Exact relations in the theory of developed hydrodynamic turbulence

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Exact relations of two types in the statistical theory of fully developed homogeneous isotropic turbulence in an incompressible fluid were found. The relations of the first type connect two-point and three-point objects of the theory which are correlation functions and susceptibilities. The second types of relations are the "frequency sum rules" which express some frequency integrals from "fully dressed" many-point objects (like vertices) via corresponding bare values. Our approach is based on the Navier-Stokes equation in quasi-Lagrangian variables and on the generating functional technique for correlation functions and susceptibilities. The derivation of these relations uses *no* perturbation expansions and *no* additional assumptions. This means that the relations are *exact* in the framework of the statistical theory of turbulence. We showed that "a many-point scaling" gives birth to the "global scaling." Here "many-point scaling" is the assumption that two-point, three-point, etc. objects of the theory of turbulence are uniform functions in the inertial interval and may be characterized by some scaling exponents. Under this assumption the only global scale-invariant model of fully developed turbulence suggested by Kolmogorov [Dokl. Akad. Nauk SSSR **32**, 19 (1941)] is consistent with the exact relations deduced.

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INTRODUCTION

Exact relations play an important role in the study of such a difficult and interesting problem as hydrodynamic turbulence. They help us to formulate and to control phenomenological models and hypotheses as well as to check the validity of approximations in analytical theories of turbulence. Moreover, exact relations give a method to check physical experiments and computer simulations of hydrodynamic turbulence.

In this paper we derive two families of exact relations in the statistical theory of fully developed homogeneous turbulence of an incompressible fluid. The theory deals with many-point many-time correlation functions of velocity field and nonlinear susceptibilities (response functions of this field to a vanishing small external force). The simplest two-point objects are the double velocity correlator F and the Green's function G. The first family consists of relations between the n-point and $(n+1)$-point objects. A well-known example is the relation between the time derivative of n-point correlators and $(n+1)$-point correlators. In Sec. IV we derive some other relations of this family.

The second family consists of "frequency sum rules." These are relations between integrals over frequency (with some weight) of "dressed" many-point objects and corresponding "bare" values. An example of such a relation is $\int G(k,\omega)d\omega = i\pi$. In Sec. V we derive relations of this family for three-point objects and present a regular procedure for deriving corresponding relations for $(n>3)$-point objects. Note that these relations reflect properties of the interaction (expressed via the bare vertex for the Navier-Stokes equations) and the causality principle.

The key word *exact* in the study of hydrodynamic turbulence has various meanings. We claim that relations obtained in this paper are exact in the framework of the statistical theory of turbulence. Actually the statistical approach is based on the assumption that turbulent solutions of the Navier-Stokes equations do exist in the statistical sense, which is the existence of correlation functions and susceptibilities. For example, the Kolmogorov famous relation [1] between the third moment is exact in the same sense. Indeed, in this paper we used (see Sec. III) the functional integration approach for a statistical description of turbulence which is based on the Navier-Stokes equations and the above-mentioned assumption of existence of their statistical solutions. *No* perturbation expansions were used. *No* simplifying assumptions concerning the value or character of the interaction and *no* additional hypotheses were made.

In our approach the sweeping effect was eliminated from the very beginning with the help of quasi-Lagrangian variables [2, 3] suggested by L'vov. As a result, the external scale of turbulence (which is the size of the largest eddies, those containing the most energy) is absent (in the explicit form) in the equations of the theory and the theory is *scale invariant*. Therefore it is natural first of all to look for a scale-invariant solution of these equations. Accordingly we assumed that two- and three-point (in space and time) objects [like second velocity correlators $F(k)$, third velocity correlator

$$F_3(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3), \tag{1}$$

the Greens function, etc.] are uniform functions in the inertial interval and may be characterized by some scaling exponents. We will call such a situation *many-point scaling*. It is necessary to distinguish the many-point scaling from *local scaling* [4] in multifractal models of turbulence [5, 6]. The local scaling is in fact "two-point simultaneous scaling." This is an assumption that the simultaneous n-order moments of velocity differences $D^{(n)}(r)$ (the two-point correlation functions) are uniform functions in the inertial interval with some exponents ζ_n. Obviously two-point simultaneous scaling is a weaker assumption than many-point scaling. We showed that many-point scaling is consistent with the exact relation deduced if scaling exponents are related according to the famous Kolmogorov-Obukhov phenomenological model of turbulence (KO model [7,8]) with $\zeta_n = n/3$. In the KO concept of turbulence there is the global scaling, characterized by the only scaling exponent of velocity field $\zeta_1 = 1/3$.

Apart from a many-point scaling leading to the global scaling, one may expect solutions of greater complexity consistent with the multifractal models of turbulence [5, 6]. We cannot reject this possibility, but postpone the question about correspondence between multifractal models of turbulence and the Navier-Stokes equation to the future.

I. BASIC EQUATIONS

The modern statistical theory of hydrodynamic turbulence began with the papers by Kraichnan [9] and Wyld [10], who suggested using a space distributed force $\mathbf{f}(t, \mathbf{r})$ to simulate excitation of stationary space-homogeneous developed hydrodynamic turbulence. According to the Kolmogorov-Obukhov universality hypotheses [7, 8] in the limit of a large Reynolds number the properties of the fine-scale part of turbulence (in the inertial interval of scales) will not depend upon details of turbulence excitation, i.e., on the type of boundary conditions for the fluid flow or on characteristics of the driving force $\mathbf{f}(t, \mathbf{r})$. Therefore one can suppose that the force $\mathbf{f}(t, \mathbf{r})$ is a random force with Gaussian statistics; it does not excite the mean flow: $\langle \mathbf{f}(t, \mathbf{r}) \rangle = 0$, and its double correlator

$$\langle f_i(t, \mathbf{r}) f_j(t, \mathbf{r}') \rangle = D_{ij}(t - t', \mathbf{r} - \mathbf{r}') \tag{2}$$

depends only upon the coordinate and time difference, which is the condition for the turbulent state to be homogeneous and stationary. Thus we shall start with the *basic model of developed turbulence* which is determined by the Navier-Stokes equation with the random force (2) in an unbounded region:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \boldsymbol{\nabla})\mathbf{v} + \boldsymbol{\nabla} P = \nu \Delta \mathbf{v} + \mathbf{f}\,,$$
$$\nabla \mathbf{v} = 0\,, \quad \nabla \mathbf{f} = 0\,. \tag{3}$$

Here $\mathbf{v}(t, \mathbf{r})$ is the velocity field of an incompressible fluid, P is the pressure, ν is the kinematic viscosity, and we have set the mass density $\rho = 1$.

