Counterbalanced interaction locality of developed hydrodynamic turbulence

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The problem of interaction locality in k space is studied in a diagrammatic perturbation approach for the Navier-Stokes equation in quasi-Lagrangian variables. Analyzing the whole diagram series we have found an exact relation between the asymptotic behavior of the triple-correlation function of velocities that governs the energy transfer over scales and the double-correlation function giving the energy distribution. Namely, at $r \ll R$, we obtain $S_3(r,R,R-r) \propto S_2(R)(r/R)[S_3(r)/S_2(r)] \propto R^{\zeta_2-1} r^{2-\zeta_2}$, where ζ_2 is the static exponent of double-velocity moment. This relation between two different physical quantities (in principle, measurable independently) is accessible to an experimental check. Also, this relation allows us to describe an energy exchange between distant scales in k space: For any steady spectrum carrying constant energy flux, the interactions of the given k-eddies with large ($k_1 \ll k$) and small eddies ($k_2 \gg k$) are shown to decrease by the same law with the distance in k space, such as $(k_1/k)^{2-\zeta_2}$ and $(k/k_2)^{2-\zeta_2}$. It means a balance of interactions for such a spectrum. Considering, in particular, the multifractal picture of developed turbulence, we analyze the range of exponents h of the velocity field $[\delta v(r) \propto r^h]$ which provides the locality of interaction in the k space. It is shown that the condition of infrared locality of interaction (with larger k_1-eddies) could give only the upper restriction for the exponent. The upper limit thus found ($h_{\max}=1$) coincides with the boundary exponent of singularity of energy dissipation. As far as an interaction locality in the ultraviolet limit ($k_2 \gg k$) is concerned, we prove that any reasonable dimension function $D(h)$ provides locality whatever small h is considered.

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I. INTRODUCTION

The theory of developed hydrodynamic turbulence deals with flows for which the external scale of motion L is much larger than the scale η of effectively damping modes. This phenomenon occurs when the Reynolds number $R = LU/\nu$ is much larger than unity. Here U is some large-scale external velocity (for example, the mean velocity of the flow past a body) and ν is the kinematic viscosity. The statistical properties of the turbulent flow in the inertial range of scales l intermediate between external and dissipation scales ($L \gg l \gg \eta$) are usually supposed to be self-similar, which means that, for example, the velocity increment over a distance l should behave as follows:

$$\delta v(r,l) \equiv v(r+l) - v(r) = \delta v(l) \propto l^h . \tag{1}$$

Here equality means identical statistical properties. Power behavior with l is also assumed for high-order momenta of the velocity field:

$$S_p(l) \equiv \langle [\delta v(l)]^p \rangle \sim U^p \left| \frac{l}{L} \right|^{\zeta_p} . \tag{2}$$

The first attempt to find exponent h was done by Kolmogorov [1], who assumed the energy dissipation rate ϵ to stay finite in the limit of infinite Reynolds numbers. Proceeding from this, Kolmogorov [1] obtained the following expression for the third-order moment of the velocity field:

$$S_3 = \langle [\delta v(l)]^3 \rangle = -\tfrac{4}{5}\epsilon l . \tag{3}$$

This relation is one of a few exact relations in turbulence theory. The triple correlator governs the behavior of the energy flux over the scales and the expression (3) means a constancy of the flux in the inertial range. Indeed, for the scale l, the rate of energy transfer is the energy $(\delta v)^2$ divided by the typical turnover time $l/\delta v$. One should remember that the statement of flux constancy is meaningful only provided the locality of interaction in the k space (which means that only motions of comparable scales effectively interact).

Two different pictures of fully developed turbulence have been suggested. The first one implies single-scaling behavior: $\zeta_p = ph$, $h = \frac{1}{3}$, which gives the famous Kolmogorov $\frac{5}{3}$ law. As far as the energy spectrum is concerned, numerous experiments [2] confirm this law. However, an anomalous scaling (i.e., a nonlinear dependence of ζ on p) seems to be observed both in natural and numerical experiments [3,4]. To take this scaling into account, the second phenomenological approach, namely the multifractality hypothesis [5,6], was suggested. It presumes a turbulent flow to possess a range of scaling exponents ($h_{\min}, h_{\max}$). Each scaling law $\delta v(l) \propto l^h$ takes place on the subset of the three-dimensional space having Hausdorff dimension $D(h)$. Such a picture gives thus the following expression for the correlation function:

$$S_p(l) \sim U^p \int_{h_{\min}}^{h_{\max}} d\mu(h) \left| \frac{l}{L} \right|^{ph+3-D(h)} . \tag{4}$$

Here the measure of the different scalings $d\mu(h)$ is supposed to be smooth. The exponent $3-D(h)$ defines the probability of finding the scaling exponent h. The multifractality concept also gives a power behavior for all correlation functions: since in (4) the smallest exponent dominates in the limit of $l\to 0$, then the correlators behave by the power laws

$$S_p(l)\sim U^p\left[\frac{l}{L}\right]^{\zeta_p}, \quad \zeta_p=\min_h[ph+3-D(h)] , \qquad (5)$$

though with an anomalous scaling defining by an unknown function $D(h)$.

Both above pictures imply (3) and are based thus on the locality assumption. The locality problem was studied previously for the single-scaling Kolmogorov spectrum only. It is done in the framework of the direct-interaction approximation [7] and in the entire perturbation series without any cutoff [8]. Here we solve the general locality problem without any assumptions about the steady spectrum except (3) and scale invariance.

To analyze the mechanism of interaction balance, one should know the triple moment for noncoinciding arguments like

$$S_3(r_{12},r_{23},r_{13})=\langle \delta v(r_{12})\delta v(r_{23})\delta v(r_{13})\rangle . \qquad (6)$$

In Sec. II, we find the asymptotic behavior of (6) for one distance being much smaller than others. To find such asymptotic behavior, one should use the exact formulas for the correlation functions which are obtained by analyzing the entire perturbation series (without any cutoff) for the Navier-Stokes equation. Neither the dimensional approach nor finite closures could give the correct answer in this case. We thus find the second [in addition to (3)] exact relation for the asymptotic behavior of the triple-correlation function. This relation allows us to give in Sec. III the whole analysis of the interaction locality. In particular, we analyze the multifractal picture directly in the framework of the Navier-Stokes equation. The possible presence of different exponents gives rise to a new question in the locality consideration. Is it possible that locality is violated in the ultraviolet region for sufficiently small h and in the infrared region for large h? We show that locality analysis gives the upper restriction for exponent h.

