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Crossover of spectral scaling in thermal turbulence

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The scaling ranges of temperature and velocity fluctuations in thermally driven turbulence are studied by analyzing the various contributions to the equations of motion. The crossover wave number k_B between Bolgiano-Obukhov and Kolmogorov-Obukhov scaling is estimated in terms of the forcings. By evaluating the thermal and buoyant stirrings and dissipations of Rayleigh-Bénard convection experiments we find k_B much larger than L^{-1}, the energy-containing scale, but smaller than $(10\eta)^{-1}$, the viscous scale. For computer simulation of randomly thermal driven turbulence we find k_B of the order of L^{-1}. This might explain why the Bolgiano-Obukhov scaling was observed in laboratory experiments whereas Kolmogorov-Obukhov scaling was found in computer simulation of thermally driven turbulence.

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I. INTRODUCTION

In recent Rayleigh-Bénard experiments [1–4] with unusually large Rayleigh numbers (Ra) up to 10^{15}, the frequency power spectra of the temperature fluctuations showed a scaling range $P_\theta(\omega) \propto \omega^{-\zeta_\theta}$ with ζ_θ near 7/5. Measurements of the velocity spectrum by photon correlation homodyne spectroscopy [5] give indication of $k^2 P_u(k) \propto k^{-\zeta_u}$ ($\zeta_u \simeq 11/5$) scaling. These findings are consistent with the so-called Bolgiano-Obukhov scaling [6,7], henceforth denoted as BO; see also [8], Sec. 21.7.

Theoretical work gives arguments that under certain assumptions BO scaling should indeed be expected [10,11] (cf. also [6–8]). On the other hand, approximate solutions of the Navier-Stokes-Boussinesq equations definitely give Kolmogorov-Obukhov (KO) scaling [12,13], i.e., $\zeta_\theta = \zeta_u = 5/3$. As was discussed in [13], an explicit introduction of plume forcing on scales as small as the boundary layer thickness makes the spectra less steep than for KO. Next, there is even an argument [14] that BO is inconsistent with global scaling in Rayleigh-Bénard flow. On the other hand, the first (and quite recent) numerical solutions of the dynamical equations on two-dimensional and three-dimensional grids seem to favor BO scaling [15], although it is not clear if one can draw conclusions from two-dimensional calculations for Rayleigh-Bénard flow, which is essentially three-dimensional, and although we do not know if the three-dimensional runs were long enough already to give reliable spectra. Thus the situation is rather confusing and deserves efforts to understand the abovementioned discrepancies and to give an explanation of the experimentally measured scaling exponents in terms of the equations of motion in addition to the work in [9–11].

We follow as closely as possible the Boussinesq equations and analyze the relative importance of their various terms. A mean-field type of argument is used to express triple correlations in terms of two-field correlation functions. This is supported by the observation that this produces a scaling behavior (28) in good agreement with numerical (approximate) solutions of the Boussinesq equations; cf. Sec. V. We also study the relative importance of the forcing terms. Volume forces are used to mimic the boundary conditions, which in experiment drive convective turbulence by large-scale wind as well as by plumes detaching from the boundary layers. Here we cannot contribute additional arguments other than the plausibility that the volume forces we use are a proper substitute for the effects of real boundaries (wind, plumes, etc.). Thus our results are under the proviso that this volume forcing is a valid assumption.

There is a crossover wave number k_B (Bolgiano scale) indicating a change in the relative importance of the nonlinear terms describing kinetic and thermal flux in comparison to the buoyant coupling $\propto \beta g$, g being the gravitational acceleration and β the thermal expansion coefficient. k_B and the corresponding length scale $l_B = k_B^{-1}$ have been introduced previously; cf. [6–8], by global dimensional arguments (comparing units). Recently L'vov and Falkovich [11] showed that the stationary spectrum of hydrodynamic thermal turbulence is defined by influxes P_S and P_E of two independent integrals of motion, entropy S, and mechanical energy E. They estimated k_B for a case of mixed excitation with energy pumping P_E as well as entropy extraction P_S caused by the environment, $k_B^4 \simeq (\beta g)^2 P_S^3 P_E^{-5}$.

In the present paper we analyze the dynamical equations and boundary conditions for the Rayleigh-Bénard convection. Emphasis is put on the role of the forcing of the thermal turbulence. In particular we estimate k_B in terms of the stirring mechanism and find it to be much larger than L^{-1}, where L is the *external* length, while in various theoretical high-Ra-flow simulations [12,13] k_B seems to be of order L^{-1}. This might explain why these simulations find KO scaling while in the boundary driven experiment BO scaling seems to prevail down to much smaller scales. k_B is found to be of the order of the inverse of the mixing layer size, or $l_B \sim l_m$.

