Comparison of the Scale Invariant Solutions of the Kuramoto-Sivashinsky and Kardar-Parisi-Zhang Equations in d Dimensions

Victor S. L'vov [a] and Itamar Procaccia

Departments of Nuclear Physics and Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel

(Received 10 July 1992)

It is shown that the scale invariant solutions of the KS and KPZ models of surface roughening are identical for dimensions $d<2$, but may differ for dimensions $d\geq 2$ in the strong coupling limit. For $d\geq 2$, these models posses two different scaling solutions, one with d-independent scaling exponents $y=z=2$, and the other with d-dependent nontrivial exponents. The first of these solutions is realizable in one of these models, but not the other. These conclusions are valid to all orders in renormalized perturbation theory.

PACS numbers: 68.35.Md, 68.35.Rh

The dynamics of growing interfaces is discussed in $d+1$ dimensions, where d is the dimension of the interface and 1 stands for time. The physics of growing interfaces becomes interesting (and complicated) when the d-dimensional interface wrinkles and forms a self-affine graph. There are many experimental examples of dynamical wrinkling of interfaces, ranging from two-phase flows in porous media [1] to bacterial colonies [2], but only a few theoretical models were discussed in any depth. The two models for surface roughening that have attracted most attention are the Kardar-Parisi-Zhang [3] (KPZ) and the Kuramoto-Sivashinsky [4,5] (KS) equations. The first one is driven by a random forcing and is written as

$$\partial h(\mathbf{x},t)/\partial t=\nu_0\nabla^2 h(\mathbf{x},t)+|\nabla h(\mathbf{x},t)|^2+\eta(\mathbf{x},t)\,. \quad (1)$$

Here $h(\mathbf{x},t)$ is the height of a growing interface and $\eta(\mathbf{x},t)$ is a white, Gaussian random noise. The second model is completely deterministic, and is driven by inherent instabilities:

$$\partial h(\mathbf{x},t)/\partial t=\nu_0\nabla^2 h(\mathbf{x},t)-\nabla^4 h(\mathbf{x},t)+|\nabla h(\mathbf{x},t)|^2\,. \quad (2)$$

In Eqs. (1) and (2) ν_0 is a parameter, which is positive in (1) and negative in (2). In both models $\mathbf{x}$ is a vector in d-dimensional space. The KPZ equation was derived as a continuum limit of models describing random particle additions to a growing interface. The KS equation was derived in the context of intrinsic instabilities, like flame propagation. It is linearly unstable, and nonlinearly chaotic, with bounded solutions roaming on a strange attractor forever.

From the theoretical point of view an interesting question is whether such models have scale invariant solutions in the long wavelength limit, and whether such solutions can be grouped in universality classes. Passing to Fourier representation in space and time one considers the correlation function of $h(\mathbf{k},\omega)$, denoted by $n(\mathbf{k},\omega)$:

$$\langle h(\mathbf{k},\omega)h^*(\mathbf{k}',\omega')\rangle=n(\mathbf{k},\omega)\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')\,. \quad (3)$$

A scale invariant solution for $n(\mathbf{k},\omega)$ is written as

$$n(\mathbf{k},\omega)=\frac{n(\mathbf{k})}{\nu k^z}f\left(\frac{\omega}{\nu k^z}\right),\qquad n(\mathbf{k})=\int n(\mathbf{k},\omega)\frac{d\omega}{2\pi}=\frac{n}{k^y}\,, \quad (4)$$

where νk^z has the dimension of ω, and (n/k^y) has the dimension of the simultaneous double correlator $n(\mathbf{k})$. One considers two models for surface roughening to be in the same universality class if for $k\rightarrow 0$ they have scale invariant solutions *with the same scaling exponents* $y(d)$ *and* $z(d)$ in all dimensions d.

It has been suggested before [6] that Eqs. (1) and (2) are in the same universality class. From the point of view of renormalized perturbation theory (and see below) this is possible, whenever the renormalized correlator $n(\mathbf{k},\omega)$ is determined predominantly by the nonlinear term in the equation of motion, which is the same for Eqs. (1) and (2). It has been shown recently [7] that the scale invariant solutions of (1) and (2) are identical in $1+1$ dimensions, in which indeed the dominance of the nonlinear term is realized, *but they can be different in* $2+1$ *dimensions*, where the linear terms retain their relevance [8]. It thus becomes extremely interesting to ask how and where the bifurcation in the solutions of these two models arises as a function of dimensionality. The aim of this Letter is to answer this question.

