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Interaction Locality and Scaling Solution in d+1 KPZ and KS Models

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Properties of correlation functions of solutions of KPZ and KS equations (that describe roughening) in the region of a strong interaction of fluctuations are considered. We prove analytically a possibility of existence of a scaling solution in this region despite the "asymptotic freedom" situation occuring near the marginal dimension $d = 2$ (corresponding to growth of an interface in a real three-dimensional space). The proof is based on the locality of the interaction of fluctuations in k-space which can be demonstrated by passing to so-called quasi-Lagrangian variables. The inequalities restricting possible values of scaling indices are found.

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Dynamic surface roughening occurs in a variety of physical contexts, like flame propagation, growth of solids, two fluid flows, etc. Two models of surface roughening have attracted much attention due to their apparent simplicity and very rich nonlinear phenomena. One is the Kardar-Parisi-Zhang (KPZ) model [1] which contains a random forcing. The second is the Kuramoto-Sivashinsky (KS) equation [2,3] which is completely deterministic. In both models a motion of an interface in d+1 dimensions is discussed. The most interesting case is of course $d = 2$ since it corresponds to the motion of the interface in three-dimensional space.

The equation for fluctuations of an interface displacement $h(t, \mathbf{x})$ ($\mathbf{x}$ being a point in d-dimentional space) in the KPZ model is

$$\partial h/\partial t = \lambda|\nabla h|^2 + \nu_0 \nabla^2 h + \eta\,, \qquad (1)$$

where $\eta(t, \mathbf{x})$ is a white, Gaussian random noise which satisfies

$$\langle \eta(t_1, \mathbf{x}_1)\eta(t_2, \mathbf{x}_2)\rangle = 2T\nu_0\delta(t_1 - t_2)\delta(\mathbf{x}_1 - \mathbf{x}_2)\,. \qquad (2)$$

Here T is a coefficient not related to the temperature but designated by the same letter since formally it plays the same role. For the KS model the equation of motion for h is

$$\partial h/\partial t = \lambda|\nabla h|^2 + \nu_0 \nabla^2 h - \mu\nabla^4 h\ . \qquad (3)$$

The quantity λ in the equations determines the strength of the nonlinear interaction and ν_0 is a parameter, which is positive in (1) and negative in (3).

Note that in terms of the "velocity" field $\mathbf{v} = -\nabla h$ the KPZ and KS equations turns into

$$\partial v_\alpha/\partial t + 2\lambda(\mathbf{v} \cdot \nabla)v_\alpha = \phi_\alpha\,, \qquad (4)$$

similar in form to the Navier-Stokes equation. For KPZ model $\phi_\alpha = \nu_0\nabla^2 v_\alpha - \nabla_\alpha \eta$ and (4) turns into the so-called noisy Burgers (NB) equation. The difference between the NB equation and the Navier-Stokes equation is that the field $\mathbf{v}$ is potential in the former one and rotational in the latter one.

The correlation functions of h in the KPZ and KS models possess a nontrivial structure since in the long-range region it is determined by a strong nonlinear interaction. The correlation functions may be investigated analytically in the framework of Wyld's diagrammatic technique [4,5]. The evaluation of the first terms of the perturbation series in λ shows that $d = 2$ is the marginal dimension for the KPZ or NB model, i.e. the logarithmic corrections to the bare correlation functions appear. The analysis of the NB or KPZ model in the framework of renormalization group approach [6,7] leads to the conclusion that "asymptotic freedom" behavior of the correlation functions occurs. Therefore we encounter the situation of strong coupling.

Computer simulations in dimensions $d = 1$ and $d = 2$ (see book [8] and references therein) show that the behavior of correlation functions in the long-wave region is governed by scaling laws. The presence of a scaling in the case of strong coupling is a surprising result and needs a theoretical support. The problem is that known exactly solvable two-dimensional models with strong coupling demonstrate the appearance of a spontaneous gap in the excitation spectrum [9,10], spoiling the scaling in the long-range region. The same behavior of excitations is assumed for the quantum chromodynamics describing the strong interaction [11].