II. EXACT RELATION FOR THE TRIPLE-CORRELATION FUNCTION

A natural way to analyze an interaction locality is to consider the Navier-Stokes equation (NSE) for incompressible fluid flow in the **k** representation [2]:

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]v_\alpha(t,\mathbf{k})=\tfrac{1}{2}\int \Gamma_{\alpha,\beta,\gamma}(\mathbf{k}|\mathbf{q},\mathbf{p})v_\beta^*(t,\mathbf{q})\times v_\gamma^*(t,\mathbf{p})d\mathbf{q}d\mathbf{p} , \qquad (7)$$

$$\Gamma_{\alpha,\beta,\gamma}(\mathbf{k}|\mathbf{q},\mathbf{p})=P_{\alpha\alpha'}(\mathbf{k})P_{\beta\beta'}(\mathbf{q})P_{\gamma\gamma'}(\mathbf{p})\times i(k_\beta\delta_{\alpha\gamma}+k_\gamma\delta_{\alpha\beta})\delta(\mathbf{k}+\mathbf{q}+\mathbf{p}) ,$$
$$P_{\alpha\beta}(\mathbf{k})=\delta_{\alpha\beta}-k_\alpha k_\beta/k^2 , \qquad (8)$$

where Γ is the *Eulerian vertex* and P is the *transverse projector.*

Equation (7) is nonlinear. Multiplying it by different powers of velocity $v(\mathbf{k},t)$ and averaging, one could obtain the equations for the correlation functions of the velocity field. Time evolution of the nth correlation function is governed by some integral with the correlation function of order $n+1$. As a first step, one could consider the simultaneous double-correlation function

$$F_{\alpha\beta}(\mathbf{k},t)\delta(\mathbf{k}-\mathbf{q})=\langle v_\alpha(\mathbf{k},t)v_\beta^*(\mathbf{q},t)\rangle , \qquad (9)$$

which describes the energy density over scales: $E(k,t)\propto k^2F(k,t)$. Here angular brackets denote turbulent ensemble averaging. The equation for $F_{\alpha\beta}$ is as follows:

$$\left[\frac{\partial}{\partial t}+\nu k^2\right]F_{\alpha\alpha'}(\mathbf{k},t)=\mathrm{Re}\int \Gamma^*_{\alpha,\beta,\gamma}(\mathbf{k}|\mathbf{q},\mathbf{p})\times F^{(3)}_{\alpha'\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p};t)d\mathbf{q}d\mathbf{p} . \qquad (10)$$

Here $F^{(3)}$ is the simultaneous triple correlator:

$$F^{(3)}_{\alpha\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p};t)\delta(\mathbf{k}+\mathbf{q}+\mathbf{p})=\langle v_\alpha(\mathbf{k},t)v_\beta(\mathbf{q},t)v_\gamma(\mathbf{p},t)\rangle . \qquad (11)$$

From the physical viewpoint, the interaction locality means that the main effect on the behavior of given k-eddies (eddies of spatial scale $1/k$) arises because of their interaction with eddies of the same order of size. Mathematically, it means that the main contribution to the integral (10) arises from the regions where q and p are of order of k which means the convergence of the integral (10) and nothing else. There exist different definitions of locality separating, for instance, the case with the main contribution from the region $(k/2,2k)$ from the case with that of the region $(k/5,5k)$, etc. To our mind, such a difference is of no principal importance for turbulence theory, whereas the difference between the case of a convergent integral in (10) (where one could define the energy flux in the k space) and the case of divergency (where energy is transferred nonlocally) is a qualitative one. Our definition of locality is usual in the general theory of turbulence [9].

So the main point here is the analysis of integral convergence and, in particular, the question about the convergence reserve that is the law of how the contribution of a distant region in the k space turns into zero while the distance increases. Interaction of k-eddies with q-eddies of much larger scales is defined by the asymptotic behavior of $F^{(3)}(\mathbf{k},\mathbf{q},\mathbf{p};t)$ at $q\ll k\simeq p$ and $p\ll k\simeq q$. On the other hand, the interaction of k- and q-eddies with $q\gg k$ is defined by the asymptotical region $k\ll q\simeq p$. Thus, in both cases (infrared and ultraviolet ones), the nonlocal part of eddy interaction is described by the triad interaction $\mathbf{k}+\mathbf{q}+\mathbf{p}=\mathbf{0}$ with one of the wave numbers being much smaller than two others. So, in order to study the problem of interaction locality, it is necessary to find the asymptotic behavior of the triple-correlation function (11) at $k\ll q\simeq p$ to obtain the range of exponents h providing convergence of the integral. It could be shown that the locality analysis gives the same results for the exponent interval whatever order $n \geq 2$ of correlation function is used.

Let us recall the relations between the scaling properties of second and third correlation functions $F^{(2)}$ and $F^{(3)}$ in the k representation and that of S_2 and S_3 in the r representation. Double correlators are related as follows:

$$S_2(l) = \int |1-\exp(i\mathbf{k}\cdot\mathbf{l})|^2 F^{(2)}(\mathbf{k}) d\mathbf{k} . \tag{12}$$

For a power function $F^{(2)}(k) \propto k^{-y}$ with $3<y<5$, the main contribution to this integral is given by the region $k \simeq 1/l$. Therefore, in the inertial interval where $S_2(l) \propto l^{\zeta_2}$ we have $y=3+\zeta_2$.

Considering the similar integral for the triple correlators $S_3(l)$ and $F^{(3)}(\mathbf{k},\mathbf{q},\mathbf{p})$ and assuming the main contribution to be provided by the region $k \simeq q \simeq p \simeq 1/l$, we obtain the scaling behavior

$$F^{(3)}(\lambda\mathbf{k},\lambda\mathbf{q},\lambda\mathbf{p}) = \lambda^{-7} F^{(3)}(\mathbf{k},\mathbf{q},\mathbf{p}) \tag{13}$$

in this region. According to the scaling hypothesis for high-order correlations, we shall assume that relation (13) is valid for any k,p,q from the inertial interval. For the third correlator, we thus know the total exponent only:

$$F^{(3)}(k,q,q) \propto k^a q^b , \quad a+b=7 , \tag{14}$$

but not the exponents a and b separately. Therefore, one should analyze the equation for $F^{(3)}$ following from (7). It is important to emphasize that to obtain the correct expression for $F^{(3)}$, one should analyze the entire series of perturbation theory (see the Appendix for details).

The main problem here is due to the fact that integrals over $\mathbf{k}$ diverge in terms of Eulerian velocity v even for the Kolmogorov spectrum. According to Kraichnan [10], such a divergency is caused by the strong sweeping of the small eddies by the largest ones, so it is irrelevant to the problem of the locality of dynamic interaction providing energy cascade due to an energy exchange between the eddies.

Following Ref. [8], we will use the *quasi-Lagrangian* (QL) velocity $\mathbf{u}(\mathbf{r}_0,\mathbf{r},t)$ in order to eliminate the sweeping in the vicinity of some space point $\mathbf{r}_0$:

$$\mathbf{v}(\mathbf{r},t) = \mathbf{u}\left[\mathbf{r}_0;\mathbf{r}-\int^t u(\mathbf{r}_0,\mathbf{r}_0,\tau)d\tau, t\right] . \tag{15}$$

If the velocities of all particles of the fluid are identical, the above QL velocity naturally coincides with the Lagrangian velocity of fluid particle along the actual trajectory. It should be noted that Eq. (15) contains no approximations, since it is just the definition of the function $\mathbf{v}(\mathbf{r},t)$. Formula (15) itself represents a precise relation between the Eulerian and quasi-Lagrangian velocities. In terms of the quasi-Lagrangian velocity, one can adequately construct a theory of turbulence [8,11], and, in particular, find the asymptotic behavior of the triple-velocity-correlation function.

