2}\mathbf{s}$, and this limit exists. Consequently, this asymptotic form is found from dimensionality and scaling considerations:

$$G^{ij}(q, \mathbf{s}) = s^{-11/3} g_\infty(\omega / s^{2/3}) \Delta_s^{ij}, \quad s \gg k, \qquad F^{ij}(q, \mathbf{s}) = s^{-22/3} f_\infty(\omega / s^{2/3}) \Delta_s^{ij}. \tag{3.6}$$

The asymptotic forms of the GF and PC $L(q,s)$ for $s \ll k$ are found from more complicated considerations. This is done most simply in the coordinate $(\omega,\mathbf{k},\mathbf{r})$ representation with $kr \gg 1$, with a subsequent transformation to the s-representation. First we shall put forward the physical arguments that fix this asymptotic form, and then we shall demonstrate how this form is determined by the diagrammatic series. By going over to the quasi-Lagrangian velocity (2.3) we have eliminated the transport at the point $\mathbf{r}_0 = 0$, and the GF and PC of interest to us pertain to a point $\mathbf{r}$ far from the coordinate origin. It is clear that those cancellations of IR divergences that are brought about by going over to the coordinate frame moving with velocity $\mathbf{v}(0)$ will be partly destroyed, since at the point $\mathbf{r}$ the vortices move with a different velocity $\mathbf{v}(\mathbf{r})$. As a result there will remain uncompensated transport with characteristic velocity

$$V_T(\mathbf{r}) = \langle (\mathbf{v}(\mathbf{r}) - \mathbf{v}(0))^2 \rangle^{1/2} \approx \varepsilon^{-1/3} r^{1/3}, \tag{3.7}$$

where $kV_T(\mathbf{r}) = \varepsilon^{-1/3} \mathbf{k}^{2/3} (kr)^{1/3}$ is the Doppler frequency of the transport over the scale r. This estimate for $V_T(\mathbf{r})$ follows from the fact that the main contribution to this velocity is made by vortex motions with scale r. It now becomes understandable why G_q and F_q for $kr \gg 1$ should have the same form as in the Eulerian DT in the transport approximation [(1.19) and (1.21)], with the replacement of V_T for the transport of L-vortices by $V_T(\mathbf{r})$ for the transport of r-vortices. It follows from this that, for the GF and PC,

$$G^{ij}(q, \mathbf{r}) = k^{-1} r^{-1/3} \tilde{g}_0(\omega / kr^{1/3}) \Delta_\mathbf{k}^{ij}, \quad kr \gg 1, \qquad F^{ij}(q, \mathbf{r}) = k^{-14/3} r^{-1/3} \tilde{f}_0(\omega / kr^{1/3}) \Delta_\mathbf{k}^{ij}. \tag{3.8}$$

Using these expressions for the GF and PC in q, $\mathbf{r}$, we can find the GF and PC in the (q,s) representation by means of Fourier transformation:.

$$G^{ij}(q, \mathbf{s}) = k^{-1} s^{-8/3} g_0(\omega s^{1/3}/k) \Delta_\mathbf{k}^{ij}, \quad s \ll k, \qquad F^{ij}(q, \mathbf{s}) = k^{-14/3} s^{-8/3} f_0(\omega s^{1/3}/k) \Delta_\mathbf{k}^{ij}. \tag{3.9}$$

We shall show that this asymptotic form of the "r-transport" is reproduced in the diagrammatic series. We shall use for this the bare-backbone representation (1.15). It is valid in the quasi-Lagrangian theory if we take into account that the sum of the momenta $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n$ flowing into the backbone is not equal to zero. The graphs containing kinematic vertices cutting the backbone are small, for $kr \gg 1$, in comparison with the main contribution. In this main contribution to the GF we can neglect the dependence of the backbone on the momenta flowing into it, and use a purely transport expression for the GF:

$$G_q(r) = \sum_{n=0}^{\infty} \int (e^{i\mathbf{k}_1\mathbf{r}} - 1)(e^{i\mathbf{k}_2\mathbf{r}} - 1) \ldots (e^{i\mathbf{k}_n\mathbf{r}} - 1) \times \Pi^{\alpha_1 \ldots \alpha_n}(q) F^{\alpha_1 \ldots \alpha_n}_{q_1 \ldots q_n} d^4 q_1 \ldots d^4 q_n. \tag{3.10}$$