II. DYNAMICAL EQUATIONS

The equations of motion for an incompressible ($\boldsymbol{\nabla}\cdot\mathbf{u} = 0$) fluid stirred by heating from below are commonly accepted to be the Navier-Stokes equation together with the heat equation, both in Boussinesq approximation [8,16],

$$\partial_t u_i(\mathbf{x},t) = -\mathbf{u}\cdot\boldsymbol{\nabla} u_i - \partial_i p + \beta g\theta\delta_{i3} + \nu\nabla^2 u_i + f_{u,i}\,, \tag{1}$$

$$\partial_t\theta(\mathbf{x},t) = -\mathbf{u}\cdot\boldsymbol{\nabla}\theta + \kappa\nabla^2\theta + f_\theta\,. \tag{2}$$

Here $\partial_t\ldots$ and $\partial_i\ldots$ denote $\partial\ldots/\partial t$ and $\partial\ldots/\partial x_i$, $\mathbf{u}(\mathbf{x},t) = (u_i)$ is the (Eulerian) velocity field, $\theta(\mathbf{x},t)$ the temperature deviation from a mean value T_0, ν and κ are the kinematic viscosity and thermal diffusivity, p is the kinematic pressure deviation from $\mathbf{g}\cdot\mathbf{r}$ (physical pressure divided by the constant density ρ_0), and $\mathbf{f}_u$ and f_θ represent the stirring mechanisms instead of using the proper boundary conditions. This substitution (volume forces instead of boundary conditions) again is common use; see, e.g., [9,17,18], [11–13]. But we think one has to very carefully check how the forcings can properly mimic the physical boundary conditions. We have in mind that the f's arise from the mean profiles, typically $U_0\cdot\nabla u_i$ or $\propto \theta_0$.

In the wave-number representation the set of equations, henceforth called the Boussinesq equations, reads

$$\begin{aligned}\partial_t u_i(\mathbf{k},t) = {}& -i\int d^3k_1 d^3k_2 \Gamma_{ijl}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) u_j^*(\mathbf{k}_1,t) u_l^*(\mathbf{k}_2,t)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2) + \beta g P_{i3}^{\perp}(\mathbf{k})\theta(\mathbf{k},t)\\ & -\nu k^2 u_i(\mathbf{k},t) + f_{u,i}(\mathbf{k},t)\,,\end{aligned} \tag{3}$$

$$\partial_t\theta(\mathbf{k},t) = -ik_i\int d^3k_1 d^3k_2 u_i^*(\mathbf{k}_1,t)\theta^*(\mathbf{k}_2,t)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2) - \kappa k^2\theta(\mathbf{k},t) + f_\theta(\mathbf{k},t)\,. \tag{4}$$

Γ is the interaction vertex proportional to $\mathbf{k}\,P^{\perp}$, properly symmetrized where $P^{\perp}$ is the transverse projector (see, for example, [8]). The wave-number δ functions are due to the assumed translation invariance.

III. BALANCE EQUATIONS

Following [11] we now consider the balance equations for energy and entropy corresponding to the Boussinesq equations. Consider first the kinetic energy per unit mass and volume,

$$E_{\text{kin}} = 2^{-1}\int \mathbf{u}^2(\mathbf{x},t)d^3x/V = 2^{-1}\int d^3k\, F_{uu}(\mathbf{k})\,.$$

$F_{uu}(\mathbf{k})$ is the trace of the simultaneous second-order velocity correlator after separation of the momentum δ function $\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)$. It is related to the one-dimensional $E_1(k)$ and three-dimensional $E_3(k)$ kinetic-energy spectrum by

$$E_1(k) = 4\pi k^2 E_3(k)\,, \quad E_3(k) = F_{uu}(k)/2\,.$$

Multiplying (3) by $u_i^*(k)$, adding the complex conjugate equation, and averaging over the ensemble yields the balance equation

$$\frac{\partial E_3(\mathbf{k})}{\partial t} + \frac{\partial\mathcal{E}(\mathbf{k})}{\partial\mathbf{k}} = \beta g\,\mathrm{Re}F_{u\theta}(\mathbf{k}) - \nu k^2 F_{uu}(\mathbf{k}) + P_{uu}(\mathbf{k})\,. \tag{5}$$

Here $\mathcal{E}(\mathbf{k})$ is the flux of kinetic energy in $\mathbf{k}$ space due to the nonlinearity. It can be determined using the relation

$$\begin{aligned}\frac{\partial\mathcal{E}(\mathbf{k})}{\partial\mathbf{k}} = \mathrm{Im}\int &\Gamma_{ijl}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)F_{uuu,ijl}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\\ &\times\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)d^3k_1 d^3k_2\,,\end{aligned} \tag{6}$$

as is shown, e.g., in [8]. $F_{uuu,ijl}$ is the simultaneous third-order velocity correlator. $F_{u\theta}$ in (5) denotes the velocity-temperature correlator, which describes the effect of gravity $\mathbf{g} = g\mathbf{e}_3$, and also the heat flux through the system. The last term in (5) is the trace of the velocity-velocity forcing correlator

$$P_{uu,ij}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') = \langle u_i(\mathbf{k}) f_{u,j}^*(\mathbf{k}')\rangle\,.$$

A corresponding balance equation holds for the intensity of the temperature fluctuations $S(k) = 2\pi k^2 F_{\theta\theta}(k)$, the definition of the temperature-temperature correlator being quite analogous to $F_{uu}(k)$. For $\theta \ll T_0$, $S(k)$ is proportional to the entropy spectrum [10] since

$$\int_0^\infty S(k)\,dk = 2^{-1}\int\theta^2(\mathbf{x},t)d^3x/V$$

describes (up to a factor) the entropy increase (per unit mass and volume) due to the thermal fluctuations. Following [8] and [11] we introduce therefore, in analogy to the flux of kinetic energy (6), the quantity $\mathcal{N}(\mathbf{k})$, which is proportional to the entropy flux in three-dimensional