The motion equation for $\mathbf{u}(\mathbf{r}_0;\mathbf{k},t)$ may be derived by substituting (15) into (7). The resulting equation has the form of the initial Navier-Stokes equation (7) but with another *dynamic* vertex V describing only the dynamic interaction of eddies in the QL approach to the theory of turbulence [8,11] (see also the Appendix):

$$\begin{aligned} V_{\alpha,\beta,\gamma}(\mathbf{k}|\mathbf{q},\mathbf{p};\mathbf{r}_0) &= P_{\alpha,\alpha'}(\mathbf{k})P_{\beta,\beta'}(\mathbf{q})P_{\gamma,\gamma'}(\mathbf{p})U_{\alpha',\beta',\gamma'}(\mathbf{k}|\mathbf{q},\mathbf{p};\mathbf{r}_0) , \\ U_{\alpha,\beta,\gamma}(\mathbf{k}|\mathbf{q},\mathbf{p};\mathbf{r}_0) &= i\{k_\gamma\delta_{\alpha,\beta}[\delta(\mathbf{k}+\mathbf{q}+\mathbf{p})-\delta(\mathbf{k}+\mathbf{q})\exp(i\mathbf{p}\mathbf{r}_0)]+k_\beta\delta_{\alpha,\gamma}[\delta(\mathbf{k}+\mathbf{q}+\mathbf{p})-\delta(\mathbf{k}+\mathbf{p})\exp(i\mathbf{q}\mathbf{r}_0)]\} . \end{aligned} \tag{16}$$

The main difference between the quasi-Lagrangian and the conventional (in terms of the Eulerian velocity) descriptions of turbulence is that the wave vector $\mathbf{k}$ is no longer preserved in the vertex V, and the functions (9) and (11) become nondiagonal in $\mathbf{k}$ [i.e., the correlation functions are not proportional to $\delta(\mathbf{k}_1-\mathbf{k}_2)$]. This is the result of the absence of spatial uniformity of the theory due to the explicit dependence of the QL velocity (15) on the coordinate of the marked point $\mathbf{r}_0$, where sweeping is precisely eliminated.

It is convenient to develop the turbulence description in ω representation supposing statistical properties to be invariant with respect to a time shift. We will study double- and triple-correlation functions of velocity field. By analogy with (9) and (11), let us introduce these functions for the QL velocities $\mathbf{u}(\mathbf{r}_0;\mathbf{k},\omega)$ in the $(\omega,\mathbf{k})$ representation:

$$\begin{aligned} &F_{\alpha\beta}(\mathbf{r}_0;\omega_1,\mathbf{k}_1,\mathbf{k}_2)\delta(\omega_1-\omega_2) \\ &\qquad = \langle u_\alpha(\mathbf{r}_0;\mathbf{k}_1,\omega_1)u^*_\beta(\mathbf{r}_0;\mathbf{k}_2,\omega_2)\rangle , \\ &F^{(3)}_{\alpha\beta\gamma}(\mathbf{r}_0;\omega_1,\omega_2,\omega_3,\mathbf{k},\mathbf{q},\mathbf{p})\delta(\omega_1+\omega_2+\omega_3) \\ &\qquad = \langle u_\alpha(\mathbf{r}_0;\mathbf{k},\omega_1)u_\beta(\mathbf{r}_0;\mathbf{q},\omega_2)u_\gamma(\mathbf{r}_0;\mathbf{p},\omega_3)\rangle , \\ &\mathbf{u}(\mathbf{r}_0;\mathbf{k},\omega) = \int \mathbf{u}(\mathbf{r}_0;\mathbf{r},t)\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})]d\mathbf{r}\,dt . \end{aligned} \tag{17}$$

It is important to emphasize that simultaneous correlation functions of the order n for QL velocities $F^{(n)}$ coincide with the same functions of Eulerian velocities $F_e^{(n)}$ and do not depend on $\mathbf{r}_0$ [8,11]. Therefore, we will consider the problem of triad interactions in the QL approach with the sweeping being eliminated completely. Using then the above-mentioned coincidence, we will obtain a conclusion about the behavior of simultaneous correlation functions of Euler velocities. In particular, we will use the following relations for the double- and triple-velocity-correlation functions:

$$\int F(\mathbf{r}_0;\mathbf{k},\mathbf{q},\omega)d\omega/2\pi=F_e(\mathbf{k},\mathbf{q})=F_e(\mathbf{k})\delta(\mathbf{k}-\mathbf{q}) , \quad (18)$$

$$F^{(3)}_{\alpha\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p})\delta(\mathbf{k}+\mathbf{q}+\mathbf{p}) = \int F^{(3)}_{\alpha\beta\gamma}(\mathbf{r}_0;\omega_1,\omega_2,\omega_3,\mathbf{k},\mathbf{q},\mathbf{p}) \times\delta(\omega_1+\omega_2+\omega_3)d\omega_1 d\omega_2 d\omega_3/(2\pi)^3 . \quad (19)$$

To obtain the simultaneous correlation function, we have integrated the expressions (18) and (19) over all of ω_j.

In order to find an asymptotic behavior of the simultaneous triple-correlation function of Eulerian velocity, we have analyzed in the Appendix the entire diagram series for the triple moment of QL velocity. In the asymptotic limit $k \ll q \simeq p$ we found the main contribution into the triple correlator. It is given by some principal subsequence of diagrams containing the double correlator of small wave vector k. The result for the simultaneous triple-correlation function of QL velocity (that coincides with the same function of Eulerian velocity) is given by the following expression:

$$F^{(3)}_{\alpha\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p})\delta(\mathbf{k}+\mathbf{q}+\mathbf{p})=\int d\mathbf{k}'d\mathbf{q}'d\mathbf{p}'d\omega\, d\omega_1 d\omega_2 d\omega_1' d\omega_2'\delta(\omega_1+\omega_2+\omega_1'+\omega_2')\delta(\omega_1'+\omega_2'+\omega)F(\mathbf{k},\mathbf{k}';\omega) \times D_{\beta\gamma\beta'\gamma'}(\mathbf{q},\mathbf{p},\mathbf{q}',\mathbf{p}';\omega_1,\omega_2,\omega_1',\omega_2')V_{\beta'\gamma'\alpha}(\mathbf{q}'|\mathbf{p}',\mathbf{k}') . \quad (20)$$

The dependence of $F^{(3)}(\mathbf{k},\mathbf{q},\mathbf{p})$ on the small wave vector k is presented in (20) both in explicit form via the $\mathbf{k}$ dependence of the double-correlation function $F(\mathbf{k},\mathbf{k}',\omega)$ and in implicit form via the $(\mathbf{k},\omega)$ dependence of D and ω dependence of F. The unknown function D is expressed in terms of some infinite diagram series. We do not know how to summarize it. However, it is not necessary for our treatment. It is enough for us to know that function D may be expanded at small κ and ω in a Taylor series with respect to the powers of k and ω (see the Appendix for the proof of this fact).