Here $\Pi^{\alpha_1 \ldots \alpha_n}(q)$ is the backbone when the inflowing momenta $q_1, q_2, \ldots, q_n$ are equal to zero. This expression differs from the transport expression (1.16) in the presence of the factors $\exp(i\mathbf{k}_j \cdot \mathbf{r}) - 1$, and therefore converges in the region of small k_j. The principal contribution to the integrals over k_j arise in the region $k_j r \cong 1$ and is proportional to $r^{n/3}$. From this there immediately follow the formula (3.7) and the above asymptotic estimates of the GF and PC for $kr \gg 1$.

3.3. Asymptotic form of the structure functions in ω. In the proof of the convergence of the integrals in the QC approximation and the quasi-Lagrangian DT a fact of great significance was the rapid decrease of the structure functions $g(\xi), f(\xi)$ as $\xi \to \infty$) where $\xi = \omega / k^{2/3}$), which is necessary for convergence of the integrals over ω. Below we shall establish the asymptotic form of the GF and PC for $\omega \gg k^{2/3}$. For this we consider the local GF and PC L_q, which possess naive Kolmogorov scaling (1.11) with $\nu = 2/3$ and $\mu = 11/3$. We find first the asymptotic form of the PC. It follows from Eq. (2.8b) that for $\omega \to \infty$ the principal contribution to the integrals is given by the region of integration over the internal variables $\omega' \cong \omega$ and $k' \cong s \cong \omega^{3/2}$. The explicit form of the structure functions (3.1) in the calculation of the integrals over the internal variables ω', $\mathbf{k}'$, and $\mathbf{s}$ is unimportant, inasmuch as the principal contribution to the integrals arises from the region $k' \cong s \cong \omega^{3/2}$. Taking into account the common dimensionality of the GF and PC in (3.1), the asymptotic form $\delta(\mathbf{s})/i\omega$ of the GF as $\omega \to \infty$, and the proportionality of the MO Σ_q and Φ_q to the square of the external momentum, we obtain

$$k^{-13/3} f^l(\omega / k^{2/3}) \approx (k^2/\omega^2)(k')^{-15/3}, \quad k' \approx \omega^{3/2}. \tag{3.11}$$

From this follows the power asymptotic form for the PC F^l_q. Analogous arguments can be applied to the Dyson equation integrated over $\mathbf{s}$. As a result, we have

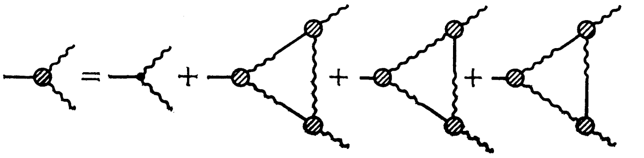
$$G_q^{\,\pi} = 1/i\omega + c_g k^{-2/3} (k^{2/3}/\omega)^4, \quad \omega \gg k^{2/3}, \qquad F_q^{\,\pi} = c_f k^{-13/3} (k^{2/3}/\omega)^{19/2}. \tag{3.12}$$

Here c_g and c_f are constants. The asymptotic forms (3.12) are easily verified from the first diagram for the MO Σ^l_q and Φ^l_q (1.10), and, in view of the power character of (3.12), the scaling relation guarantees that these asymptotic forms are reproduced in the entire diagrammatic series. On the basis of the same arguments concerning the dominant contribution of the regions of integration $\omega' \sim k'^{2/3} \sim s'^{2/3} \sim \omega$ we can find the asymptotic forms of the GF and PC for $\omega \gg k^{2/3}$ in the quasi-Lagrangian theory:

$$G^{ij}(q,\, s) = \Delta_{\mathbf{k}}^{ij}\delta(s)/i\omega + k^{-11/3}\varphi_g^{ij}(\mathbf{k}/k,\, \mathbf{s}/k)\,(k^{2/3}/\omega)^{17/2},$$
$$F^{ij}(q,\, s) = k^{-22/3}\varphi_f^{ij}(\mathbf{k}/k,\, \mathbf{s}/k)\,(k^{2/3}/\omega)^{14}, \quad \omega \gg k^{2/3}. \qquad (3.13)$$