The aim of the letter is to present a theoretical foundation of a scaling in the long-wave region both for KPZ and KS models. We will examine scaling form of a correlation function $n(t,\mathbf{r}) = \langle h(t,\mathbf{r})h(0,0)\rangle$. In the Fourier representation it takes the form

$$n(\omega,\mathbf{k}) = \frac{n(\mathbf{k})}{ak^z} f\left(\frac{\omega}{ak^z}\right), \quad n(\mathbf{k}) = \frac{n_0}{k^y} \ . \tag{5}$$

Here $n(\mathbf{k})$ is the simultaneous pair correlation function in k-space, f is a dimensionless function of a dimensionless argument, n_0 and a being dimensional constants, and y and z are the static and dynamic exponents, the values of which are dependent on d (remember that the roughness exponent χ is equal to $(y-d)/2$).

We will show that under following conditions

$$z < 2; \ z < (d+4)/3; \ z > 0 \tag{6}$$

there occures a concelation of both ultraviolet and infrared divergences between terms of different orders of Wyld's perturbation series for KPZ and KS models. Earlier it was shown [12] that for $d = 1$ there is order by order cancellation of logarithmic divergences. An absence of diverging contributions to integral equations for $n(\mathbf{k})$ means a *locality of interaction*: a fluctuation of a given wave vector is affected mainly by fluctuations of wave vectors of the same order.

For the proof we shall use a *quasi-Lagrangian* (qL) variable $\mathbf{u}$ suggested by L'vov in connection with the problem of turbulence (see [13,14]):

$$\mathbf{v}(t,\mathbf{r}) = \mathbf{u}(t, \mathbf{r} - \mathbf{R}(t)) \,, \tag{7}$$

where

$$\mathbf{R}(t) = 2\lambda \int_{t_0}^{t} d\tau \mathbf{u}(\tau, \mathbf{r}_0) \ . \tag{8}$$

Here the function $\mathbf{u}(t,\mathbf{r})$ depends on auxilary parameters: a marked time t_0 and coordinates of a marked point $\mathbf{r}_0$. An equation for the qL-variable can be derived by substituting (7) into the equation (4):

$$\frac{\partial u_\alpha}{\partial t} + \lambda \nabla_\alpha \left((u_\beta - u_{0,\beta})^2 \right) = \tilde{\phi}_\alpha \ . \tag{9}$$

Here u_α, $\tilde{\phi}$ are qL-variables depending on t, $\mathbf{r}$ and also on t_0, $\mathbf{r}_0$, the relation between ϕ and $\tilde{\phi}$ is similar to that (7) between $\mathbf{v}$ and $\mathbf{u}$. The value $u_{0\alpha}$ in (9) is $u_\alpha(\mathbf{r}_0)$. This equation differs from the equation (4) in the term $u_{0\alpha}$ subtracting the "sweeping" in the marked point $\mathbf{r}_0$.

For our purpose it is convenient to use $\mathbf{k}$-representation. The equation of motion for $u_\alpha(\mathbf{k})$ follows from (9):

$$i\frac{\partial}{\partial t} u_\alpha(t,\mathbf{k}) = \frac{1}{2} \int \frac{d^d q\, d^d p}{(2\pi)^{2d}} V_{\alpha\beta\gamma}(\mathbf{k};\mathbf{q},\mathbf{p}) u_\beta(t,-\mathbf{q}) u_\gamma(t,-\mathbf{p}) + \tilde{\phi}_\alpha(t,\mathbf{k}) \,, \tag{10}$$

where

$$V_{\alpha\beta\gamma}(\mathbf{k};\mathbf{q},\mathbf{p}) = (2\pi)^d 2\lambda k_\alpha \delta_{\beta\gamma} \Big(\delta(\mathbf{k}+\mathbf{q}+\mathbf{p}) - \delta(\mathbf{k}+\mathbf{q}) \exp(i\mathbf{p}\cdot\mathbf{r}_0) - \delta(\mathbf{k}+\mathbf{p}) \exp(i\mathbf{q}\cdot\mathbf{r}_0) + \delta(\mathbf{k}) \exp(i(\mathbf{q}+\mathbf{p})\cdot\mathbf{r}_0) \Big) \ . \tag{11}$$

The quantity $V_{\alpha\beta\gamma}$ is no other than the bare *dynamic interaction vertex*. The main technical difference between the quasi-Lagrangian description and the conventional one is that the wave vector $\mathbf{k}$ is no longer preserved in the dynamic vertex V since this vertex is not proportional to $\delta(\mathbf{k}+\mathbf{q}+\mathbf{p})$. This is a result of the absence of spatial homogeneity of the theory due to the explicit dependence of the equation (10) on the coordinate $\mathbf{r}_0$. Meanwhile (10) does not contain t_0 explicitly, so homogeneity in time does remain.