The main contribution to integral (20) over k',ω is from the region $k'\simeq k$, $\omega\simeq k^x$ (x being the dynamic exponent that is the frequency scaling index), where function $F(k,k',\omega)$ is not small. On the other hand, the main contribution into the integral over ω_1' and ω_2' is from the region $\omega_1'\simeq\omega_2'\simeq q^x \gg k^x \simeq \omega$. It allows us to neglect the value of ω and put $\omega_2'=-\omega_1'$ (instead of $\omega_2'=-\omega_1'-\omega$) in the argument of function D. As a result, only the function $F(k,k',\omega)$ depends on ω in (20). It allows us to integrate (20) over ω with the help of (18) and then to integrate (20) over $\mathbf{k}'$, ω_2, and ω_2' using δ functions. One thus has

$$F^{(3)}_{\alpha\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p})\delta(\mathbf{k}+\mathbf{q}+\mathbf{p})=F(\mathbf{k})\int d\mathbf{q}'d\mathbf{p}'d\omega_1 d\omega_1' D_{\beta\gamma\beta'\gamma'}(\mathbf{q},\mathbf{p},\mathbf{q}',\mathbf{p}';\omega_1,-\omega_1,\omega_1',-\omega_1')V_{\beta'\gamma'\alpha}(\mathbf{q}'|\mathbf{p}',\mathbf{k}) . \quad (21)$$

Here V is the *dynamic vertex* (16). Substituting (16) into (21) and integrating over $\mathbf{p}'$ one has

$$F^{(3)}_{\alpha\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p})\delta(\mathbf{k}+\mathbf{q}+\mathbf{p}) = -iP_{\alpha\alpha'}(\mathbf{k})F(\mathbf{k})\int d\mathbf{q}'d\omega_1 d\omega_1'\{k_{\gamma''}P_{\gamma''\gamma'}P_{\gamma'\beta'}D_{\beta\gamma\beta'\gamma'}(\mathbf{q},\mathbf{p},\mathbf{q}',-(\mathbf{q}'+\mathbf{k});\omega_1,-\omega_1,\omega_1',-\omega_1') + q_{\alpha'}'P_{\beta\beta''}(\mathbf{q}')[P_{\gamma'\beta''}(\mathbf{q}'+\mathbf{k})D_{\beta\gamma\beta'\gamma'}(\mathbf{q},\mathbf{p},\mathbf{q}',-(\mathbf{q}'+\mathbf{k});\omega_1,-\omega_1,\omega_1',-\omega_1') - P_{\gamma'\beta''}(\mathbf{q}')D_{\beta\gamma\beta'\gamma'}(\mathbf{q},\mathbf{p},\mathbf{q}',-\mathbf{q}';\omega_1,-\omega_1,\omega_1',-\omega_1')]\} . \quad (22)$$

Now let us expand D in the Taylor series with respect to $\mathbf{k}$: $D(\mathbf{k})\simeq D(0)+D_\delta' k_\delta+D_{\delta\sigma}'' k_\delta k_\sigma/2$. The following expansion of the triple correlator (22) thus arises in the limit of small k:

$$F_{\alpha\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p}) = \left[\Phi_{1,\alpha\beta\gamma\delta}(\mathbf{q})\frac{k_\delta}{q}+\Phi_{2,\alpha\beta\gamma\delta\sigma}(\mathbf{q})\frac{k_\delta k_\sigma}{q^2}+\cdots\right]F(k) . \quad (23)$$

Here $F(k)\propto k^{-y}$ is the simultaneous double-correlation function of QL velocity that coincides with the same function of Eulerian velocity. So, the scaling exponent y is determined by the scaling exponent ζ_2 of double-velocity moment (2) by the relation $y=\zeta_2+3$, which follows from (12). It is worth noting that in (21) we replaced $\omega_2'=-\omega_1'-\omega$ by $\omega_2'=-\omega_1'$. In order to take into account the dependence of D on ω, let us expand $D(\omega)$ at small ω, since the main contribution in the integral (20) stems from the region of small ω if k is chosen to be small. Now we substitute the expansion $D(\omega)\simeq D(0)$

$+\omega\partial D/\partial\omega+\ldots$ into (20) and integrate the additional terms arising in (21) in the same manner. Finally, we obtain the following corrections to (23):

$$F_{\alpha\beta\gamma}(\mathbf{k},\mathbf{q},\mathbf{p})=\left\{\Phi_{1,\alpha\beta\gamma\delta}(\mathbf{q})\frac{k_\delta}{q}\left[1+a_1\left[\frac{k}{q}\right]^x\right]+\Phi_{2,\alpha\beta\gamma\delta\sigma}(\mathbf{q})\frac{k_\delta k_\sigma}{q^2}\times\left[1+a_2\left[\frac{k}{q}\right]^x\right]+\cdots\right\}F(k) . \quad (24)$$

Here a_1 and a_2 are some dimensionless constants, x being the dynamic scaling exponent (for the Kolmogorov solution, $y=\frac{11}{3}$, $x=\frac{2}{3}$). For multifractal subsets with an arbitrary h, one could obtain $x=1-h$, since the typical frequency scales with l as $\delta v(l)/l$. We shall demonstrate below that $h\le 1$, so $x>0$. The account of the frequency dependence gives indeed the small corrections in each order of the expansion in k/q. Therefore, we shall use (23) in the further calculations.

It is important to emphasize here that δ functions of the frequencies in (20) are the same in both terms arising in (20) after substituting the vertex V. This leads to the absence of a term proportional to $k^xF(k)$ in the expansion (23). So the main term in the asymptotic expansion of triple-correlation function (24) is $\propto k^{1-y}\Phi_1(q)/q$. The total scaling exponent of the triple-correlation function is $y_3=7$ according to (14). The scaling exponent of $\Phi_{1,2}(q)$ may be thus expressed in terms of the scaling exponent y of the double correlator

$$\Phi(q)\propto q^z , \quad z=7-y . \quad (25)$$

Let us allow a small digression. It is interesting to compare z, which one would obtain in the direct-interaction approximation [7] for the triple-correlation function, with the above correct answer. In that approximation (see the Appendix for details)

$$D_{\alpha\beta\gamma\delta}(\mathbf{q},\mathbf{p},\mathbf{q}',\mathbf{p}';\omega_1,\omega_2,\omega_1',\omega_2')=G_{\alpha\gamma}(\mathbf{q},\mathbf{q}';\omega_1)F_{\beta,\delta}(\mathbf{p},\mathbf{p}';\omega_2)\delta(\omega_1+\omega_1') . \quad (26)$$

Using (20), one has in the same way the estimation for functions Φ:

$$\Phi(q)\sim qF(q)G(q,0)\propto q^{z_0} , \quad z_0=x+y-1 . \quad (27)$$

Due to the scaling relation $2x+y=5$ [8,11], the same result for z_0 could be easily obtained in any (finite) order of diagrammatic perturbation theory or under the assumption that the dynamic vertex V is not renormalized. The value of z_0 coincides with the correct result $z=7-y$ only on the Kolmogorov solution $x=\frac{2}{3}$, $y=\frac{11}{3}$. For example, in the multifractality approach, $y>\frac{11}{3}$, which reflects the renormalization of the vertices for a multifractal turbulence. We do not know now how it occurs, but we know the total exponent of the triple correlators. So, in order to find the scaling exponent z of large wave vectors in the triple-velocity correlator, it is necessary to take into account the entire diagrammatic series which leads to a renormalization of the vertices.

The difference between the initial result of Kraichnan [7] and the present approach is due to the fact that the direct-interaction approximation is an uncontrolled procedure. On the contrary, we shall analyze in the Appendix the whole series, find the main contribution into the triple correlator for $q\gg k$, and prove that the neglected terms are small by parameter $(k/q)^{11/3}$. Our approach is thus controlled with a really small parameter. We get a result different from that of [7]: interactions with distant scales are counterbalanced for a real spectrum with a constant energy flux (whatever ζ_2 it has), not only for the Kolmogorov solution as in the direct-interaction approximation.