$\mathbf{k}$ space and can be determined from

$$\frac{\partial \mathcal{N}(\mathbf{k})}{\partial \mathbf{k}} = \mathrm{Im} \int k_i F_{\theta\theta u_i}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2)\delta(\mathbf{k}+\mathbf{k}_1+\mathbf{k}_2)d^3k_1 d^3k_2$$

with the third-order correlator $\delta(\mathbf{k}-\mathbf{k}')F_{\theta\theta u_i}(\mathbf{k},\mathbf{k}_1,\mathbf{k}_2) = \langle\theta(\mathbf{k})\theta(\mathbf{k}_1)u_i(\mathbf{k}_2)\rangle$. Equation (4) implies the balance

$$\frac{1}{2}\frac{\partial F_{\theta\theta}(\mathbf{k})}{\partial t} + \frac{\partial \mathcal{N}(\mathbf{k})}{\partial \mathbf{k}} = -\kappa k^2 F_{\theta\theta}(\mathbf{k}) + P_{\theta\theta}(\mathbf{k})\,. \quad (7)$$

This equation corresponds to (5) and contains the temperature-temperature forcing correlator,

$$P_{\theta\theta}(\mathbf{k})\delta(\mathbf{k}-\mathbf{k}') = \langle\theta(\mathbf{k})f_\theta^*(\mathbf{k})\rangle\ .$$

We shall use these equations after integrating over a wave-number sphere of radius k, containing all large scales from L to k^{-1}. This d^3k integration also averages over the directions. It is assumed that the force densities $\mathbf{f}_u$ and f_θ are concentrated on the large scales of order L and that we are interested in wave numbers $k > L^{-1}$. Equations (5) and (7) imply for the stationary case

$$\varepsilon(k) = \beta g H(k) + P_{uu} - \epsilon_E(k)\,, \quad (8)$$
$$n(k) = P_{\theta\theta} - \epsilon_S(k)\,. \quad (9)$$

Here

$$\varepsilon(k) = k^2 \int \mathcal{E}(k,\Omega)d\Omega\,, \quad n(k) = k^2 \int \mathcal{N}(k,\Omega)d\Omega \quad (10)$$

are the one-dimensional kinetic energy and entropy flux,

$$H(k) = \int_0^k dk_1 \int d\Omega\, k_1^2 F_{u\theta}(k_1,\Omega) \quad (11)$$

is the turbulent heat flux through the system by motions of all eddies with wave numbers k_1 in the interval $0 < k_1 < k$. The value

$$H \equiv H(\infty) = \int d^3k_1 F_{u\theta}(\mathbf{k}_1) = \int \frac{d^3x}{V}\langle u_3(\mathbf{x})\theta(\mathbf{x})\rangle \quad (12)$$

has the physical meaning of the total heat flux, averaged over the volume. Next

$$\epsilon_E(k) = \nu k^2 \int_0^k dk_1 \int d\Omega\, k_1^2 F_{uu}(k_1,\Omega)\,, \quad (13)$$

$$\epsilon_S(k) = \kappa k^2 \int_0^k dk_1 \int d\Omega\, k_1^2 F_{\theta\theta}(k_1,\Omega)\,, \quad (14)$$

are the rates of dissipation of kinetic energy and entropy by motions with $k_1 < k$. The values

$$\epsilon_E = \epsilon_E(\infty) = \nu \int \frac{d^3x}{V}\langle\mathbf{u}\cdot\Delta\mathbf{u}\rangle\ ,$$
$$\epsilon_S = \epsilon_S(\infty) = \kappa \int \frac{d^3x}{V}\langle\theta\Delta\theta\rangle\ , \quad (15)$$

are the total dissipation rates. Since the forces are assumed to have large scales only,

$$\int^k P_{uu}(\mathbf{k}')d^3k' \equiv P_{uu} = \int \frac{d^3x}{V}\langle\mathbf{u}(\mathbf{x})\cdot\mathbf{f}_u(\mathbf{x})\rangle\ ,$$
$$\int^k P_{\theta\theta}(\mathbf{k}')d^3k' \equiv P_{\theta\theta} = \int \frac{d^3x}{V}\langle\theta(\mathbf{x})f_\theta(\mathbf{x})\rangle\ ,$$

are both independent of k. In Sec. VII we shall estimate P_{uu} and $P_{\theta\theta}$ for the Rayleigh-Bénard experiment.

The heat flux H via buoyancy serves as an input in the total-energy balance. This is closely related to the role of the potential energy $E_{\rm pot}$ whose balance we consider now:

$$E_{\rm pot} = \int \frac{d^3x}{V}\beta g z\theta(\mathbf{x})$$

is the potential energy per unit mass and volume. Its time derivative, which due to stationarity is zero, depends on $\partial_t\theta(\mathbf{x},t)$, i.e., on the dynamical equation (2),

$$\frac{dE_{\rm pot}}{dt} = 0 = \beta g \int \frac{d^3x}{V} z\partial_t\theta(\mathbf{x},t)$$
$$= \beta g \int \frac{d^3x}{V}\left[-z\nabla\cdot(\mathbf{u}\theta) + z f_\theta(\mathbf{x},t)\right]\ ,$$

$$\beta g H = P_{u\theta}\,, \quad P_{u\theta} = -\beta g \int \frac{d^3x}{V}\langle z f_\theta(\mathbf{x},t)\rangle\ . \quad (16)$$

This is an important equation relating the mean heat flux through the system to the potential-energy input $P_{u\theta}$.