The qL-variables are useful since we achieved the property of *locality of the vertex* in the k-space: the vertex V tends to zero if any wave vector (k, q or p) goes to zero. To make it clear we have saved the last term in (11) which really gives no contribution. Note that the original "Eulerian vertex" (the first term in (11)) has another asymptotic regimes: it does not tend to zero if q or p goes to zero.

We can use the equation (10) as a starting point in developing the Wyld diagrammatic expansion. The natural objects in the expansion are *dressed propagators* $G_{\alpha\beta}$ and $F_{\alpha\beta}$. The former one called the Green's function is defined as an average susceptibility of the qL-velocity field u_α to a vanishingly small "force" $\delta\phi_\alpha$ to be added to the right-hand side of the equation (9):

$$G_{\alpha\beta}(t_1 - t_2, \mathbf{r}_1, \mathbf{r}_2) = \langle \delta u_\alpha(t_1,\mathbf{r}_1) / \delta\phi_\beta(t_2,\mathbf{r}_2) \rangle \ . \tag{12}$$

The second propagator is the double correlation function of the qL-variables:

$$F_{\alpha\beta}(t_1 - t_2, \mathbf{r}_1, \mathbf{r}_2) = \langle u_\alpha(t_1,\mathbf{r}_1) u_\beta(t_2,\mathbf{r}_2) \rangle \ . \tag{13}$$

One can derive by the conventional way Dyson–Wyld equations which are introduced by the following relations [4,5]:

$$G = G_0 + G_0 * \Sigma * G, \quad F = G * (\Phi_0 + \Phi) * G^* \,, \tag{14}$$

where G_0 and Φ_0 are the bare values of the Green's function and of the noise function; the quantities $\Sigma_{\alpha\beta}(t,\mathbf{r}_1,\mathbf{r}_2)$ and $\Phi_{\alpha\beta}(t,\mathbf{r}_1,\mathbf{r}_2)$ are called *self-energy function* and *intrinsic noise function* respectively. They may be considered as a sum of one-particle irreducible diagrams representing Σ and Φ in terms of the bare dynamic vertex and dressed propogators G and F.

The star "$*$" in (14) designates summation over repeated indices *plus* integration over the corresponding $\mathbf{r}$, t variables. Let us examine the scaling solutions of the diagrammatic equations of the type (5) for the functions G, F. Introduce scaling indices determining such a solution: $\omega \propto r^{-z_1}$, $F \propto r^{y_1-d}$, where ω is a characteristic frequency and r is a characteristic scale. The Dyson–Wyld equation (14) for G shows that $G \propto r^{-d}$. If there are **no** divergencies in the expressions for Σ and Φ we may assert that the scaling solution exists since there will not be any dimensional parameter in the theory. Then the index of the vertex will not be renormalized, which leads to the scaling relation

$$y_1 + 2z_1 = d + 2. \tag{15}$$

The locality of dynamic interaction causes the excellent convergence of the integrals determining the contributions to Σ and Φ both in the ultraviolet and infrared regions. The analysis of diagrams for the quantities Σ and Φ in the one-loop approximation shows that such a situation (the absence of divergencies) occers under the following conditions:

$$z_1 < 2;\ z_1 < (d+4)/3;\ z_1 > 0. \tag{16}$$

The first two inequalities are the conditions of the absence of ultraviolet divergencies in the diagrams for Σ and Φ respectively, and the latter one ensures the absence of infrared divergencies (in both Σ and Φ). In the same way it is possible to understand (for more detail see [13,14]) that because of the locality of the vertex V (11) there are **no** infrared and ultraviolet divergencies in higher order diagrams under the condititions (16).