Now let us briefly explain the physical meaning of the final expressions (23) and (24). The triple-correlation function (11) with one wave number being small in comparison with two others corresponds to the triple-correlation function (26) of velocity increments between three points in r space, with one distance being smaller than two others: $r_{23}=r\ll R=r_{13}\approx r_{12}$. The points 2 and 3 are thus close, being distant from the point 1. The main contribution in the correlation of velocity increments is due to the pair correlator between distant points 1 and 2 (or 3). It gives $F(k)$ in (20)–(23). The small factor k/q in (23) turns into r/R in the r representation. It describes a decorrelating influence of the spatial structure of a small r-eddy on a large R-eddy. The term $(k/q)^x$ describes a decorrelating influence of the motion of the small eddy. The rest of the dependence of S_3 on r could be restored with the help of (3) [or (14)] using the knowledge of the total exponent:

$$S_3(r,R,R)\propto S_2(R)\frac{r}{R}\frac{S_3(r)}{S_2(r)}\propto R^{\zeta_2-1}r^{2-\zeta_2} . \quad (28)$$

This formula corresponds to the first term of the expansion (24). The diagrammatic approach allows us to also obtain the next three terms, which have different causes and give different contributions into the locality analysis.

III. LOCALITY ANALYSIS

Now we are ready to substitute the triple-correlation function (23) into Eq. (10) and check the integral convergence. The procedure is divided into two parts: infrared ($q\ll k$) and ultraviolet ($q\gg k$). In the first case, the integral in (10) behaves as follows (we drop all constant factors):

$$\int_{1/L}dq\int d\Omega\, F^{(3)}(q,k,k)q^2V(q,k,k) , \quad (29)$$

while in the second case it is

$$\int^{1/\eta}dq\int d\Omega\, F^{(3)}(k,q,q)q^2V(k,q,q) . \quad (30)$$

Here L and η are external and internal scales, respectively, and Ω is a solid angle of the vector q.

In the triple-correlation function (23), the first term is odd with respect to the change of the sign of the small wave number. Therefore, due to the integration over $d\Omega$, this term does not contribute to the infrared region of energy exchange (29), and the next term becomes essential. This term is even under $k \to -k$ and is proportional to $\Phi_2(k)q^{2-y}$. Therefore, the infrared convergence is defined by the integral

$$\Phi_2(k)k^{-1}\int_{1/L} F(q)q^4 dq \ . \tag{31}$$

However, for the ultraviolet integral (30), the first term in (23) gives a nonvanishing value:

$$F(k)k^2\int^{1/\eta}\Phi_1(q)q\,dq \ . \tag{32}$$

The vertex $\Gamma(k,q,q) \propto k$ was substituted in both (31) and (32).

Dividing these integrals by $F(k)$, one obtains the estimate for a typical time of the nonlinear interaction of the k-eddy with the largest eddies

$$\tau_{\rm ir}^{-1}(k,1/L) \simeq \frac{\Phi_2}{kF(k)}\int_{1/L} F(q)q^4 dq \ , \tag{33}$$

and a typical time of the interaction with the smallest eddies

$$\tau_{\rm uv}^{-1}(k,1/\eta) \simeq k^2\int_{1/L}\Phi_1(q)q\,dq \ . \tag{34}$$

As one can see, the interaction with small scales produces some turbulent viscosity since $\tau_{\rm uv} \propto k^2$.

First, we substitute the simple single-scaling Kolmogorov solution into (33) and (34) to demonstrate how it converges. In this case, $F(k) \propto k^{-11/3}$ and $\Phi(q) \propto q^{-10/3}$ by virtue of (25). Calculating (33) and (34), we determine that the contributions of distance regions in the k space into the integral (10) governing energy exchange behave as $(kL)^{-4/3}$ and $(k\eta)^{-4/3}$. That law was previously obtained by Kraichnan, and it demonstrates the locality of the simple Kolmogorov solution. It is worthwhile to note that the contributions of small and large scales decrease with the distance by the same law. The same is true for a general local steady spectrum (not necessarily with the Kolmogorov exponent), carrying constant energy flux. Indeed, in the general multifractal picture, we have $y = \zeta_2 + 3$. The exponent of Φ is defined by (25) as $z = 7 - y = 4 - \zeta_2$ due to the flux constancy, which fixes the exponent of the triple-correlation function. Therefore, $F(q) \propto q^{\zeta_2+3}$ while $\Phi(q) \propto q^{4-\zeta_2}$. Substituting these into (33) and (34), we obtain integrals that behave in the same way independently of the value of ζ_2: $(kL)^{\zeta_2-2}$ and $(k\eta)^{2-\zeta_2}$. The coincidence of the laws of decreasing contributions from the largest and smallest scales was first pointed out in [12] for Kolmogorov-like spectra of wave turbulence. Such a coincidence means that in a stationary spectrum, the contributions to interaction of all scales, from small to large ones, are balanced [12,9]. It also means that for a spectrum with constant flux, the dynamical interaction either converges at $k \to 0$ and ∞ or diverges at both limits. It is worth noting that such a balance does not necessarily hold for any cascade steady spectra. For example, it is not the case for a turbulence system with two (or more) motion integrals giving different steady spectra, as in two-dimensional turbulence.

The locality condition is thus

$$\zeta_2 < 2 \ , \tag{35}$$

which takes place for any reliable model of turbulence (the exponents of low-order correlators are close to those of the single-scaling Kolmogorov picture $\zeta_p \approx p/3$).

IV. INTERACTION LOCALITY IN THE MULTISCALING MODEL

We assumed the interaction to be local in k space for the multifractal picture. Now it is necessary to check that it is true for each h subset. Mathematically, the integral convergence means that one could change the order of integrations over dh and dq. One may represent in (33) the double-correlation function as an integral over exponents h as in (4):

$$F(q) \sim \int_{h_{\min}}^{h_{\max}} d\mu(h)q^{-2h-3}(qL)^{D(h)-3} \ . \tag{36}$$

Substituting this into (33), we find that the integral is proportional to the negative power of L (i.e., it converges) if $h_{\max} < 1$. It is interesting to note that the dimension $D(h)$ does not arise in the final expression, since $qL = 1$ for the lower limit of integration. The probability of finding the largest eddy is equal to unity for all h subsets. Multifractality plays its role for small scales only. The condition $h < h_{\max} = 1$ means that the maximal velocity gradients $\delta v(\eta)/\eta \propto \eta^{h-1}$ become singular at $R \to \infty$ for all h subsets.

For the ultraviolet integral (34), we could not represent $\Phi(q)$ as an integral over dh, since it is proportional to the ratio of two correlation functions $F_3(q)/F(q)$ [see (25) and (28)]. So the locality analysis for the interaction of k-eddies with much smaller scales could give us only the condition (35) and nothing else. It gives no restrictions for the possible values of exponent h. Indeed, it deals only with the exponent of the double-correlation function, which is defined as $\min[2h - D(h) + 3]$ and not by the minimal value of h. The point is that $D(h)$ decreases while h decreases, so however slow (for small h) the drop of the correlation function is in k space, the decrease in the probability of finding the small eddy from that h subset provides the interaction locality.

One could explain the physical meaning of the above locality analysis in the following way. The aforementioned locality means that a typical interaction time $\tau(k,k')$ for the eddies of the same order of size should be much larger than $\tau_{\rm ir}(k,1/L)$ and $\tau_{\rm uv}(k,1/\eta)$. The first time could be estimated as a turnover time for the k-eddy from the h subset (we put $k \simeq k'$)

$$\tau(k,h) \simeq [k\delta v(k,h)]^{-1} \simeq \frac{L}{U}(kL)^{h-1} \ . \tag{37}$$

The time of an infrared interaction is estimated as being necessary for the k-eddy to be distorted noticeably by the velocity difference $\Delta v_L(k)$ of the large-eddy flow on the scale of the small one. It is thus as follows:

$$\tau_{\rm ir} \simeq [k\Delta v_L(k)]^{-1} \simeq \frac{L}{U} \ .$$

It is important that this time is defined by the scale of the k-eddy only and is independent of its velocity (that is, which h subset the eddy belongs to). Comparing $\tau(k)$ and $\tau_{\rm ir}$, we obtain the above locality criterion $h < h_{\max}=1$.