Now let us consider the total balance in the system. In the limit $k\to\infty$ (k now much larger than the dissipation wave number) Eqs. (8) and (9) take the form

$$\epsilon_E = P_{uu} + P_{u\theta}\,, \quad \epsilon_S = P_{\theta\theta} \quad (17)$$

with the help of Eqs. (11)–(16). We also take into account that nonlinear terms (energy and entropy fluxes) do not contribute after integration over the whole $\mathbf{k}$ space. Note that the physical dimension of the terms in the first of Eqs. (17) is $[(\text{length})/(\text{time})]^2$ $(\text{time})^{-1}$ $=(\text{length})^2$ $(\text{time})^{-3}$ and differ from those in the second one, which is $(\text{time})^2$ $(\text{time})^{-1}$.

The thermal dissipation ϵ_S according to (17) is to be supplied by the thermal forcing $P_{\theta\theta}$ while $\epsilon_E = P_{uu}+P_{u\theta}$ dissipates the sum of input by kinetic P_{uu} *and* thermal stirring $P_{u\theta}$; the latter can be described by the forcing f_θ according to (16) or, equivalently, by the heat flux (12).

Now let us estimate the various terms in the balance equations (8) and (9). Consider first the flux $\varepsilon(k)$ in (8) which is determined by (6) and (10). Kraichnan [19] has shown in Lagrangian-history-direct-interaction approximation that the integrals over $\mathbf{k}_1$ and $\mathbf{k}_2$ in (6) converge. Later on, Belinicher, L'vov, and Falkovich [20, 21] proved that these integrals do indeed converge in *each* order of the diagrammatic perturbation theory. Therefore, when $\mathbf{k}$ is in the inertial subrange the main contribution to these integrals occurs where $k_1 \sim k_2 \sim k$. (As usual equality in the sense of *order of magnitude* is indicated by $\sim$ instead of $=$.) Thus by power counting one finds $\partial\mathcal{E}(\mathbf{k})/\partial\mathbf{k} \sim k^4 F_{uu}u(k)$. (The fourth k power originates from the k_1 in Γ, k_1^3 from d^3k_1.) Then Eq. (10) implies immediately that $\varepsilon(k) \sim k^7 F_{uu}u(k)$.

Next we consider the buoyancy contribution $H(k)$ in (8). The naive power-counting estimate for $H(k)$ leads to a wrong result because the main contribution to the integral (11) occurs for $k_1 \sim 1/L$. In order to find the k dependence of $H(k)$ let us take the complete **k**-space integral in (11) and subtract the integral over the complement of the k ball. This latter contributes mostly at the lower bound k and is evaluated by power counting. The resulting total contribution is

$$\beta g H(k) - \beta g H \sim -\beta g k^3 F_{u\theta}(k)\,.$$

Quite analogously the thermal balance is treated. The following balance equations result:

$$k^7 F_{uuu}(\mathbf{k}) \sim -\beta g k^3 F_{u\theta}(k) + P_{u\theta} + P_{uu}\,, \tag{18}$$

$$k^7 F_{u\theta\theta}(\mathbf{k}) \sim P_{\theta\theta}(k)\,. \tag{19}$$

In order to see the relative weight of the various terms in the balance equations we evaluate them in a mean-field-type style.

IV. ESTIMATE OF TRIPLE CORRELATORS AND HEAT FLUX

To determine the triple correlators in terms of the kinetic and thermal spectra, i.e., in terms of $F_{uu}(k)$ and $F_{\theta\theta}(k)$, we use the equations of motion. Multiplying (1) by $u^*(k)$ and remembering that ∂_t is of the order of the turnover frequency on scale k

$$\begin{aligned}\omega(k) &\sim k[u(x+k^{-1}) - u(x)] \\ &\sim k\sqrt{k^3 F_{uu}(k)} \sim k^{5/2}\sqrt{F_{uu}(k)}\,,\end{aligned} \tag{20}$$

we obtain as an order of magnitude estimate

$$\begin{aligned}&\omega(k) F_{uu}(k) \sim k^4 F_{uuu}(k)\,, \\ &\omega(k) F_{\theta\theta}(k) \sim k^4 F_{u\theta\theta}(k)\,.\end{aligned} \tag{21}$$

In (20) we have used the fact that it is the velocity difference which is responsible for the Lagrangean motion. It is dominated by the Fourier amplitudes in the shell k, since the larger scales are subtracted in the relative velocity and the smaller scales do not contain significant energy. From (21) and (20) the triple correlators can be estimated as

$$\begin{aligned}&F_{uuu}(k) \sim k^{-3/2} F_{uu}(k)^{3/2}\,, \\ &F_{u\theta\theta}(k) \sim k^{-3/2} F_{\theta\theta}(k)\sqrt{F_{uu}(k)}\,.\end{aligned} \tag{22}$$