In the inertial interval ν and D may be taken to be zero.

A. Quasi-Lagrangian approach to theory of turbulence

The problem of developed turbulence involves two completely different interactions. The *dynamic* interaction of turbulent eddies with the characteristic scale $1/k$ (k eddies) leads to an exchange of energy between eddies and is responsible for the energy distribution among the scales. The *sweeping* interaction is simply the sweeping of small k eddies without any shape variation by the velocity of the largest eddies (of the energy containing scale L). In the inertial interval of scales ($kL \gg 1$) the sweeping interaction is substantially stronger than the dynamic one. However, the sweeping interaction does not change the energy of k eddies. In order to overcome the masking effect of sweeping it is convenient to eliminate sweeping from the very beginning by an appropriate choice of variables. To do this we shall use the *quasi-Lagrangian velocity* $\mathbf{u}(t_0, \mathbf{r}_0|t, \mathbf{r})$ (see [2, 3]):

$$\mathbf{v}(t, \mathbf{r}) = \mathbf{u}(t_0, \mathbf{r}_0 | t, \mathbf{r} - \mathbf{R})\,, \tag{4}$$

$$\mathbf{R} = \mathbf{R}(t_0, \mathbf{r}_0|t) = \int_{t_0}^{t} \mathbf{u}(t_0, \mathbf{r}_0|\tau, \mathbf{r}_0) d\tau\ . \tag{5}$$

Here the function $\mathbf{u}(t_0, \mathbf{r}_0|t, \mathbf{r})$ has additional arguments, the marked time t_0 and the coordinate of the marked point $\mathbf{r}_0$. If the velocities of at all points are uniform, the velocity $\mathbf{u}(t_0, \mathbf{r}_0|t, \mathbf{r})$ naturally coincides with the Lagrangian velocity of fluid particles along the actual trajectory $\mathbf{R}(t_0, \mathbf{r}_0|t)$. As the real velocities in some volume of size $1/k$ vary only slightly, the velocities $\mathbf{u}(t_0, \mathbf{r}_0|t, \mathbf{r})$ will differ little in this volume from Lagrangian ones. Therefore it would be reasonable to call them *quasi-Lagrangian velocities* (qL velocities).

It should be stressed that the relations (4) and (5) contain no approximations. All physical considerations reaffirm such a choice of variables as being reasonable, promising success for the theory which will make use of them. The formulas (4) and (5) represent a precise relationship between Eulerian and quasi-Lagrangian velocities. One can adequately build the theory in terms of qL velocities. They are as much a physical reality as Eulerian or Lagrangian velocities, and may be experimentally measured and effectively used in numerics.

An equation for the qL velocity may be derived by substituting (4) into the Navier-Stokes equation (3)

$$\frac{\partial u_\alpha}{\partial t} + \boldsymbol{\nabla}_\beta[(u_\beta - u_{0,\beta})(u_\alpha - u_{0,\alpha})] + \nabla_\alpha \tilde{P}$$
$$= \nu \Delta u_\alpha + \tilde{f}_\alpha\,, \quad \nabla_\alpha u_\alpha = 0\ . \tag{6}$$

Here u_α, $\tilde{P}$, and $\tilde{f}_\alpha$ are qL variables depending on t, $\mathbf{r}$ and also on t_0, $\mathbf{r}_0$, with the relations between P and $\tilde{P}$, $\mathbf{f}$ and $\tilde{\mathbf{f}}$ are similar to that between $\mathbf{v}$ and $\mathbf{u}$. The value

$u_{0,\alpha}$ in (6) is $u_\alpha(t_0, \mathbf{r}_0|t, \mathbf{r}_0)$. This equation differs from the Navier-Stokes equation in that the term $\mathbf{u}_0$ subtracts the sweeping in the marked point $\mathbf{r}_0$. Note that sweeping persists at all other points $\mathbf{r} \neq \mathbf{r}_0$.

Equation (6) depends on the coordinate $\mathbf{r}_0$ explicitly, via the last argument in $u_{0,\alpha}$. Therefore the mathematical formulation of the problem loses its spatial homogeneity. However, Eq. (6) does not contain t_0 explicitly, so homogeneity in time does remain and one can omit the index t_0 when describing a steady turbulent state.

The next important step, conventional investigations (see, e.g., [11]), is to transform to the $\mathbf{k}$ representation, that is, to expand the turbulent fluid velocity field in plane waves:

$$\mathbf{u}(\mathbf{r}_0|t, \mathbf{k}) = \int d\mathbf{r} \exp(-i\mathbf{k}\cdot\mathbf{r})\mathbf{u}(\mathbf{r}_0|t, \mathbf{r}) \ . \tag{7}$$

Such an expansion does not readily reflect the qualitative knowledge of hydrodynamic turbulence as a system of interacting localized eddies. On the other hand, it will enable us to use the detailed and powerful technique of analysis of the perturbation series in the $\mathbf{k}$ space. The equation of motion for $u_\alpha(\mathbf{r}_0|t, \mathbf{k})$ follows from (6):

$$i\left(\frac{\partial}{\partial t} + \nu k^2\right)u_\alpha(\mathbf{r}_0|t, \mathbf{k}) = \frac{1}{2}\int \frac{d^3q\, d^3p}{(2\pi)^6} V_{\alpha\beta\gamma}(\mathbf{r}_0|\mathbf{k}; \mathbf{q}, \mathbf{p}) u_\beta(\mathbf{r}_0|t, -\mathbf{q}) u_\gamma(\mathbf{r}_0|t, -\mathbf{p}) + i\tilde{f}_\alpha(\mathbf{k})\,, \tag{8}$$

$$V_{\alpha\beta\gamma}(\mathbf{r}_0|\mathbf{k}; \mathbf{q}, \mathbf{p}) = (2\pi)^3\left(k_\gamma\delta_{\alpha\beta} + k_\beta\delta_{\alpha\gamma} - 2\frac{k_\alpha k_\beta k_\gamma}{k^2}\right)[\delta(\mathbf{k}+\mathbf{q}+\mathbf{p}) - \delta(\mathbf{k}+\mathbf{q})\exp(i\mathbf{p}\cdot\mathbf{r}_0) - \delta(\mathbf{k}+\mathbf{p})\exp(i\mathbf{q}\cdot\mathbf{r}_0)] + \delta(\mathbf{k})\exp[i(\mathbf{q}+\mathbf{p})\cdot\mathbf{r}_0 r]. \tag{9}$$