The consideration of an ultraviolet interaction is a bit more complicated. The interaction of the given k-eddy with small k_1-eddies (with $\eta \geq k_1 \gg k$) should provide some turbulent viscosity ν_T:

$$\tau_{\rm uv}(k,k_1,h) \simeq [k^2 \nu_T(k_1,h)]^{-1} .$$

The value of viscosity should depend on the turbulence level at the place and the time under consideration. This means that the viscosity depends on the velocity of the k-eddy which generates the small eddies. The rate of energy dissipation

$$\nu_T k_1^2 [\delta v(k_1)]^2 \simeq \nu_T \epsilon^{2/3} k_1^{4/3} \left[\frac{k_1}{k} \right]^{2/3-\zeta_2}$$

should be equal to the local energy flux $\epsilon \simeq k \delta v^3(k,h)$. It gives the viscosity

$$\nu_T(k_1) \simeq \epsilon^{1/3} k_1^{-4/3} \left[\frac{k_1}{k} \right]^{2/3-\zeta_2}$$

$$\simeq k^{1/3} k_1^{-4/3} \delta v(k,h) \left[\frac{k_1}{k} \right]^{2/3-\zeta_2} ,$$

and the following estimate for $\tau_{\rm uv}$:

$$\tau_{\rm uv}(k,k_1,h) \simeq \frac{L}{U} \left[\frac{k_1}{k} \right]^{2-\zeta_2} (kL)^{h-1} .$$

Determining the ultraviolet locality, we obtain $\tau/\tau_{\rm uv} \simeq (k_1/k)^{2-\zeta_2} \ll 1$, which gives criterion (35) independent of h under consideration. The locality analysis thus gives only an upper restriction for the values of exponents h.

In conclusion, we would like to say that the range of exponents evidently should be restricted from below. We suppose that the high-order correlation functions behave as if they are determined by the positive exponents only [3,4]. What we have shown here is that such a restriction should be given not by locality analysis but by the other conditions (e.g., incompressibility [13]).

APPENDIX: DIAGRAMMATIC APPROACH TO THE ASYMPTOTIC BEHAVIOR OF THE HIGH-ORDER-VELOCITY-CORRELATION FUNCTIONS

1. The quasi-Lagrangian diagram technique

Following [8,11], we use the quasi-Lagrangian (QL) velocity $\mathbf{u}(\mathbf{r}_0,\mathbf{r},t)$ (15) in order to eliminate the sweeping in the vicinity of some space point $\mathbf{r}_0$. In the $(t,\mathbf{k})$ representation these equations reproduce the original Navier-Stokes equations (7) but with another vertex V [Eq. (16)], which was called *dynamic* [8].

The detailed analysis of the Wyld diagram technique (DT) for quasi-Lagrangian velocities was given in [8,11]. Such a diagram technique was formulated [8] in terms of the double-velocity-correlation function for the QL velocities $\mathbf{u}(\mathbf{r}_0;\mathbf{k},\omega)$ (17) and Green's function:

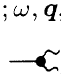
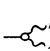
$$G_{\alpha\beta}(\mathbf{r}_0;\omega_1,\mathbf{k}_1,\mathbf{k}_2)\delta(\omega_1-\omega_2) = \langle \delta u_\alpha(\mathbf{r}_0;\mathbf{k}_1,\omega_1)/\delta f_\beta(\mathbf{r}_0;\mathbf{k}_2,\omega_2) \rangle , \quad \text{(A1)}$$

where f is an extraneous force. Let us introduce the graphical notations for Green's function and for the double-velocity-correlation function for the QL velocities (A1) and for the dynamic vertex (16), which is necessary in our treatment:

$G_{\alpha\beta}(\boldsymbol{r}_0;\omega,\boldsymbol{q},\boldsymbol{p}) =$ $\alpha,\boldsymbol{k}_1$ $\beta,\boldsymbol{k}_2$

$G^*_{\alpha\beta}(\boldsymbol{r}_0;\omega,\boldsymbol{q},\boldsymbol{p}) =$ $\alpha,\boldsymbol{k}_1$ $\beta,\boldsymbol{k}_2$

$F_{\alpha\beta}(\boldsymbol{r}_0;\omega,\boldsymbol{q},\boldsymbol{p}) =$ $\alpha,\boldsymbol{k}_1$ $\beta,\boldsymbol{k}_2$

$\Gamma_{\alpha\beta\gamma}(\boldsymbol{k}|\boldsymbol{q},\boldsymbol{p}) =_{\alpha,\boldsymbol{k}}$  $_{\gamma,\boldsymbol{p}}^{\beta,\boldsymbol{q}}$ $\quad V_{\alpha,\beta,\gamma}(\boldsymbol{k}|\boldsymbol{q},\boldsymbol{p};\boldsymbol{r}_0) =_{\alpha,\boldsymbol{k}}$  $_{\gamma,\boldsymbol{p}}^{\beta,\boldsymbol{q}}$ (A2)

These symbols are dictated by the functional approach [2] to deriving Wyld's DT. They automatically specify all topological properties of the diagrams for the mass operators described in [14].

As usual, in order to obtain the equations for Green's functions and double-correlation functions (A1), it is necessary to summarize one-particle irreducible diagrams. As a result, one obtains the system of Dyson-Wyld equations, which has the usual graphic form [15]. Being analytically written in our nondiagonal diagram technique, these equations take an integral form [8,11] but not an algebraic one:

$$G_{\alpha\beta}(\mathbf{r}_0;\omega,\mathbf{q},\mathbf{p}) = G^0_{\alpha\beta}(\omega,\mathbf{q})\delta(\mathbf{q}-\mathbf{p}) + \int G^0_{\alpha\gamma}(\omega,\mathbf{q})\Sigma_{\gamma\delta}(\mathbf{r}_0;\omega,\mathbf{q},\mathbf{k})G_{\delta\beta}(\mathbf{r}_0;\omega,\mathbf{k},\mathbf{p})d\mathbf{k} ,$$
$$F_{\alpha\beta}(\mathbf{r}_0;\omega,\mathbf{q},\mathbf{p}) = \int G_{\alpha\gamma}(\mathbf{r}_0,\omega,\mathbf{q},\mathbf{k}_1)\Phi_{\gamma\delta}(\mathbf{r}_0;\omega,\mathbf{k}_1,\mathbf{k}_2)G^*_{\delta\beta}(\mathbf{r}_0;\omega,\mathbf{k}_2,\mathbf{p})d\mathbf{k}_1 d,\mathbf{k}_2 . \quad \text{(A3)}$$

Here $\Sigma_{\gamma\delta}(\mathbf{r}_0;\omega,\mathbf{q},\mathbf{p})$ and $\Phi_{\gamma\delta}(\mathbf{r}_0;\omega,\mathbf{q},\mathbf{p})$ are mass operators with inflowing momentum $\mathbf{q}$ and outflowing momentum $\mathbf{p}$. These mass operators are represented by an infinite series of one-particle irreducible diagrams, which have the same form as in the Wyld diagram technique for Eulerian velocities [15,2] but the Eulerian vertices Γ (7) have to be replaced by the dynamic vertices V (16):

$$\Sigma=\Sigma_2+\Sigma_4+\Sigma_6+\cdots\ ,\quad \Phi=\Phi_2+\Phi_4+\Phi_6+\cdots\ . \tag{A4}$$

Here Σ_{2n} is some functional of $2n$ vertices V, n double-velocity-correlation functions F, and $(2n-1)$ Green's functions G and G^*; Φ_{2n} is a functional of $2n$ vertices V, $(n+1)$ functions F, and $2(n-1)$ functions G and G^*.