Analogously, the cross correlation between θ and u_3 can be expressed in terms of the spectral power of the kinetic and thermal fluctuations. Note that a naive factorization of $F_{u\theta}(k)$ as $\sqrt{F_{uu}(k)F_{\theta\theta}(k)}$ would be wrong, as can be seen, for example, in the simple case of a passive scalar (when $\beta = 0$) where $F_{u\theta}(k) = 0$ because of symmetry [11]. βg is the coupling strength between the temperature and velocity fields. Therefore the $u_3\theta$ correlation should be proportional to some power of βg. In order to evaluate this correlator we multiply the equation (1) by $\theta(k)$ and after averaging obtain

$$\omega(k) F_{u\theta}(k) \sim k^4 F_{uu\theta}(k) + \beta g F_{\theta\theta}(k)\,. \tag{23}$$

Keeping on the right-hand-side (rhs) of (23) only the term $\propto \beta g$ but neglecting the nonlinear term and using $\omega(k)$ from (21) results in

$$F_{u\theta}(k) \sim -\beta g k^{-5/2} F_{\theta\theta}(k) F_{uu}^{-1/2}(k)\,, \tag{24}$$

up to the sign, which is discussed later. The nonlinear term in (23) can be taken into account by splitting $F_{uu}t(k)$ into double correlators analogously to (22). Apparently, this factorization is ambiguous, namely $F_{uu\theta}(k) \sim k^{-3/2} F_{uu}(k)\sqrt{F_{\theta\theta}(k)}$ or

$$F_{uu\theta}(k) \sim k^{-3/2}\sqrt{F_{uu}(k)}F_{u\theta}(k)\,. \tag{25}$$

The first alternative must, however, be ruled out as can be understood again by considering the case $\beta = 0$. $F_{uu}t(k)$ has to vanish in this case. This fact is only consistent with the second factorization (25). It is easy to see now that $k^4 F_{uu}t(k)$ in (23) is of the same order as the linear term $\omega(k)F_{u\theta}(k)$ [see Eq. (20)]. Hence the nonlinear term in the equation of motion does not invalidate (up to the sign) the estimate leading to (24). These approximations for the triple and the cross correlators were already stated by L'vov and Falkovich in [11], who argue that the sign in (24) is negative.

Of course, the considerations in this section do not constitute a proof. They are a plausible demonstration. It should therefore be pointed out that the above results find a far more stringent support by a consistent theory of fully developed convective turbulence based on a diagrammatic perturbative approach to the Boussinesq equations (1) and (2) in terms of quasi-Lagrangean variables as put forward in [22].

To get a feeling for the quality of (22) and (24) we consider the case of KO scaling. Then, $F_{uu}(k) \sim F_{\theta\theta}(k) \sim k^{-11/3}$ [8]. The cross correlator $F_{u\theta}(k)$, which is responsible for the heat flux, then behaves as $F_{u\theta}(k) \sim k^{-13/3}$, according to (24). This is steeper than $\sqrt{F_{\theta\theta}(k)F_{uu}(k)}$. It is well consistent with the approximate solution obtained numerically in [12], where instead of $-13/3$ ($= -4.33$) the exponent -4.52 was found. There is very fast isotropization by eddy decay. The main contribution to the heat current stems from the large scales. The triple correlators according to (22) decay as k^{-7} as it has to be in the case of KO scaling [8].

V. KO AND BO SPECTRA AS SOLUTIONS OF THE BALANCE EQUATIONS

Now we have expressed all terms in the balance equations by only two correlators, the kinetic one $F_{uu}(k)$ and the thermal one $F_{\theta\theta}(k)$. This allows us to compare the relative importance of the various contributions. Substituting (22) and (24) into (18) and (21) we obtain the following balance equations

$$k^{11/2}F_{uu}(k)^{3/2} \sim (\beta g)^2 k^{1/2} F_{uu}(k)^{-1/2} F_{\theta\theta}(k) + P_{u\theta} + P_{uu} \,, \tag{26}$$

$$k^{11/2}F_{\theta\theta}(k)F_{uu}(k)^{1/2} \sim P_{\theta\theta} \,. \tag{27}$$

Let us consider these balance equations (26) and (27) in more detail. There are interesting limiting cases which follow from these approximate and order-of-magnitude balance equations.

If the total thermal input is small, $P_{\theta\theta} \sim 0$, the spectral power $F_{\theta\theta}(k)$ should be small too, according to (27), so the buoyant (first) term on the rhs of (26) should be small and $k^{11/2}F_{uu}(k)^{3/2} \sim P_{uu}$. Thus there is constant energy flux, i.e., $F_{uu}(k) \sim k^{-11/3}$, and KO is found. Taking $F_{uu}(k) \sim k^{-11/3}$ we get from (27) that also $F_{\theta\theta}(k) \sim k^{-11/3}$, and from (24) that $F_{u\theta}(k) \sim k^{-13/3}$, as already discussed.

If, on the other hand, $P_{\theta\theta}$ is large, and so is $F_{\theta\theta}(k)$ according to (27), the buoyant term in (26) can dominate the velocity input $P_{u\theta} + P_{uu}$, and therefore

$$k^{11/2}F_{uu}(k)^{3/2} \sim (\beta g)^2 k^{1/2} F_{uu}(k)^{-1/2} F_{\theta\theta}(k) \,.$$

Together with (27) this leads to $F_{uu}(k) \sim k^{-21/5}$. Inserting into (27) gives $F_{\theta\theta}(k) \sim k^{-17/5}$ and finally using (24) results in $F_{u\theta}(k) \sim k^{-19/5}$. These are the BO scalings.