The main technical difference between the quasi-Lagrangian and the conventional (in terms of Eulerian velocity) description of turbulence is that the wave vector $\mathbf{k}$ is no longer preserved in the *dynamic vertex* V since it is *not* proportional to $\delta(\mathbf{k}+\mathbf{q}+\mathbf{p})$. This is a result of the absence of spatial uniformity of the theory due to the explicit dependence of the qL velocity (5) on the coordinate of the marked point $\mathbf{r}_0$ where sweeping is precisely eliminated. This is a very high but necessary price for the elimination of the sweeping from the theory. Formally the absence of the sweeping is reflected in the property of *locality of the vertex* V in k space: in asymptotic regimes where one of the wave vectors (k, q, or p) goes to zero, the vertex V tends to zero. To make it clear we have saved the last term in (9) which really gives no contribution. Note that initial Eulerian vertex [the first term in (9)] is proportional to k but does not tend to zero if $\mathbf{q}$ or $\mathbf{p}$ goes to zero.

B. Short notations

The analytical expressions in our treatment are rather cumbersome. In order to make these more observable we introduce the following short notation: the asterisk "$*$". Appearing between two functions of $x = (t, \mathbf{r})$ it designates summation over repeated indices *plus* integration over the corresponding time-space variables. For example, in the expression

$$A_{\alpha\alpha'\beta\beta'} = B_{\alpha\alpha'\alpha''}(x, x', x'') * C_{\alpha''\beta\beta'}(x'', x_1, x_1')$$

it designates summation over α'' and integration $\int dt''\, d^3r''$. In the $(t, \mathbf{k})$ representation one has to perform integration over the time and wave vector corresponding to the repeated indices: $\int dt'' \int d^3q''/(2\pi)^3$.

Using this notation one may rewrite the qL equation of motion (8) in the inertial interval (ν and $\tilde{f}$ are omitted) as

$$i\frac{\partial}{\partial t}u_\alpha = \tfrac{1}{2}V_{\alpha\beta\gamma} * u_\beta * u_\gamma \ . \tag{10}$$

Note that vertex V here should be considered as a function of times t, t', and t'' corresponding to three indices α, β, and γ. It is clear that $V(t, t', t'') \propto \delta(t-t')\delta(t-t'')$. Comparing (8) and (10) one may formulate the following rule: wave vectors which belong to different functions and correspond to the same repeated index have opposite signs in the arguments of these functions. In this example $\mathbf{q}$, $\mathbf{p}$ are arguments of V and $-\mathbf{q}$, $-\mathbf{p}$ are arguments of u_β, u_γ. In the $(\omega, \mathbf{k})$ representation instead of integration over the times one has to use the following rule for the frequencies: each propagator depends on ω_j, each vertex V is proportional to $\delta(\omega_i + \omega_j + \omega_k)$, and one has to perform all of the integrations $\int d\omega_j/2\pi$. This is conventional for diagrammatic techniques.

II. DIAGRAMMATIC APPROACH

The diagrammatic perturbation theory suggested by Wyld [10] is a regular procedure for investigating hydrodynamic turbulence in the framework of the basic model (3). This technique was later generalized by Martin, Siggia, and Rose [12], who demonstrated that it may be used to investigate the fluctuation effects in the low-frequency dynamics of any condensed-matter system, fluid or not. In fact this technique is also a classical limit of the Keldysh diagrammatic technique [13] which is applicable to any physical system described by interacting Fermi and Bose fields. Zakharov and L'vov [14] extended the Wyld technique to the statistical description of Hamiltonian nonlinear-wave fields including hydrodynamic turbulence in the Clebsch variables [3]. The diagrammatic perturbation theory of the Wyld type for the qL equation of motion (6) was developed by Belinicher and L'vov [2].

A. Basic objects: correlators and susceptibilities

The natural objects in the Wyld diagrammatic expansion are *dressed propagators* which are the *Green's function* $G_{\alpha\beta}$ and the *double correlator* $F_{\alpha\beta}$. The former is defined as the susceptibility of the average qL velocity field u_α to a force $\phi_\beta(t,\mathbf{r})$ which would be added to the right-hand side of the equation of motion (10) (the notation ϕ was introduced to distinguish it from the external force $\tilde{f}$ which drives the turbulence). Namely, for a vanishing small force $\delta\phi$

$$G_{\alpha\beta}(\mathbf{r}_0|t,\mathbf{r}_1,\mathbf{r}_2) = -i\frac{\delta\left\langle u_\alpha(\mathbf{r}_0|t,\mathbf{r}_1)\right\rangle}{\delta\phi_\beta(0,\mathbf{r}_2)}\,. \tag{11}$$

The latter is the double correlator of the qL velocity,

$$F_{\alpha\beta}(\mathbf{r}_0|t,\mathbf{r}_1,\mathbf{r}_2) = \left\langle u_\alpha(\mathbf{r}_0|t,\mathbf{r}_1)u_\beta(\mathbf{r}_0|0,\mathbf{r}_2)\right\rangle\,. \tag{12}$$

When considering a state which is stationary in time and homogeneous in space, it is useful to pass into the $(\omega,\mathbf{k})$ representation:

$$G_{\alpha\beta}(\mathbf{r}_0|\omega,\mathbf{q}_1,\mathbf{q}_2) = \int dt d^3r_1 d^3r_2 G_{\alpha\beta}(\mathbf{r}_0|t,\mathbf{r}_1,\mathbf{r}_2) \times\exp(i\omega t - i\mathbf{q}_1\cdot\mathbf{r}_1 - i\mathbf{q}_2\cdot\mathbf{r}_2)\,. \tag{13}$$

An analogous representation for the double correlator is

$$F_{\alpha\beta}(\mathbf{r}_0|\omega,\mathbf{q}_1,\mathbf{q}_2) = \int dt d^3r_1 d^3r_2 F_{\alpha\beta}(\mathbf{r}_0|t,\mathbf{r}_1,\mathbf{r}_2) \times\exp(i\omega t - i\mathbf{q}_1\cdot\mathbf{r}_1 - i\mathbf{q}_2\cdot\mathbf{r}_2)\,. \tag{14}$$