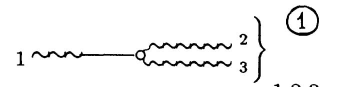
2. Diagram series for the triple-correlation function

There are many various approximate methods for calculating the double-velocity-correlation function, which gives a reasonable result in the inertial interval: the Kolmogorov-Obukhov spectrum [2,16]. The triple-velocity-correlation function is the first object that cannot be calculated in the conventional approaches without complete elimination of the sweeping. Therefore, it is reasonable to use the quasi-Lagrangian approach for analyzing triple or higher momenta of the velocity field.

In the QL diagram technique, the triple-correlation function (17) is represented by an infinite series of diagrams that has the same form as in the Wyld diagram technique for Eulerian velocities [15,2], though with the dynamic vertices V (16):

$$F^{(3)}=F^{(3)}_1+F^{(3)}_3+F^{(3)}_5+\cdots\ . \tag{A5}$$

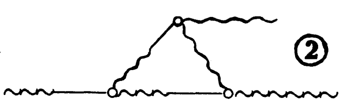
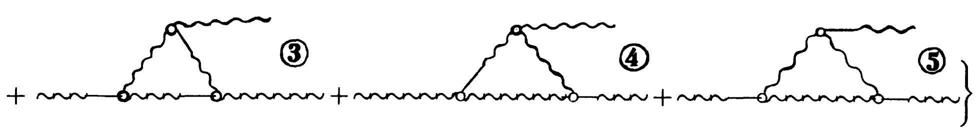
Here $F^{(3)}_{2n+1}$ is some functional of $(2n+1)$ vertices V, $(n+2)$ double-velocity-correlation functions F, and $(2n+1)$ Green's functions G and G^*. In Kraichnan's direct-interaction approximation $F^{(3)}=F^{(3)}_1$, where

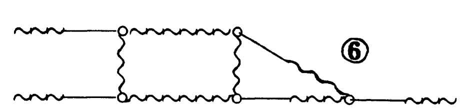
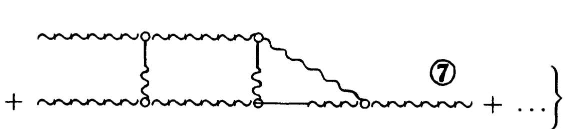
$$F^{(3)}_{1,\alpha\beta\gamma}(\boldsymbol{r}_0;\omega_1,\omega_2,\omega_3,\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=\sum\Big\{$$

$$\Big\}_{1,2,3} \tag{A6}$$

Here $\sum\{\ \}_{1,2,3}$ denotes the sum of three terms with the following replacement of the exponents: $1\to2\to3\to1$ and $1\leftarrow2\leftarrow3\leftarrow1$. The rules for reading these diagrams are as usual. They may be recollected by comparing the diagrams for $F^{(3)}_1$ with the corresponding analytical expressions:

$$\begin{aligned}F^{(3)}_{\alpha\beta\gamma}(\mathbf{r}_0;\omega_1,\omega_2,\omega_3,\mathbf{k},\mathbf{q},\mathbf{p})=\int d\mathbf{k}'d\mathbf{q}'d\mathbf{p}'\{[G_{\alpha\alpha'}(\mathbf{r}_0,\omega_1,\mathbf{k},\mathbf{k}')F_{\beta\beta'}(\mathbf{r}_0,\omega_2,\mathbf{q},\mathbf{q}')F_{\gamma\gamma'}(\mathbf{r}_0,\omega_3,\mathbf{p},\mathbf{p}')V^{(3)}_{\alpha'\beta'\gamma'}(\mathbf{k}'|\mathbf{q}',\mathbf{p}';\mathbf{r}_0)]\\ +[\text{term with } 1\to2\to3\to1\ ,\ \mathbf{k}\to\mathbf{q}\to\mathbf{p}\to\mathbf{k}]\\ +[\text{term with } 1\leftarrow2\leftarrow3\leftarrow1\ ,\ \mathbf{k}\leftarrow\mathbf{q}\leftarrow\mathbf{p}\leftarrow\mathbf{k}]\}\ .\end{aligned} \tag{A7}$$

The diagrams for $F^{(3)}_3$ and $F^{(3)}_5$ look like

$$F^{(3)}_{3,\alpha\beta\gamma}(\boldsymbol{r}_0;\omega_1,\omega_2,\omega_3,\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=\sum\Big\{$$


$$\Big\}_{1,2,3}$$

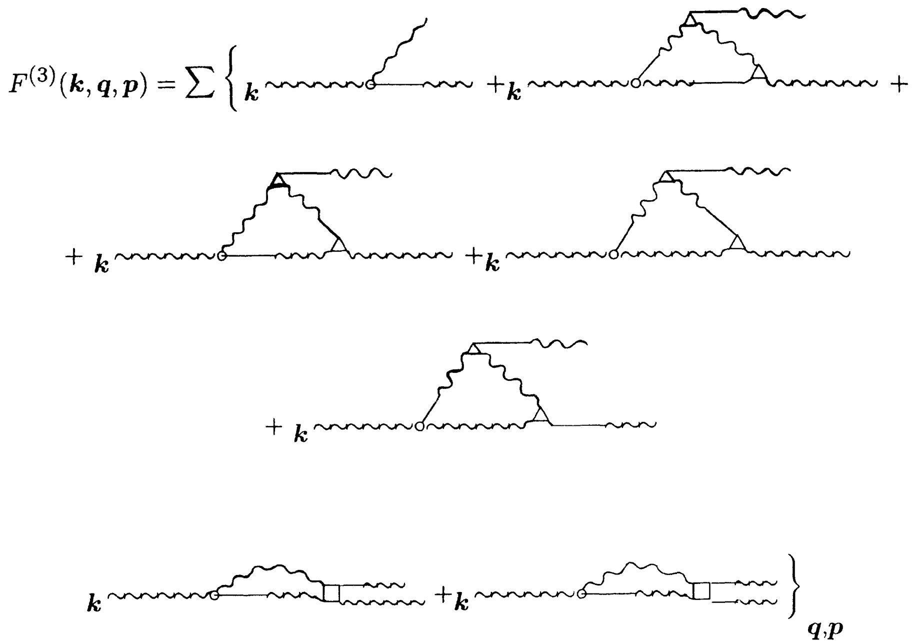
$$F^{(3)}_{5,\alpha\beta\gamma}(\boldsymbol{r}_0;\omega_1,\omega_2,\omega_3,\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)=\sum\Big\{$$