Note that $P_{u\theta}$ is the integral over all k of the buoyant term kept here. Therefore, the buoyant term can dominate $P_{u\theta} + P_{uu}$ only if it changes sign with k or if P_{uu} is very small or even negative, which means a loss of kinetic energy on the large scales.

It may be useful to summarize both limiting cases more completely, i.e., including the relevant prefactors. First, KO scaling,

$$\begin{aligned} F_{uu}(k) &= K_u \epsilon_E^{2/3} k^{-11/3} \,, \\ F_{\theta\theta}(k) &= K_\theta \epsilon_S \epsilon_E^{-1/3} k^{-11/3} \,, \\ F_{u\theta}(k) &= K_{u\theta} \beta g \epsilon_S \epsilon_E^{-2/3} k^{-13/3} \,. \end{aligned} \tag{28}$$

The constants K_u, K_θ, and $K_{u\theta}$ are dimensionless. We furthermore have used Eqs. (16) and (17) to express $P_{\theta\theta}$, P_{uu}, and H by the dissipation rates.

For BO scaling the correlators are

$$\begin{aligned} F_{uu}(k) &= B_u (\beta g)^{4/5} \epsilon_S^{2/5} k^{-21/5} \,, \\ F_{\theta\theta}(k) &= B_\theta (\beta g)^{-2/5} \epsilon_S^{4/5} k^{-17/5} \,, \\ F_{u\theta}(k) &= B_{u\theta} (\beta g)^{1/5} \epsilon_S^{3/5} k^{-19/5} \,. \end{aligned} \tag{29}$$

Again, B_u, B_θ, and $B_{u\theta}$ are dimensionless constants. Note that all correlators are expressed in terms of βg and ϵ_S powers. ϵ_S does not appear in BO, according to the physical idea that the last two terms on the rhs of (26) are small in comparison to the first term, representing buoyancy. KO scaling, instead, depends on ϵ_S, ϵ_S, and βg.

The very possibility of realization of KO and BO spectra depends on the signs of the terms in the balance equations (26) and (27) since the spectra are determined by the direction of the respective conserved fluxes of mechanical energy and of temperature fluctuations, which in turn are determined by the signs of the forcing terms $(P_{u\theta} + P_{uu})$ and $P_{\theta\theta}$. According to the total-balance equations (17) these signs are both positive. It corresponds to the almost trivial physical picture that the forcing introduces energy and temperature fluctuations (negative entropy) into the system which are then dissipated by viscosity and thermal conductivity.

Since $P_{\theta\theta} > 0$ the sign of the temperature fluctuation flux $n(k)$, which is the lhs of Eq. (27), is also positive. This agrees with the intuitive picture of an initially smooth temperature field produced by the large-scale force $f_\theta(\mathbf{x}, t)$ which is convoluted into increasingly smaller structures by the action of the turbulent velocity field.

The energy flux in the case of the BO spectrum, whose power-law exponent is $\zeta_u = 11/5$, is expected to be positive as the following general argument shows: There are two other known types of spectra. In both cases the direction of the energy flux is known. One is the spectrum of the velocity fluctuations in thermal equilibrium. It has the exponent $\zeta_u = -2$ and the energy flux is zero by definition. The other one is the KO spectrum. It has $\zeta_u = 5/3$ and a positive-energy flux, as it describes the flow of energy from small to large wave numbers. Let us consider now the direction of the flux as a function of the spectral exponent. ζ_u increases from -2 (thermal equilibrium), passes through 5/3 (KO), and reaches 11/5 (BO). If one assumes that the corresponding fluxes change continuously and vanish only in the thermal equilibrium, one concludes that the direction of the flux should be the same in the BO-case as it is in the KO situation, i.e., positive.

In the two limiting cases KO and BO discussed above either the forcing or the first term on the rhs of Eq. (26) is dominant. Since both terms are positive they provide an energy flux of the desired direction. Thus the negative sign of the cross correlator $F_{u\theta}(k)$, which was arbitrarily chosen in Eq. (24), is determined here by the requirement that the balance equation (26) shall describe the BO spectrum. Another argument for $F_{u\theta}(k) < 0$ in the inertial interval was given by L'vov and Falkovich in [10, 11].

The quantity $4\pi k^2 F_{u\theta}(k)$ describes the k density of the heat flux between the top and bottom plates carried by eddies of typical size $1/k$. Therefore, the above considerations lead to the conclusion that in the inertial interval heat is transported from the top to the bottom plate. For unstable stratification this leads to the remarkable situation of a countergradient heat flux from the cold to the hot plate. Nevertheless, the total heat flux $H = 4\pi \int k^2 F_{u\theta}(k)\, dk$ seems to be positive (or close to zero). Thus the function $F_{u\theta}(k)$ has to change sign at some k_* near $1/L$. So there are two regions with different behavior of $F_{u\theta}(k)$. In the pumping range ($k < k_* \simeq 1/L$) the heat flux is directed along the mean temperature gradient as is commonly expected. In the inertial subrange ($k \gg k_*$) the direction of heat flux is fully determined by the strong nonlinearities in the system and is negative. Since the mean temperature profile $\theta_0(\mathbf{x})$ does not contribute in the equations of motion for $u(\mathbf{k})$ and $\theta(\mathbf{k})$ in this k range a direction of heat flux independent of the sign of the stratification is fully compatible with the observations.