Therefore we proved the existence of the scaling solution of the NB-equation but for the qL-variable $\mathbf{u}$. To find the structure of the correlation function of $\mathbf{v}$ (say $\langle v_\alpha\, v_\beta\rangle$) we should substite (7) and then expand $\mathbf{u}(\mathbf{r}-\mathbf{R})$ in $\mathbf{R}$ which gives $\langle v_\alpha\, v_\beta\rangle$ as a series of correlation functions of u. The analysis of the terms of the series shows that all terms have the same dimensionality and that there are no divergences (in contrast with the turbulent case [13,14]). Therefore the correlation function $\langle v\, v\rangle$ possesses the scaling behavior with the same exponents as $\langle u\, u\rangle$. The analogous assertion is valid for all higher order correlation functions of v.

Returning now to the correlation function $\langle h\, h\rangle$ of the solutions of the KPZ or KS equations and recalling $\mathbf{v} = -\nabla h$ we conclude that the solution (5) may be realized and has the exponents $y = y_1 + 2$, $z = z_1$. As a consequence of the relation (15) we find

$$y + 2z = d + 4, \tag{17}$$

and consequently that the restrictions imposed on z are (6). Note that a relation of the type (17) was obtained in [7] as a consequence of Galilean invariance but *under the assumption* that scaling exists, whereas the aim of our work was a proof of the existance of scaling.

Let us stress that our proof of interaction locality is valid for any dimension d. Therefore in the region of strong coupling the scaling behavior of correlation functions of KPZ and KS models may be realized in the space of any dimension under conditions (6). In the dimension $d = 1$ one has $y = 2$, which is a consequence of a conservation law following from the KPZ-equation (1) in this dimension [6]. Therefore $z = 3/2$ as it follows from (17).

It is naturally to expect that because of locality of interaction the structure of correlation functions in the region of strong fluctuations will be determined only by the form of nonlinearity. Therefore the correlation functions for KPZ and KS models (having the same nonlinear term) should coincide in the long-wave limit. This is actually observed for $d = 1$. However in $d = 2$ this is not the case. It is related to the existance of different solution of the diagramme equations in $d = 2$ which does not supply the locality of interaction [15]. The problem why in $d = 2$ KS model does not possess scaling solution as it happens with KPZ model remains open.

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[1] M. Kardar, G. Parisi and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[2] G. Sivashinsky, Acta Austronautica, **4**, 1177 (1977).
[3] Y. Kuramoto, Suppl. Prog. Theor. Phys. **64**, 346 (1978).
[4] H.W. Wyld, Ann. Phys. (N.Y.) **14**, 143 (1961).
[5] P. C. Martin, E. D. Siggia and H. A. Rose, Phys. Rev., **A8**, 423 (1973).
[6] D. Forster, D.R. Nelson and M.J. Stephen, Phys. Rev. **A16**, 732, (1977).
[7] E.Medina, T.Hwa, M.Kardar and Y.-C.Zhang, Phys.Rev. **A39** , 3053, (1989).
[8] F. Family and T. Vicsek, *Dynamics of Fractal Surfaces*, World Scientific, Singapore (1991).
[9] P. B. Wiegmann, Pis'ma Zh. Eksp. Teor. Fiz, **39**, 180 (1984) [Sov.Phys.–JETP Letters, **39**, 214, (1984)].
[10] P. B. Wiegmann, Pis'ma Zh. Eksp. Teor. Fiz, **41**, 79, (1985) [Sov. Phys.–JETP Letters, **41**, 95, (1985)].
[11] A.M.Polyakov, *Gauge Fields and Strings*, Harwood Academic Pablishers, N.Y. (1987).
[12] V.S. L'vov, V.V. Lebedev, M. Paton and I. Procaccia, Nonlinearity, submitted.
[13] V.I. Belinicher, V.S. L'vov: Sov. Phys. JETP **66**, 303 (1987).
[14] V. S. L'vov, Phys. Rep. **207**, 1, (1991).
[15] I. Procaccia, M.P. Jensen, V.S. L'vov, K. Sneppen, and R. Zeitak, Phys. Rev. A, accepted.