A convenient formal device to investigate the scale invariant solutions of equations like (1) and (2) is the renormalized perturbation theory *à la* Wyld [9]. In order to employ this approach one introduces the Green's function $G(\mathbf{k},\omega)$, defined as the response of the nonlinear system to a vanishingly small external perturbation $\delta f(\mathbf{k}',\omega')$:

$$G(\mathbf{k},\omega)\delta(\mathbf{k}-\mathbf{k}')\delta(\omega-\omega')=\langle\delta h(\mathbf{k},\omega)/\delta f(\mathbf{k}',\omega')\rangle\,.$$

In terms of $G(\mathbf{k},\omega)$ and the dressed correlator $n(\mathbf{k},\omega)$ one derives the Dyson-Wyld equations:

$$G(\mathbf{k},\omega)=[\omega-i\gamma_{\mathbf{k}}-\Sigma(\mathbf{k},\omega)]^{-1}\,, \quad (5a)$$

$$n(\mathbf{k},\omega)=|G(\mathbf{k},\omega)|^2[\Phi(\mathbf{k},\omega)+\eta^2]\,. \quad (5b)$$

In these equations $\gamma_{\mathbf{k}}$ is the bare linear part, which is $\gamma_{\mathbf{k}}=-\nu_0 k^2$ for Eq. (1) and $\gamma_{\mathbf{k}}=(|\nu_0|k^2-k^4)$ for Eq. (2). The term η^2 is the strength of the noise in KPZ, and is zero in the case of KS. The mass operators $\Sigma(\mathbf{k},\omega)$ and $\Phi(\mathbf{k},\omega)$ are the "self-energy" and "intrinsic noise," respectively, which are infinite expansions in terms of $G(\mathbf{k},\omega)$ and $n(\mathbf{k},\omega)$. The first diagrams appearing in the expansion of $\Sigma(\mathbf{k},\omega)$ and $\Phi(\mathbf{k},\omega)$ are the same for KPZ and KS. They are shown in Fig. 1. We reiterate that at this level of description the only difference between KPZ and KS as it appears in Eqs. (5) is the form of $\gamma_{\mathbf{k}}$, and the existence of an external random forcing in the case of KPZ.

We shall examine the scale invariant solutions of Eqs. (5) using first the lowest-order diagrams; after deriving our results, we shall demonstrate their validity to all orders. The main point to understand is that there can be two different types of scale invariant solutions, that we term "local" and "nonlocal." The local solutions are obtained when the contributions to a given k behavior come mainly from fluctuations whose wave vector is of the order of k. Nonlocal solutions are dominated by interactions of fluctuations of a given k vector with short wavelength fluctuations close to the ultraviolet (UV) cutoff. Mathematically, the local solutions are obtained when all the integrals in the series converge. The nonlocal solutions are obtained in our case when the integrals diverge in the UV. We discuss now these two classes of solutions in the lowest-order diagrams.

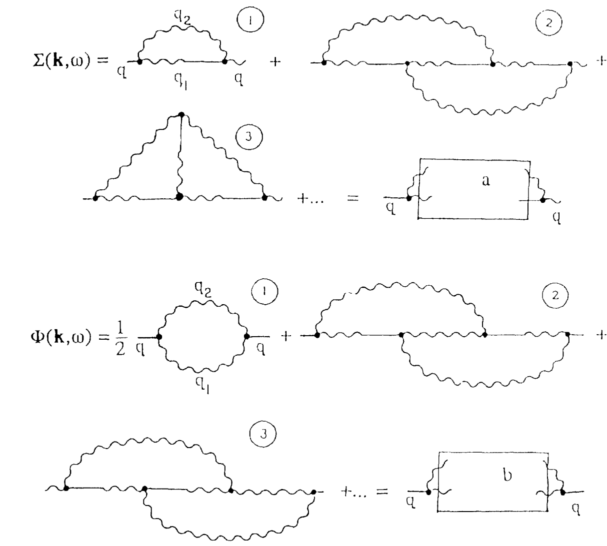

FIG. 1. The first diagrams in the infinite series expansion of the self-energy Σ and the intrinsic noise strength Φ. If the series is cut after the diagrams denoted by 1, one gets the one-loop approximation. The symbolic resummation of the series is also shown. The structure of this symbolic diagram is used in the argument extending our results to all orders in perturbation theory.