VI. CROSSOVER SCALE K_B

In the general case where a mechanical forcing as well as a thermal forcing is present a crossover from KO behavior to BO behavior can occur at a certain wave number k_B, which we discuss now.

The two limiting cases in the balance equation (26) have been that on the rhs either the first or the last two terms are dominant. Since the buoyancy term depends on k, its size varies with scale, while the input of mechanical energy (equal to ϵ_S) is independent of k. In both limiting cases, KO as well as BO, the buoyancy contribution decreases more or less steeply with k. Using (28) or (29) we obtain

$$(\beta g)^2\sqrt{\frac{k}{F_{uu}(k)}}F_{\theta\theta}(k) \sim \beta g k^3 F_{u\theta}(k) \sim \begin{cases} k^{-4/3}, & \text{KO}\,, \\ k^{-4/5}, & \text{BO}\,. \end{cases}$$

For sufficiently *small* k the buoyancy will thus always be dominant, while for *large* k it will fade away. The limiting case of BO scaling thus can show up for smaller k or larger scales, while the large-k or small-scale behavior (but still k in the inertial range) will be KO. The crossover between both cases occurs if the buoyant term is of the order of the kinetic forcing, i.e.,

$$(\beta g)^2 k^{1/2} F_{uu}(k)^{-1/2} F_{\theta\theta}(k) \sim \epsilon_E = P_{u\theta} + P_{uu}\,.$$

This determines the crossover wave number k_B. To express k_B in terms of $P_{\theta\theta}$, P_{uu}, and $P_{u\theta}$ we make use of the full balance equations. The result is

$$k_B^4 \sim (\beta g)^6 \epsilon_S^3/\epsilon_E^5 \sim (\beta g)^6 P_{\theta\theta}^3/\left(P_{uu}+P_{u\theta}\right)^5\,. \tag{30}$$

The corresponding length scale $l_B = k_B^{-1}$ reads

$$l_B \sim \epsilon_E^{5/4}\epsilon_S^{-3/4}(\beta g)^{-3/2}\,. \tag{31}$$

These estimates coincide with the expressions found from dimensional analysis [6–8], as far as the βg, ϵ_S, and ϵ_E dependence is considered. The additional use of the balance equations, however, provides via Eqs. (17) the relation with the input mechanism, represented by P_{uu}, $P_{u\theta}$, and $P_{\theta\theta}$, as was also discussed in [11]. We shall estimate k_B in Sec. VII.

Clearly, for $k < k_B$ one expects BO scaling, and for $k > k_B$ one should find KO behavior, provided k_B is located in the inertial range. This was the assumption in our estimates of the various terms in the balance equations. If k_B is of the order of L^{-1} or even less, there will be *only* KO scaling; if k_B is of the order of η^{-1} or more, there will *only* be BO scaling, followed by the viscous range with exponential decay of the correlators.

In the simulations [12,13] the rhs of (30) is of order L^{-4}, so *no* BO scaling can be expected. Experimentally [1–4], the rhs of (30) seems to be larger; therefore BO scaling might occur. We estimate the value of k_B in the Rayleigh-Bénard experiment in the following section.

VII. THE EXPERIMENTAL CROSSOVER SCALE

The most convenient possibility to estimate l_B is to express ϵ_S and ϵ_E directly in terms of the Nusselt and Rayleigh numbers. Such expressions have been derived by Shraiman and Siggia [14]:

$$\epsilon_S = \kappa\left\langle|\nabla\theta|^2\right\rangle = \kappa\Delta^2 L^{-2}\text{Nu}\,, \tag{32}$$

$$\epsilon_E = \nu\left\langle u_{i,j}^2\right\rangle = \kappa^3 L^{-4}\text{Pr}\,\text{Ra}(\text{Nu}-1)\,. \tag{33}$$

Here, Δ is the temperature difference between top and bottom plate, $\text{Pr} = \nu/\kappa$ is the Prandtl number (near 1), and Nu is the dimensionless heat flux, the Nusselt number (being large, so $\text{Nu} - 1 \approx \text{Nu}$). These useful expressions (32) and (33) can be derived exactly from the equations of motion together with the correct physical boundary conditions but without any explicit forcing, i.e., $\mathbf{f}_u = \mathbf{0}\,,\ f_\theta = 0$.

Note that ϵ_S and ϵ_E in the exact expressions (32) and (33) denote the *total* volume average of the dissipation rates, including the contribution of the boundary layers. For the estimate of l_B according to (31) we need instead the *bulk* values of ϵ_S and ϵ_E.