We recall some properties of the propagators:

- The propagators (11) and (12) do not depend on coordinates $\mathbf{r}_0$, $\mathbf{r}_1$, and $\mathbf{r}_2$ separately but only on the two differences $(\mathbf{r}_1-\mathbf{r}_0)$ and $(\mathbf{r}_2-\mathbf{r}_0)$ [3]. This reflects the space homogeneity of the initial problem. It allows us to omit the index $\mathbf{r}_0$ in the all of following formulas by putting the origin in this point.
- The simultaneous double correlators of qL velocity F and Euler velocity $F_{\rm E}$ are identical [3]:

$$F_{\alpha\beta}(\mathbf{r}_0|t=0,\mathbf{r}_1,\mathbf{r}_2) = F_{{\rm E},\alpha\beta}(t=0,\mathbf{r}_1-\mathbf{r}_2)\,. \tag{15}$$

Obviously the Euler correlators in the $\mathbf{r}$ representation $F_{\rm E}$ depend only on the coordinate difference $\mathbf{r}_1-\mathbf{r}_2$. In the $\mathbf{k}$ representation it means that

$$F_{{\rm E},\alpha\beta}(\mathbf{k}_1,\mathbf{k}_2) = (2\pi)^3F_{{\rm E},\alpha\beta}(\mathbf{k}_1)\delta(\mathbf{k}_1+\mathbf{k}_2)\,.$$

It follows from (14) that the simultaneous correlator is the integral over frequency of the correlator in ω representation. As a result, one obtains an important *frequency sum rule for the double qL correlator* [2]:

$$\int\frac{d\omega}{2\pi}F_{\alpha\beta}(\mathbf{r}_0|\omega,\mathbf{k}_1,\mathbf{k}_2) = (2\pi)^3F_{\rm E}(\mathbf{k}_1)\delta(\mathbf{k}_1+\mathbf{k}_2). \tag{16}$$

B. Dyson-Wyld equations for propagators

Using the Wyld technique one may derive [10,14] a system of equations for the dressed propagators, known as the Dyson-Wyld equations. In our short notation these may be written in the inertial interval as

$$\omega G_{\alpha\beta}(\omega,\mathbf{q}_1,\mathbf{q}_2) - \Sigma_{\alpha\gamma}(\omega,\mathbf{q}_1,\mathbf{q}_3)*G_{\gamma\beta}(\omega,-\mathbf{q}_3,\mathbf{q}_2) = (2\pi)^3\delta(\mathbf{q}_1+\mathbf{q}_2)P_{\alpha\beta}(\mathbf{q}_1)\,, \tag{17}$$

$$F_{\alpha\beta}(\omega,\mathbf{q}_1,\mathbf{q}_2) = -\;G_{\alpha\gamma}(\omega,\mathbf{q}_1,-\mathbf{q}_3)*\Phi_{\gamma\delta}(\omega,\mathbf{q}_3,\mathbf{q}_4) *G_{\delta\beta}(-\omega,-\mathbf{q}_4,\mathbf{q}_2)\,. \tag{18}$$

Here $P_{\alpha\beta}(\mathbf{q}) = \delta_{\alpha\beta}-q_\alpha q_\beta/q^2$ is the transverse projector. The mass operators Σ and Φ are the *self energy* and *intrinsic noise functions*, respectively. Let us stress that the expressions (17), (18) should be considered as exact relations in which the terms Σ,Φ are supplied by the interaction.

In the framework of perturbation theory the functions Σ,Φ are given by infinite series of one-particle irreducible blocks:

$$\Sigma = \Sigma_2+\Sigma_4+\Sigma_6+\cdots\,,\qquad \Phi = \Phi_2+\Phi_4+\Phi_6+\cdots\,.$$

In these expressions Σ_{2p} is a functional of $2p$ vertices V, p double correlators F and $2p-1$ Green's functions G; Φ_{2p} is a functional of $2p$ vertices V, $p+1$ correlators V, and $2(p-1)$ functions G. The conventional way to deduce such a series may be found in [10, 12, 14, 15]. The Wyld perturbation series appears to be similar to the Feynman perturbation series in quantum electrodynamics [16].

At high frequencies the corrections provided by the interaction are small. Therefore the asymptotic expression of the Green's function for $\omega\to\infty$ will be

$$G_{\alpha\beta}(\omega,\mathbf{q}_1,\mathbf{q}_2)\to G_{0,\alpha\beta}(\omega,\mathbf{q}_1,\mathbf{q}_2)\,,$$
$$G_{0,\alpha\beta}(\omega,\mathbf{q}_1,\mathbf{q}_2) = (2\pi)^3\delta(\mathbf{q}_1+\mathbf{q}_2)\omega^{-1}P_{\alpha\beta}(\mathbf{q}_1)\,. \tag{19}$$

Here G_0 is the bare Green's function. The above relation follows from (17) by putting $\Sigma=0$ in this limit.

III. GENERATING FUNCTIONAL

In our paper we use functional integration techniques which give the same perturbation expansions as suggested in [10, 12]. We shall use the functional representation of these diagrammatic series in order to analyze them as a whole, without any truncations. We shall assume familiarity with the technique of functional integration, which in this context originates with the well-known ideas of Feynman [17]. A textbook description of functional integration methods closely related to the present problem may be found in the book by Popov [18]. In this section we explain the relation between the original Wyld expansion and the functional integral representation of the propagators. The representation of correlation functions appearing in the Wyld technique in the form of functional integrals was developed by de Dominicis [19] and Janssen [20].