Here φ_g and φ_f are polynomials of second order in $\mathbf{k}/k$ and $\mathbf{s}/k$. The GF and PC (3.13) fall off rapidly with ω, and this guarantees the regularity of the functions y (2.10), (3.2) introduced in the proof of the convergence of the diagrams.

3.4. The dressed vertex function. The above investigation of the quasi-Lagrangian theory was carried out in terms of the bare vertex function (2.5). From the scaling relations it is clear that the homogeneity index of the vertex is not changed by the interaction. In fact, in lowest order the equation for the vertex



(3.14)

adds to the bare vertex the scaling combination $\gamma^2 G^2 F \omega k^3$. It would be interesting to know whether the bare δ-function structure remains in the vertex or whether the interaction washes it out completely. To answer this question we can make use of the method of r-transport. Going over, in Eq. (3.14), to the r-representation with respect to the sum of the momenta entering the vertex, we can convince ourselves (see Ref. 20) that for $kr \gg 1$ the vertex tends to a constant, implying that the δ-function feature is preserved. In particular, the lower-order diagrams in (3.14) do not depend on r for $kr \gg 1$, since each of the GF in the ring contribute a factor $[V_T(r)]^{-1}$ and a PC gives $[V_T(r)]^2$.

We shall discuss the possibility that the diagrammatic equations of fully developed turbulence have solutions leading to a large renormalization of the vertex (2.5). It may be assumed that as a result of the strong interaction the renormalized vertex (3.14) becomes much larger than the bare vertex. Then, as in the theory of phase transitions, in Eqs. (3.14) we can omit the bare vertex function and obtain a self-consistent system of equations for the GF, the PC, and the vertex. It is natural to seek the solution of this system of equations in the form of the scaling functions (3.1), and also

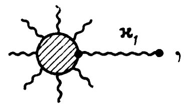
$$\gamma_{q12}^{ijl} = k^{\sigma}\Gamma^{ijl}\left(\frac{\omega_1}{k_1^{\nu}}, \frac{\omega_2}{k_2^{\nu}}, \frac{\mathbf{k}}{k}, \frac{\mathbf{k}_1}{k}, \frac{\mathbf{k}_2}{k}\right)\delta(\omega-\omega_1-\omega_2). \qquad (3.15)$$

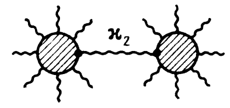
With the assumption that the interaction is local the scaling relation and energy conservation are replaced by

$$2(\nu-\sigma)+\mu=3, \quad \nu-\sigma+2\mu=7. \qquad (3.16)$$

It follows from this that $\mu = 11/3$ and $\nu = \sigma - 1/3$. Thus, for this class of solutions too, the scaling and the local nature of the interaction require the index of the equal-time PC (and of the higher equal-time VC) to have the Kolmogorov values. But the index of the frequency ν and of the GF may be changed. This situation is a consequence of the distinctive renormalization invariance of the theory[18]: the dressed GF and dressed vertex appear in the form of a product in the Wyld equations, the equation (3.14) for the vertex, and the expressions for the higher velocity correlators. The question of the determination of the index σ should be solved, in our opinion, on the basis of the requirements for self-consistency of the solution, as is done in the renormalization-group approach in the theory of phase transitions, and also on the basis of the condition for stability of the solution. At the present time we see no serious arguments involving divergences of the diagrams or of the series as a whole that would dictate the need to consider the class of solutions with strong renormalization of the vertex.