Consider the diagrams in Fig. 1 in d dimensions, and substitute the scaling form (4) for $n(\mathbf{k},\omega)$, and for $G(\mathbf{k},\omega)$ the form $G(\mathbf{k},\omega)=(1/\nu k^z)g(\omega/\nu k^z)$. The result is

$$\Sigma^{(2)}(\mathbf{k},\omega)=-\int \mathbf{k}_1\cdot(\mathbf{k}+\mathbf{k}_1)(\mathbf{k}+\mathbf{k}_1)\cdot\mathbf{k}\frac{1}{\nu k_1^z}g\left(\frac{\omega_1}{\nu k_1^z}\right)\frac{n}{\nu|\mathbf{k}+\mathbf{k}_1|^{z+y}}f\left(\frac{\omega+\omega_1}{\nu|\mathbf{k}+\mathbf{k}_1|^z}\right)d^dk_1d\omega_1\,, \tag{6}$$

$$\Phi^{(2)}(\mathbf{k},\omega)=\frac{1}{2}\int[\mathbf{k}_1\cdot(\mathbf{k}-\mathbf{k}_1)]^2\frac{n}{\nu k_1^{z+y}}f\left(\frac{\omega_1}{\nu k_1^z}\right)\frac{n}{\nu|\mathbf{k}+\mathbf{k}_1|^{z+y}}f\left(\frac{\omega+\omega_1}{\nu|\mathbf{k}+\mathbf{k}_1|^z}\right)d^dk_1d\omega_1\,. \tag{7}$$

The nonlocal solutions are obtained by asserting that the integrals are dominated by the high end of the k_1 range, i.e., $k_1\sim k_{\max}\gg k$, and of $\omega_1\sim\omega_{\max}\sim k_{\max}^z\gg\omega$; then we find

$$\Sigma^{(2)}(\mathbf{k},\omega)\sim C_1nk^2(k_{\max})^a/\nu\,, \tag{8}$$

$$\Phi^{(2)}(\mathbf{k},\omega)\sim C_2n^2(k_{\max})^b/\nu\,, \tag{9}$$

where

$$a=2+d-y-z,\quad b=4+d-2y-z\,. \tag{10}$$

The nonlocal solution required that a and b were positive. On the other hand, if a and b were negative, the main contribution to the integrals would arise from the local region $k_1\sim k$. It is easy to see by power counting that the result in that case must be

$$\Sigma^{(2)}(\mathbf{k},\omega)\sim C_3nk^{2+a}/\nu\,, \tag{11}$$

$$\Phi^{(2)}(\mathbf{k},\omega)\sim C_4n^2k^b/\nu\,. \tag{12}$$

When the local solutions hold, one finds well-known scaling relations [10]. Use the fact that $\Sigma^{(2)}(\mathbf{k},\omega)\sim k^z$ together with (11) to derive

$$2z+y=4+d\,. \tag{13}$$

Of course, this single relation is not sufficient to evaluate y and z in all dimensions. In 1+1 dimensions it is known theoretically that $y=2$ and therefore $z=\frac{3}{2}$. In 2+1 dimensions there are only numerical results [11] which indicate that $y\approx 2.8$, and therefore z has to be about 1.6. On the other hand, in the nonlocal case, Eqs. (8) and (9) together with (5) lead to

$$G(\mathbf{k},\omega)=\frac{1}{\omega-i\nu k^2},\quad n(\mathbf{k},\omega)=\frac{\Phi}{\omega^2+(\nu k^2)^2}\,. \tag{14}$$