A. Effective action in qL approach

Following the work by de Dominicis and Peliti [21] we may assert that the correlation functions of the solutions of (6) are generated by the following functional:

$$\mathcal{Z}(\mathbf{l},\hat{\mathbf{l}}) = \int \mathcal{D}\mathbf{u}\,\mathcal{D}\mathbf{p}\exp\Big(iI + \int dt\,d\mathbf{r}(\mathbf{l}\cdot\mathbf{u} + \hat{\mathbf{l}}\cdot\mathbf{p})\Big). \tag{20}$$

Here $\mathbf{p}$ is the auxiliary vector field conjugated to $\mathbf{u}$ ($\nabla\mathbf{p} = 0$) and the effective action is $I = I_0 + I_{\rm int}$ where

$$I_0 = \int dt\,d\mathbf{r}\Big(p_\alpha\frac{\partial u_\alpha}{\partial t} + \nu\nabla_\beta p_\alpha\nabla_\beta u_\alpha + ip_\alpha D_{\alpha\beta}p_\beta\Big), \tag{21}$$

$$I_{\rm int} = -\int dt\,d\mathbf{r}\,\nabla_\beta p_\alpha(u_\alpha - u_{0,\alpha})(u_\beta - u_{0,\beta}). \tag{22}$$

Here the integration is performed over all functions $\mathbf{u}(t,\mathbf{r})$ and $\mathbf{p}(t,\mathbf{r})$ for fixed coordinate of marked point $\mathbf{r}_0$. The variables $\mathbf{l}(t,\mathbf{r})$ and $\hat{\mathbf{l}}(t,\mathbf{r})$ are arbitrary functions of time t and vector $\mathbf{r}$. The coefficients of the expansion of $\mathcal{Z}$ in $\mathbf{l}(t,\mathbf{r})$ and $\hat{\mathbf{l}}(t,\mathbf{r})$ are the correlation functions of the fields $\mathbf{u}(\mathbf{r}_0|t,\mathbf{r})$ and $\mathbf{p}(\mathbf{r}_0|t,\mathbf{r})$.

Note that in the expression of Ref. [21] there appeared a functional determinant which may be represented in the form of an integral over auxiliary Fermi fields [22, 23]. It can be demonstrated that in the present case the determinant is equal to unity because of causality properties of the Green's functions (which will be discussed below). Therefore we shall omit the determinant.

B. Propagators

The first term in the expansion of $\mathcal{Z}$ in $\mathbf{l}(t,\mathbf{r})$, $\hat{\mathbf{l}}(t,\mathbf{r})$ has the following form:

$$\mathcal{Z}^{(2)} = \int dt_1 d\mathbf{r}_1\,dt_2 d\mathbf{r}_2[\,\tfrac{1}{2}l_\alpha(t_1,\mathbf{r}_1)\,\langle u_\alpha(\mathbf{r}_0|t_1,\mathbf{r}_1)u_\beta(\mathbf{r}_0|t_2,\mathbf{r}_2)\rangle\,l_\beta(t_2,\mathbf{r}_2) + l_\alpha(t_1,\mathbf{r}_1)\,\langle u_\alpha(\mathbf{r}_0|t_1,\mathbf{r}_1)p_\beta(\mathbf{r}_0|t_2,\mathbf{r}_2)\rangle\,\hat{l}_\beta(t_2,\mathbf{r}_2)]\,, \tag{23}$$

where the angular brackets $\langle\,\rangle$ mean an average with the weight $\exp(iI)$. Note that $\langle u_\alpha u_\beta\rangle$ is the double correlator of qL velocity $F_{\alpha\beta}$ (12), whereas

$$G_{\alpha\beta}(\mathbf{r}_0|t,\mathbf{r}_1,\mathbf{r}_2) = -\,\langle u_\alpha(\mathbf{r}_0|t,\mathbf{r}_1)p_\beta(0,\mathbf{r}_2)\rangle \tag{24}$$

is the Green's (response) function of the system. Indeed, if an additional external force ϕ_β is added to the right-hand side of Eq. (6), the effective action I would acquire a new term $-\phi_\beta p_\beta$ and therefore $\langle\mathbf{u}\rangle$ would be

$$\langle u_\alpha(\mathbf{r}_0|t_1,\mathbf{r}_1)\rangle = -i\int dt_2\,d\mathbf{r}_2\,\langle u_\alpha(\mathbf{r}_0|t_1,\mathbf{r}_1)p_\beta(t_2,\mathbf{r}_2)\rangle \times\phi_\beta(t_2,\mathbf{r}_2)\,. \tag{25}$$

The double correlator $\langle p_\alpha p_\beta\rangle$ is exactly zero and therefore does not appear in the technique.

By expanding the functional (20) with respect to the action $I_{\rm int}$ (22) and performing the Gaussian integration in all the terms one can reproduce the Wyld perturbation series for the Green's function and double correlator mentioned above. The technical details of such a procedure may be found in the book by Popov [18]. We have thus established the connection between the two approaches and are ready to make use of the second formalism.

C. Definitions

We are going to derive some relations between many-point correlation functions or between vertices. First we introduce designations for the three-point (dressed) vertices. One of the vertices Γ is connected with the following three-point correlation function:

$$\langle u_\alpha(t_1,\mathbf{r}_1)p_\beta(t_2,\mathbf{r}_2)p_\gamma(t_3,\mathbf{r}_3)\rangle = G_{\alpha\delta}(t_1-t_4,\mathbf{r}_1,\mathbf{r}_4) * \Gamma_{\delta\eta\xi}(t_4,\mathbf{r}_4,t_5,\mathbf{r}_5,t_6,\mathbf{r}_6) \tag{26}$$

$$*G_{\eta\beta}(t_5-t_2,\mathbf{r}_5,\mathbf{r}_2) * G_{\xi\gamma}(t_6-t_3,\mathbf{r}_6,\mathbf{r}_3). \tag{27}$$

Passing into Fourier representation, we get

$$\Gamma_{\alpha\beta\gamma}(t_1,\mathbf{r}_1,t_2,\mathbf{r}_2,t_3,\mathbf{r}_3) = (2\pi)^{-11}\int d\omega_1\,d\omega_2\,d\omega_3\,d^3q_1\,d^3q_2\,d^3q_3\delta(\omega_1+\omega_2+\omega_3) \times\exp(+i\omega_1t_1 - i\mathbf{q}_1\cdot\mathbf{r}_1 + i\omega_2t_2 - i\mathbf{q}_2\cdot\mathbf{r}_2 + i\omega_3t_3 - i\mathbf{q}_3\cdot\mathbf{r}_3) \times\Gamma_{\alpha\beta\gamma}(\omega_1,\mathbf{q}_1,\omega_2,\mathbf{q}_2,\omega_3,\mathbf{q}_3) \tag{28}$$

The bare value of the vertex $\Gamma_{\alpha\beta\gamma}$ is no other than the interaction vertex (9). The vertex $\Gamma_{\alpha\beta\gamma}$ may be represented as an infinite series of one-particle irreducible diagrams.