3.5. The higher-order velocity correlators. In the framework of this approach, the equal-time n-VC $F_{\mathbf{k}_1\ldots\mathbf{k}_n}^{\alpha_1\ldots\alpha_n}$ can be calculated. The general dimension of the equal-time n-VC can be found easily from the first diagram and is reproduced in higher orders of the DT as a result of the scaling relation: $F_{\mathbf{k}_1\ldots\mathbf{k}_n}^{\alpha_1\ldots\alpha_n} \propto k^{10n/3}$, which coincides with the results of the Kolmogorov-Obukhov phenomenological approach. In our approach we can determine the asymptotic forms of the n-VC in the case when one momentum $\mathbf{k}_1 = \varkappa_1$ is much smaller than the others: $\varkappa_1 \ll k_j$ ($j = 1,\ldots,n, j \neq i$), and also when the sum $\varkappa_2 = \mathbf{k}_\alpha + \mathbf{k}_\beta + \ldots + \mathbf{k}_\gamma$ of a group of momenta is much smaller than the momenta themselves: $\varkappa_2 \ll k_j$ ($j = 1,\ldots,n$). The principal contribution to the n-VC then arises from the graphs (3.17), where a small momentum flows through the pair correlator:

A 

, B: 

(3.17)

Taking into account that the vertex is proportional to $\varkappa$ and $F_\varkappa \propto \varkappa^{-11/3}$ we easily find $F^A \propto \varkappa_1^{-8/3}$ and $F^B \propto \varkappa_2^{-5/3}$. This result is valid is all orders of the quasi-Lagrangian DT and cannot be obtained in the Eulerian DT if we confine ourselves to a finite number of diagrams for the n-VC.

CONCLUSION

We shall summarize the results of the paper. The principal difficulties of the current stage of development of the theory of turbulence that are due to the problem of transport can be solved successfully by going over to a coordinate frame moving with the velocity of the liquid at a certain spatial point $\mathbf{r}_0$. Naturally, the theory that then arises depends on the point $\mathbf{r}_0$, and therefore is nonlocal in the momentum. The scale-invariant pair correlator and scale-invariant response are the solution of the corresponding Dyson equations and generalized kinetic equation.

The interaction of vortices of scale k^{-1} turns out to be local, i.e., only vortices of similar scales interact dynamically, and the interaction of vortices of different scales reduces to their mutual transport. The requirement of scale invariance (scaling) and energy conservation fixes the energy distribution over the scales, which corresponds to the Kolmogorov-Obukhov law: $E_k \propto k^{-5/3}$. The present theory also gives constructive machinery for investigating more specialized questions.

It is necessary to note that the mathematical objects of the theory are asymptotic series that depend on the external parameters. After the elimination of transport effects they do not contain small parameters as such. It is clear that partial summations of such series, which we have repeatedly carried out, are a poorly defined mathematical procedure. Our quasi-Lagrangian diagram technique in terms of dressed pair correlators and Green's functions and bare vertex functions can be regarded as a way of supplementing the original Eulerian diagram technique. The method yields results that are physically rich in content and permits one to hope that the approximations given by the first few diagrams, and the methods of asymptotic summation of series (e.g., Borel summation), will turn out to be adequate for the derivation of quantitative results. Of course, the question of the one-to-one correspondence of observable physical quantities to asymptotic series of the theory of developed hydrodynamic turbulence remains open.

Nonperturbative contributions (giving zero in the expansion in a perturbation-theory series) to the observable quantities of developed turbulence are possible. We note also that in our solutions of the diagrammatic equations the homogeneity index (scaling dimension) of the vertex is not renormalized. However, the theory may also contain other solutions, in which the scaling dimension of the vertex is changed but, by virtue of the scaling relations and the locality of the interaction, the dimension of the equal-time velocity correlators remains as before. A number of important questions remain unstudied. These concern the uniqueness of the solution obtained, the stability and establishment of the solution, the approach of the nonuniversal solution in the energy-containing interval to a scale-invariant solution in the inertial interval, etc. Thus, there is a large field for further investigations.

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