Accordingly, in this case $y=z=2$, independent of d. The unknown constant ν becomes an effective viscosity. Equation (8) furnishes an estimate for ν and n, in the one-loop approximation:

$$\nu(\nu-\nu_0)=C_1 n k_{\max}^a, \quad n=C_2 n^2 k_{\max}^b/\nu^2 . \tag{15}$$

At this point we can find at which dimensions the nonlocal solutions can exist. Substituting $y=z=2$ in Eqs. (10) we find $a=b=d-2$. Since it is necessary for a and b to be positive to allow nonlocal solutions, we discover that these solutions become allowed exactly at two dimensions, and continue to exist at all higher dimensions. Exactly at two dimensions one expects logarithmic corrections that were treated explicitly in Ref. [8]. For $d<2$ only local scale invariant solutions are allowed for both KPZ and KS.

Similarly, it is instructive to find the window of locality as a function of dimension. Since local solutions are obtained when a and b are negative, we can find the boundaries of existence of the local solution by solving simultaneously the scaling relation (13) together with $a<0$ and $b<0$. The results are the two constraints for z,

$$z<2, \quad z<(4+d)/3 . \tag{16}$$

It is easy to see that infrared locality requires $z>0$. The resulting window of locality is shown in Fig. 2, together with the known results for z for KPZ at $d=1$ and $d=2$, which both belong to the window of locality. In the same figure we also indicate the trajectory of $z(d)$ for the nonlocal solution. It gets born exactly at $d=2$, and joins immediately the $z=2$ line.

Note that the window of locality (16) was obtained using the lowest-order diagrams. Using scaling indices in this window to compute higher-order diagrams results in divergences. However, these divergences can be eliminated by changing variables to so-called "quasi-Lagrangian variables"; see Ref. [12]. As a result the window of locality remains unchanged when higher-order diagrams are taken into account.

Finally, we need to discuss the role of the higher-order diagrams for the nonlocal solutions. Consider, for example, the higher-order diagrams for the imaginary part of $\Sigma(\mathbf{k},\omega)$, $\Sigma''(\mathbf{k},\omega)$. All these diagrams end with a vertex proportional to $(\mathbf{k}_j\cdot\mathbf{k})$ where $\mathbf{k}_j$ is some internal integration variable; see Fig. 1. Accordingly, we can resum the series symbolically (to an unknown result) that must have the structure shown in Fig. 1, i.e., again with a last vertex proportional to $(\mathbf{k}_j\cdot\mathbf{k})$. Therefore, for small k, the expansion of $\Sigma''(\mathbf{k},\omega)$ in k starts at least with $O(k)$, without $O(k^0)$. However, $\Sigma''(\mathbf{k},\omega)$ is even in $\mathbf{k}$, since the term proportional to $\mathbf{k}$ must vanish for $k=0$, by symmetry. Indeed, the integrand inside the diagram is odd with respect to all the inner $\mathbf{k}_j$ vectors in the limit $k=0$. Thus the expansion of $\Sigma(\mathbf{k},\omega)$ begins with k^2. For small k

$$\Sigma''(\mathbf{k},\omega)=\nu k^2 , \tag{17}$$

where ν is the resummed effective viscosity. We shall show now that the estimate (15) for ν and n remains unchanged at any finite order.

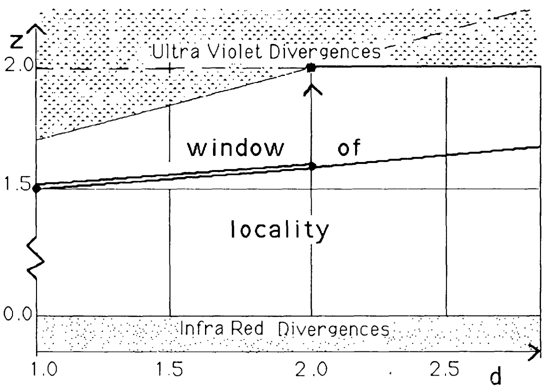

FIG. 2. The schematic trajectories of the static exponent z for the local and nonlocal solutions as a function of dimensionality d. Between $d=1$ and $d=2$ the trajectories of KPZ and KS are glued together: Only local solutions exist. Exactly at $d=2$ the nonlocal solution with $z=2$ for all $d>2$ gets born. The figure shows the window of locality bounded from below by $z=0$ (below this line there exist ir divergences). The window is bounded from above by the line $z=(4+d)/3$ for $d<2$ and by $z=2$ for $d>2$. The known results for the local and nonlocal values of z are shown as dots and as a square at $d=1$ and $d=2$, respectively.