To present the definitions of the type of (27) in a more compact form we shall omit the arguments of the functions. Therefore the definition (27) will be rewritten in the following form:

$$\langle u_\alpha p_\beta p_\gamma \rangle = G_{\alpha\delta} * \Gamma_{\delta\eta\xi} * G_{\eta\beta} * G_{\xi\gamma} \; . \tag{29}$$

This expression may be considered as a definition of $\Gamma_{\delta\eta\xi}$ both in the real and in the reciprocal spaces.

In addition to the vertex $\Gamma_{\alpha\beta\gamma}$, two extra three-point vertices exist. We introduce these vertices (which we designate Λ and Y) in accordance with the following relations:

$$\langle u_\alpha u_\beta p_\gamma \rangle = G_{\alpha\delta} * G_{\beta\eta} * \Lambda_{\delta\eta\xi} * G_{\xi\gamma} - (G_{\alpha\delta} * F_{\beta\eta} * + F_{\alpha\delta} * G_{\beta\eta} *)\Gamma_{\delta\eta\xi} * G_{\xi\gamma} \, , \tag{30}$$

$$\begin{aligned}\langle u_\alpha u_\beta u_\gamma \rangle = {} & (G_{\alpha\delta} * F_{\beta\eta} * F_{\gamma\xi} * + G_{\beta\delta} * F_{\gamma\eta} * F_{\alpha\xi} * + G_{\gamma\delta} * F_{\alpha\eta} * F_{\beta\xi} *)\Gamma_{\delta\eta\xi} \\ & -(G_{\alpha\delta} * G_{\beta\eta} * F_{\gamma\xi} * + G_{\beta\delta} * G_{\gamma\eta} * F_{\alpha\xi} * + G_{\gamma\delta} * G_{\alpha\eta} * F_{\beta\xi} *)\Lambda_{\delta\eta\xi} + G_{\alpha\delta} * G_{\beta\eta} * G_{\gamma\xi} * Y_{\delta\eta\xi} \, . \end{aligned} \tag{31}$$

The above definitions are constructed so that the bare values of the vertices Λ, Y are equal to zero. They appear only as a consequence of the interaction.

IV. RELATIONS FOR TWO-POINT AND THREE-POINT OBJECTS

The representation of correlators in the form of functional integrals enables us to deduce some exact relations for the correlators. Let us illustrate the idea with a simple example. Consider the correlator

$$\left\langle \frac{\delta I}{\delta u_\alpha(t_1, \mathbf{r}_1)} u_\beta(t_2, \mathbf{r}_2) \right\rangle \, ,$$

where I is the effective action. The correlator may be written as the following functional integral:

$$\int \mathcal{D}\mathbf{u}\, \mathcal{D}\mathbf{p} \exp(iI) \frac{\delta I}{\delta u_\alpha} u_\beta \, . \tag{32}$$

It is clear that

$$\exp(iI) \frac{\delta I}{\delta u_\alpha} = -i \frac{\delta \exp(iI)}{\delta u_\alpha} \, .$$

Substituting the expression in the functional integral and performing integration by parts leads to the conclusion that the integral (32) is equal to $i\delta_{\alpha\beta}\delta(t_1 - t_2)\delta(\mathbf{r}_1 - \mathbf{r}_2)$. Therefore

$$\left\langle \frac{\delta I}{\delta u_\alpha(t_1, \mathbf{r}_1)} u_\beta(t_2, \mathbf{r}_2) \right\rangle = i\delta_{\alpha\beta}\delta(t_1 - t_2)\delta(\mathbf{r}_1 - \mathbf{r}_2) \, . \tag{33}$$

Let us clarify the essence of the deduced relation. For this purpose we substitute the effective action I by the expression $I_0 + I_{\rm int}$ where the term with the time derivative in I_0 was saved only in accordance with the above discussion. Then we find

$$\left\langle \frac{\delta I}{\delta u_\alpha} u_\beta \right\rangle = -\langle \dot{p}_\alpha u_\beta \rangle + iV_{\gamma\alpha\eta} * * \langle p_\gamma u_\eta u_\beta \rangle \, , \tag{34}$$

where $\dot{p} \equiv \partial p / \partial t$. Here (and below) the number of asterisks corresponds to the number of integrations over space-time variables. We see that the correlator on the left side of (33) is divided into two parts: a two-point block and a three-point block. The two-point block is the time derivative of the Green's function and the three-point block is in accordance with (30) reduced to a combination of propagators and triple dressed vertices. Comparing the result with the definition (17) we deduce

$$\begin{aligned}\Sigma_{\alpha\beta} = {} & V_{\alpha\gamma\delta} * G_{\gamma\xi} * * F_{\delta\eta} * \Gamma_{\xi\eta\beta} \\ & -\tfrac{1}{2} V_{\alpha\gamma\delta} * G_{\gamma\xi} * * G_{\delta\eta} * \Lambda_{\xi\eta\beta} \, . \end{aligned} \tag{35}$$

This is none other than the well-known relation of the Dyson type between the self-energy function, propagators, and dressed vertices [10, 16]. Let us stress that the relation, which is usually derived in the framework of the perturbation expansion, was derived here using the functional integration technique and *no assumptions concerning the character of interaction or features of the perturbation expansion were made.*

Similarly the following relations may be deduced:

$$\left\langle \frac{\delta I}{\delta u_\alpha(t_1, \mathbf{r}_1)} p_\beta(t_2, \mathbf{r}_2) \right\rangle = 0 \, , \tag{36}$$

$$\left\langle \frac{\delta I}{\delta p_\alpha(t_1, \mathbf{r}_1)} p_\beta(t_2, \mathbf{r}_2) \right\rangle = i\delta_{\alpha\beta}\delta(t_1 - t_2)\delta(\mathbf{r}_1 - \mathbf{r}_2) \, , \tag{37}$$

$$\left\langle \frac{\delta I}{\delta p_\alpha(t_1, \mathbf{r}_1)} u_\beta(t_2, \mathbf{r}_2) \right\rangle = 0 \, . \tag{38}$$

Equation (36) is satisfied as a consequence of causality properties of the Green's function G (see the discussion below). Equation (37) is equivalent to the relation (35). Equation (38) after the substitution $I = I_0 + I_{\rm int}$ gives

$$\langle \dot{u}_\alpha u_\beta \rangle + \frac{i}{2} V_{\alpha\eta\gamma} * * \langle u_\eta u_\gamma u_\beta \rangle = 0 \, . \tag{39}$$

This relation is trivial and immediately follows from (10). Using now (31), we may deduce the following exact expression for the internal noise function:

$$\begin{aligned}\Phi_{\alpha\beta} = {} & -\tfrac{1}{2} V_{\alpha\gamma\delta} * F_{\gamma\xi} * * F_{\delta\eta} * \Gamma_{\xi\eta\beta} \\ & +2V_{\alpha\gamma\delta} * G_{\gamma\xi} * * F_{\delta\eta} * \Lambda_{\xi\eta\beta} \\ & -V_{\alpha\gamma\delta} * G_{\gamma\xi} * * G_{\delta\eta} * Y_{\xi\eta\beta} \, , \end{aligned} \tag{40}$$

which is extracted by comparing (17), (18), (35) and the relation (39). This relation is similar to (35). Note again that the procedure of its derivation does *not* use the perturbation approach.