$$+\ \ldots\Big\}_{1,2,3} \tag{A8}$$

Note that the integral of the expression (A7) over ω_1, ω_2, and ω_3 gives [according to (19)] the simultaneous correlation function and it has to be proportional to $\delta(\mathbf{k}+\mathbf{q}+\mathbf{p})$. Unfortunately, this common property is not satisfied in (A7). The same problem arises under the calculation of the simultaneous double-correlation function for QL velocity. To solve the problem, it is necessary to discuss some general properties of the QL technique in more detail. We have already mentioned that all simultaneous correlation functions of Eulerian and quasi-Lagrangian velocities coincide. This fact is reflected in (18) and (19) for double- and triple-velocity-correlation functions. Fulfillment of these general requirements for n-order-velocity-correlation functions is not so obvious and must be considered as additional sum rules that resemble the Ward identities. These sum rules have to be fulfilled for every specific solution of the complete equation in the QL theory, which takes into account the entire diagram series. Nevertheless, the sum rules are not necessarily fulfilled for solutions obtained by arbitrary reduction of the series. In order to obtain the calculation method, we must pass to the $\mathbf{r}$ representation with respect to the common wave vectors. For triple-correlation functions, this means that we introduce the quantity

$$\tilde{F}^{(3)}_{\alpha\beta\gamma}(\mathbf{r}-\mathbf{r}_0;\mathbf{k},\mathbf{q},\mathbf{p}) = \int F^{(3)}_{\alpha\beta\gamma}(\mathbf{r}_0;\mathbf{k}+\mathbf{g}/3,\mathbf{q}+\mathbf{g}/3,\mathbf{p}+\mathbf{g}/3)\exp(i\mathbf{g}\cdot\mathbf{r})d\mathbf{g} \ . \tag{A9}$$

In fact, $\tilde{F}^{(3)}$ is independent of $\mathbf{r}$. However, as was stated above, this property must be fulfilled for the entire series, but not for each of its terms. In principle, $\tilde{F}(3)$ can be calculated with any $\mathbf{r}$, but in the case of a cutoff of the diagram series, it is reasonable to do this at $\mathbf{r}=\mathbf{r}_0$ because this is the point where the sweeping has been eliminated in the best way.

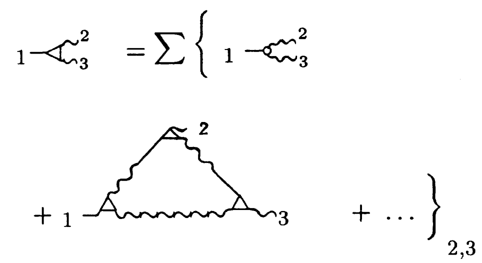
Now let us consider the diagram series (A6) for $F^{(3)}$ in the asymptotic limit $k \ll q \simeq p$. It is possible to find the principal subsequence of diagrams for $F^{(3)}$ in this limit. At every order of perturbation theory, these are the diagrams with small wave vector $\mathbf{k}$ "flowing into diagram" via *double momenta* F [like (4) and (7)]. The other diagrams, e.g., (6), etc. (with small $\mathbf{k}$ "flowing into diagram" via Green's function) are smaller due to the smallness of (q/k). This smallness stems from the simple fact that double moment F falls with k much faster than Green function G. So diagrams with the Green function of small wave number are less by a factor of $(k/q)^y$, where y is the exponent of F. As was mentioned above, the exponent is close to the Kolmogorov one: $y \approx \frac{11}{3}$. So we have a good small parameter $(k/q)^{11/3}$. In the asymptotical region, the principal subsequence of diagrams may be represented in a partially summed form:



(A10)

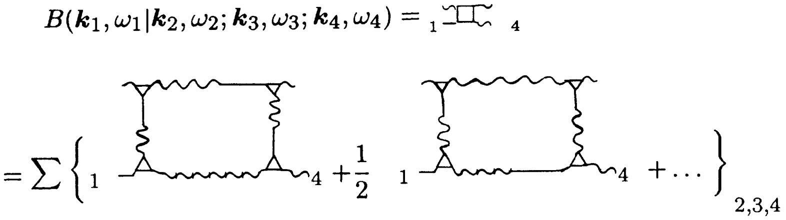
Here $\sum\{\ \}_{\mathbf{q},\mathbf{p}}$ denotes the sum of two terms with the following replacement of the indices: $\mathbf{q}\leftrightarrow\mathbf{p}$. Triple (three points) complete vertex A and quadruple (four points) complete vertices B and C are the following infinite sum of two-particle irreducible diagrams:

$A(\boldsymbol{k}_1,\omega_1|\boldsymbol{k}_2,\omega_2;\boldsymbol{k}_3,\omega_3) =$

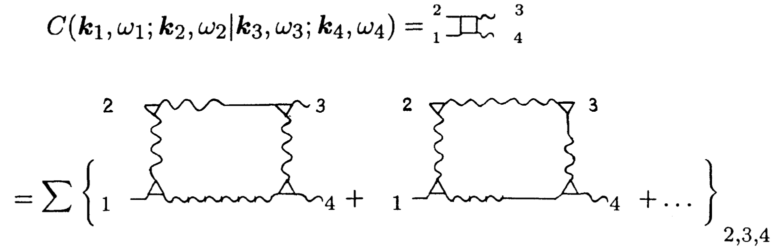


(A11)

$B(\boldsymbol{k}_1,\omega_1|\boldsymbol{k}_2,\omega_2;\boldsymbol{k}_3,\omega_3;\boldsymbol{k}_4,\omega_4) =$



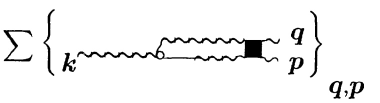
$C(\boldsymbol{k}_1,\omega_1;\boldsymbol{k}_2,\omega_2|\boldsymbol{k}_3,\omega_3;\boldsymbol{k}_4,\omega_4) =$



(A12)

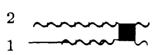
The diagrams (A6), (A8), and (A10) for $F^{(3)}(\mathbf{k},\mathbf{q},\mathbf{p})$ may be represented in the following form:

$F^{(3)}(\boldsymbol{k},\boldsymbol{q},\boldsymbol{p}) = \sum$



(A13)

where



$D_{\alpha\beta\gamma\delta}(1,2,3,4)$

is some function of $\mathbf{k}_j,\omega_j$ $(j=1-4)$ expressed according to (A10) via complete vertices A, B, and C. In an analytical form the relation (A13) is represented by the expression (20). It is very important to note that all lines F depending only on $\mathbf{k}$ and ω are presented in (A13) explicitly. The diagrams for D do not contain such lines. Functions F and G in these diagrams depend on $\mathbf{k}$ and ω via sums like $\mathbf{k}+\mathbf{q}'\simeq\mathbf{q}\simeq\mathbf{p}$ and $\omega+\omega_1\simeq\omega_2$. Function D thus depends on $\mathbf{k}$ and ω analytically and may be expanded at small k and ω into series with respect to powers of k and ω. This point allows us to obtain the asymptotical limit of the triple correlator in the explicit form (24).

The final relations (24) or (28) are exact, since we made no uncontrolled approximation while analyzing the Navier-Stokes equation. Of course, using diagrammatic series (as well as functional integral) is a poorly defined procedure from the viewpoint of rigorous mathematics. Nevertheless, in theoretical physics it could give exact results in fluid mechanics as well as in quantum-field theory or statistical mechanics if one analyzes series as a whole (or, equivalently, the function integral). For example, the Dyson equation or Ward identities are usually obtained in that way. Note that the relation obtained is some analog of Ward identities in the field theory.

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