Since for the nonlocal solution the main contribution from every diagram comes from the UV region $k_j\sim k_{\max}$, power counting (cf. Fig. 1) leads to the estimate $\Sigma^{(2n)}(\mathbf{k},\omega)\approx\Sigma^{(2)}(\mathbf{k},\omega)\Lambda^{n-1}$, where Λ is an expansion parameter,

$$\Lambda\approx[k_{\max}]^c n/\nu^2, \quad c=4+d-y-2z=d-2 . \tag{18}$$

Using the estimate (15) one finds that $\Lambda=\sim 1$. Similarly, one can show (cf. Fig. 1) that

$$\Phi^{(2n)}(\mathbf{k},\omega)\approx\Phi^{(2)}(\mathbf{k},\omega)\Lambda^{n-1}\approx\Phi^{(2)}(\mathbf{k},\omega) .$$

Thus the nonlocal solution (14) is self-consistent with the diagrammatic perturbation theory order by order.

It is easy to see that the nonlocal solution at $d\geq 2$ is only available to one, but not the other model. The proof follows exactly the one for $d=2$ in Ref. [8], and it shows that the sign of the bare value of the parameter ν_0 distinguishes the two equations. Thus if the nonlocal solution is realized by KS, it cannot be realized by KPZ, and vice versa. The numerical evidence [8] in two dimensions points towards the KS model as the one that realizes the nonlocal solution. We conjecture that KS indeed realizes the nonlocal solution $z=y=2$ for all dimensions higher than two.

We also know from numerical simulations [13] that at $d=2$ and $d=3$ KPZ realizes the local scaling solution. We have no proof, however, that this excludes the possibility that KS would also have a local solution. In fact, from the point of view of the theory, it remains an open question whether KS can or cannot realize the local solution at $d\geq 2$. It is our feeling that if indeed it would turn out that the numerical evidence about KS realizing the nonlocal solution is correct, then the theoretical reason for the exclusion of the local solution for this model may be related to the matching of the local solution in the ir region to the UV part of the spectrum. In KS the UV spectrum is characterized by a local maximum at the fastest growing mode, followed by a fast decay due to the hyperviscosity term.

In summary, we showed that KPZ and KS have two types of strong coupling scaling solutions, with different scaling exponents. The window of locality in the z-d plane has been calculated. The nonlocal solution does not exist below two dimensions. Above two dimensions, both solutions exist, but the nonlocal one can be realized by one model but not the other. We have no proof that the local solution cannot be realized by KS, and therefore at this point we can only refer to numerical simulations to suggest that KS realizes the nonlocal solutions at dimensions $d=2$ and higher.

This work has been supported in part by the German–Israeli Foundation GIF and the U.S.-Israel Bi-National Science Foundation BSF. V.L. is grateful for the Professorial Fellowship granted by the Weizmann Institute of Science.

[a]Permanent address: Institute of Automation and Electrometry, Academy of Science of Russia, 630090 Novosibirsk, Russia.

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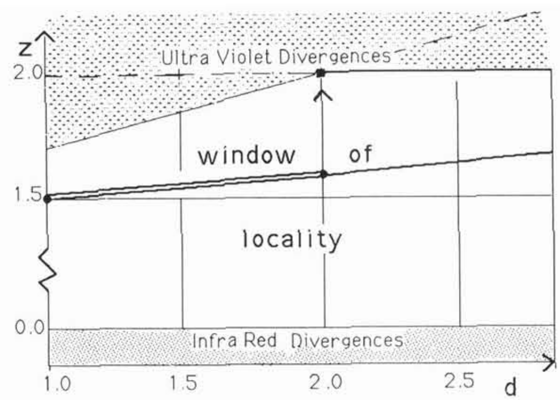

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