The above procedure may be generalized. Consider the three-point object

$$\left\langle \frac{\delta I}{\delta p_\alpha(t_1,\mathbf{r}_1)} u_\beta(t_2,\mathbf{r}_2) u_\gamma(t_3,\mathbf{r}_3) \right\rangle .$$

Performing the same integration by parts we deduce

$$\left\langle \frac{\delta I}{\delta p_\alpha} u_\beta u_\gamma \right\rangle = 0 . \tag{41}$$

Using now the substitution $I = I_0 + I_{\rm int}$, we find

$$\langle \dot u_\alpha u_\beta u_\xi \rangle + \frac{i}{2} V_{\alpha\eta\gamma} ** \langle u_\eta u_\gamma u_\beta u_\xi \rangle = 0 . \tag{42}$$

This is the relation between the three-point and the four-point objects. Other connections between the three-point and the four-point objects may be derived by varying of fields u, p and derivatives of I in the triple average of the type of (41).

It is clear that analogously relations between many-point objects may be deduced.

V. CONSEQUENCES OF ANALYTICAL PROPERTIES

In this section we intend to exploit the analytical properties of the Green's function to deduce some "sum rules." We have noted that the Green's function is the susceptibility of the system. As a consequence of the causality principle the function $G(t)$ has to be zero for $t < 0$. Therefore in the ω representation the Green's function is analytic in the upper half-plane.

The simplest of the "sum rules" has the following form:

$$\int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} G_{\alpha\beta}(\omega,\mathbf{q}_1,\mathbf{q}_2) = -\frac{i}{2}(2\pi)^3 \delta(\mathbf{q}_1+\mathbf{q}_2) P_{\alpha\beta}(\mathbf{q}_1). \tag{43}$$

To prove the relation let us deform the contour of integration over ω in the upper half-plane which is the region of analyticity of the function $G(\omega)$. Namely we shall deform the contour so that it will go far enough from the origin. Then at each point of the contour $|\omega|$ will be large and the asymptotic value of the Green's function (19) may be used. Substituting the value into the integral we arrive at the result (43).

The method of deducing (43) may be extended to more complicated cases. For this purpose nonlinear susceptibilities of the system should be considered. Namely if the external force ϕ_α is added to the right-hand side of Eq. (10) the average value $\langle \mathbf{u} \rangle$ appears, which can be represented as a series over ϕ_α:

$$\langle u_\alpha \rangle = -i \langle u_\alpha p_\beta \rangle * \phi_\beta - \tfrac{1}{2} \langle u_\alpha p_\beta p_\gamma \rangle * \phi_\beta * \phi_\gamma + \cdots . \tag{44}$$

This series generalizes the expression (25) determining the linear response of the system.

The expression enables us to assert that as a consequence of causality principle the correlator $\langle u_\alpha(t_1) p_\beta(t_2) p_\gamma(t_3) \rangle$ should be equal to zero if $t_1 < t_2$ or $t_1 < t_3$. Recalling now the definition (29) and the causality properties of the Green's function we conclude that the vertex $\Gamma_{\alpha\beta\gamma}$ possesses the same property: $\Gamma_{\alpha\beta\gamma}(t_1,t_2,t_3) = 0$, if $t_1 < t_2$ or $t_1 < t_3$. It means that the function $\Gamma_{\alpha\beta\gamma}$ in the Fourier representation determined by (28) possesses an analytical property. Namely the function

$$\Gamma_{\alpha\beta\gamma}(\omega,\omega_1,-\omega-\omega_1) \tag{45}$$

is analytic in the upper ω-half-plane.

Now we may prove the validity of the relation (36). After the substitution $I = I_0 + I_{\rm int}$ the quantity on the left-hand side of (36) reduces to a combination of the triple vertex Γ and Green's functions G (since the correlator $\langle pp \rangle$ is equal to zero). It is easy to check that as a consequence of noted properties of Γ and G the combination is equal to zero.

The analytical properties of Γ and G enable us to deduce a new sum rule. Let us consider the integral

$$\int \frac{d\omega}{\omega + i0} \Gamma_{\alpha\beta\gamma}(\omega,\omega_1,-\omega-\omega_1) . \tag{46}$$

The quantity $+i0$ in the denominator means as usual that the path of the integration should go above the pole of $(\omega + i0)^{-1}$. Since the integrand here is analytic in the upper ω-half-plane the contour of integration may be as above deformed to go far enough from the origin. Then the vertex $\Gamma_{\alpha\beta\gamma}$ may be replaced by its bare value $V_{\alpha\beta\gamma}$ for all points of the integration curve (at high frequencies the corrections to the vertex due to the interaction are negligible). Therefore the integral appears to be equal to $-i\pi V_{\alpha\beta\gamma}$. Separating the function $(\omega+i0)^{-1}$ into its real and imaginary parts we conclude

$$V_{\alpha\beta\gamma} = \Gamma_{\alpha\beta\gamma}(\omega=0,\omega_1,-\omega_1) + \frac{i}{\pi} \fint \frac{d\omega}{\omega} \Gamma_{\alpha\beta\gamma}(\omega,\omega_1,-\omega-\omega_1) . \tag{47}$$

The crossed integral designates here the principle value of the integral. This equation will be very important for us further on because it relates the dressed vertex Γ and bare vertex V. It is possible to derive similar relations for higher-order vertices.

VI. SCALING BEHAVIOR

Now we are going to find some physical consequences of the deduced exact relations. We shall examine the scaling solution for correlation functions describing the turbulence. Recall that such a solution for the Euler variable $\mathbf{v}$ cannot exist because of sweeping which in the framework of a diagram technique leads to infrared divergences. However, for qL variables a scaling solution exists which is connected with the locality of the vertex V in k space (see Sec. I) enabling excellent convergence of integrals both in infrared and in ultraviolet regions [2, 3]. Since single-time correlation functions of Euler and qL velocities coincide (see Sec. II) the dynamic scaling for

qL variables supplies the scaling behavior of single-time correlation functions of Euler velocities observed experimentally (see, e.g., [11]).

A. Scaling of two-point qL propagators

Scaling implies that the propagators for qL variables $G(t, \mathbf{r}_1 - \mathbf{r}_2)$, $F(t, \mathbf{r}_1 - \mathbf{r}_2)$ introduced by (12) and (24) are homogeneous functions

$$G(\lambda^z t, \lambda(\mathbf{r}_1 - \mathbf{r}_2)) = \lambda^{y_1 - d} G(t, (\mathbf{r}_1 - \mathbf{r}_2)) ,$$
$$F(\lambda^z t, \lambda(\mathbf{r}_1 - \mathbf{r}_2)) = \lambda^{y_2 - d} F(t, (\mathbf{r}_1 - \mathbf{r}_2)) . \quad (48)$$

Here $d = 3$ is the dimension of the space, z is the dynamic exponent, and y_1, y_2 are static exponents characterizing the scaling behavior of single-time propagators. In the following we shall use more compact notation instead of (48), (52):

$$G \sim r^{y_1 - d}, \quad F \sim r^{y_2 - d}, \quad t \sim r^z . \quad (49)$$

The convergence of all integrals in expressions containing correlation functions of qL variables means that the characteristic scales and time intervals in the integrals are of the order of external scales and time intervals. For example, the characteristic frequency ω in the integral (43) is determined by the values of $\mathbf{q}_1$, $\mathbf{q}_2$. If they lie in the inertial interval then in accordance with (49) $\omega \sim q^z$. Hence the relation (43) leads to the conclusion

$$y_1 = 0 . \quad (50)$$

As a consequence of (17), (18), and (50) we find that in the $(t, \mathbf{r})$-representation

$$\Sigma \sim r^{-d-2z}, \quad \Phi \sim r^{-d-2z+y_2} . \quad (51)$$

Note that the scaling dimensions of the two terms on the left-hand side of (17) are the same.

B. Scaling of three-point qL objects

Assume that the three-point qL objects (like Γ, Λ, and Y) are homogeneous functions of their argument in the inertial interval [similar to the case of two-point objects (48)]

$$\Gamma(\lambda^z(t_1 - t_2), \lambda^z(t_2 - t_3), \lambda(\mathbf{r}_1 - \mathbf{r}_2)\lambda(\mathbf{r}_2 - \mathbf{r}_3)) = \lambda^{y_3 - 2z - 2d} \Gamma((t_1 - t_2), \lambda^z(t_2 - t_3), (\mathbf{r}_1 - \mathbf{r}_2)(\mathbf{r}_2 - \mathbf{r}_3)) , \quad (52)$$

etc. *for any relation between the arguments.*

One may introduce also the exponents of vertices Λ, Y, taken in the $(t, \mathbf{r})$ representation

$$\Lambda \sim r^{y_4 - 2z - 2d}, \quad Y \sim r^{y_5 - 2z - 2d} . \quad (53)$$

Up to now scaling exponents of two, three, four, etc. objects one can consider as independent. We call such an assumption as *many-point scaling.*

It follows from (47) that the exponent y_3 of the dressed vertex Γ *coincides* with the exponent of the bare vertex V. It means that under the situation of many-point scaling the exponent of the interaction is *not renormalized.* Now it follows from (9) that

$$y_3 = -1 . \quad (54)$$

C. Scaling relations

Comparing the scaling dimensions of different terms of (30), (31) we conclude that

$$y_4 = y_2 - 1, \qquad y_5 = 2y_2 - 1 . \quad (55)$$

The equalities (35), (40) are now satisfied if

$$y_2 + 2z = d + 2 . \quad (56)$$

This is the first basic scaling relation. It was proved order by order in the framework of perturbation expansion [2].

The second scaling relation in the inertial interval of scales may be derived from (39). The scaling exponent of the triple correlator $\langle uuu \rangle$ may be extracted from (31), (49), (50), (53), (54), and (55):

$$\langle u\,u\,u \rangle \sim r^{2y_2 + z - 1 - 2d} . \quad (57)$$

Taken into account also (9), (10) we conclude that the relation (39) leads to the equation

$$2y_2 + z = 2 + 2d . \quad (58)$$

Comparing now (56) and (58) we find

$$z = \tfrac{2}{3}, \quad y_2 = d + \tfrac{2}{3} = \tfrac{11}{3} . \quad (59)$$

These are the Kolmogorov's value of indices.

Now using relations of the type of (39), (42) we may find the scaling indices of many-point correlators. For example,

$$\langle u(\mathbf{r}_1)\, u(\mathbf{r}_2) \cdots u(\mathbf{r}_n) \rangle \sim r^{n/3} . \quad (60)$$

This is the set of well-known relations given by Kolmogorov [7] and Obukhov [8] in their initial KO phenomenological model of turbulence. Thus the only indexes permitted by these sum rules are those of KO. The generalization of the indexes permitted by the β model is ruled out by these considerations.

CONCLUSION

We found the exact relations of two types in the statistical theory of fully developed turbulence. In principle these relations may be checked experimentally. They also allow us to make some important conclusions about the structure of statistical theory of turbulence. In particular we showed that the scaling exponent of fully dressed vertex of interaction $\Gamma_{\alpha,\beta,\gamma}$ (if it exists) is not renormalized. Together with other exact relations it leads to the Kolmogorov-Obukhov's values of exponents of moments of velocity differences, $\zeta_n = n/3$.

Existence of scaling exponent of vertex Γ means that this function is a uniform function in the inertial interval. This is a natural assumption. However, one can assume that $\Gamma_{\alpha,\beta,\gamma}$ and other three-point objects are not uniform functions at arbitrary relations between their arguments. It is not excluded that these functions are uniform only if some relations are satisfied: at $k_1 \sim k_2 \sim k_3$ they have one set of exponents, whereas at $k_1 \ll k_2 \sim k_3$, etc. they have other sets of exponents. This is the only way in analytical theory of turbulence which may lead to multifractality. We cannot reject this possibility, but postpone the question about correspondence between multifractal models of turbulence and the Navier-Stokes equation to the future.

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