We assume that these bulk values are represented by the total volume averages (32) and (33) to a sufficient accuracy. One can check this by decomposing the total averages into the sum of the contributions from the bulk and the boundary layer. For ϵ_S the ratio of these two terms is estimated as

$$\frac{\epsilon_{S,\text{bulk}}}{\epsilon_{S,\text{BL}}} \simeq \frac{\kappa\left(\dfrac{\Delta_c}{10\eta}\right)^2\dfrac{L-2\lambda_{\text{BL}}}{L}}{\kappa\left(\dfrac{\Delta}{\lambda_{\text{BL}}}\right)^2\dfrac{2\lambda_{\text{BL}}}{L}} \sim \left(\frac{\Delta_c}{\Delta}\right)^2\Big/\frac{\lambda_{\text{BL}}}{L}$$

$$\propto \text{Ra}^{-2/7}\Big/\text{Ra}^{-2/7} \sim O(1)\,.$$

Here we used that the boundary layer thickness l_{BL} is of the order of the viscous length 10η. Analogously, it is assumed that also for ϵ_E the smaller bulk dissipation rate and the smaller volume of the boundary layer compensate.

If we then use Eqs. (32) and (33) to estimate the crossover length scale l_B of (31) we obtain

$$l_B/L \sim \text{Pr}^{-1/4}\text{Ra}^{-1/4}\text{Nu}^{1/2}\,. \tag{34}$$

The Ra dependence of the Nusselt number has been derived in [2] using a boundary layer together with a mixing layer theory. In [14] it was shown that one does not necessarily need the notion of a mixing layer to obtain the Nu-vs-Ra dependence,

$$\text{Nu} \propto \text{Ra}^\beta\,, \quad \beta = 0.290 \approx 2/7\,. \tag{35}$$

In the $L = 40$ cm cell with aspect ratio 1/2 the prefactor is 0.165±0.005; cf. [4]. The mixing layer thickness is characterized by the property that if the plumes have grown to this size l_m they on average lose their contact to the boundary layer and detach into the bulk of convective turbulence. The scaling theory [2] leads to

$$l_m/L \propto \text{Ra}^{-\gamma}\,, \quad \gamma = 0.146 \approx 1/7\,, \tag{36}$$

the prefactor being 2 according to [4].

Inserting (35) into (34) gives

$$l_B/L \propto \text{Ra}^{-1/4+1/7} = \text{Ra}^{-3/28}\,. \tag{37}$$

This $-3/28$ scaling of l_B with Ra was also obtained in [9] from $\epsilon_E \sim u_c^3 L^{-1}$ (with $u_c \sim \mathrm{Ra}^{3/7}\kappa L^{-1}$ according to the scaling theory [2]) and ϵ_S according to (32). The Ra dependence is rather weak, $\mathrm{Ra}^{-0.107}$. In the range $\mathrm{Ra} = 10^8$–10^{15}, the hard turbulence regime, l_B/L decreases by a factor of $1/10^{7\times 3/28} = 1/5.6$ only. Beginning with the onset of turbulence, namely, with the transition to a state with spatial decorrelation at $\mathrm{Ra} = 5 \times 10^5$, the crossover length l_B shrinks by a factor of 1/10.

Clearly, l_B is less than the external scale L, decreasing even with increasing Ra. But, clearly, also l_B is larger than the characteristic inner scale, which scales as

$$10\eta/L \propto \mathrm{Ra}^{-0.32} \propto \mathrm{Ra}^{-9/28};$$

cf. Ref. [23]; the prefactor is 50 for the Rayleigh-Bénard cell mentioned above. The same scaling exponent and a prefactor of 180 is estimated in [4]. Comparing l_B from (37) with the viscous cutoff length we obtain

$$l_B/10\eta \propto \mathrm{Ra}^{0.21},$$

increasing in the hard turbulence regime by a factor of 30. Therefore, for large Ra the scale l_B is less than L but larger than 10η,

$$10\eta \ll l_B \ll L,$$

the only proviso being the unknown constant in l_B according to (34) or (37). If it is large, l_B might still be near L, and no BO scaling would occur. If it is small, l_B might be near 10η, and only BO (but no KO) scaling is realized.

Since 10η shrinks much faster than l_B, there might be a crossover between l_B and 10η. Then $l_B < 10\eta$ for moderately large Ra and $l_B > 10\eta$ for very large Ra, so a Kolmogorov range could develop, starting from some intermediate Ra.

The width of the mixing layer l_m scales almost with the same power of Ra as l_B; compare (36) and (37). It is

$$l_B/l_m \propto \mathrm{Ra}^{1/28},$$

which is compatible with $l_B \approx l_m$. If this can be confirmed including the prefactors one would have a surprising interpretation of the mixing length even in the bulk of convective turbulence or, vice versa, another interpretation of the mixing length as the crossover scale from Bolgiano (on larger scales) to Kolmogorov (on smaller scales) behavior.

The hard-I to hard-II transition, advanced in Ref. [4] and explained by the onset of restrictions in the responsiveness of the probe due to its own boundary layer in Ref. [23], might thus obtain a new aspect. According to our discussion it could be the onset of a Kolmogorov scaling range between the Bolgiano spectrum for smaller ω and the viscous range for larger ω. This makes a new experiment with a smaller probe even more exciting, as it could clarify both, the probe effect as well as the true spectral exponent for $\omega > \omega_B$. We define ω_B, the Bolgiano frequency, using the Taylor hypothesis,

$$\omega_B = u_c/l_B \propto \kappa L^{-2}\mathrm{Ra}^{3/7+3/28} \propto \mathrm{Ra}^{15/28}.$$

Clearly, ω_B increases slower with Ra (the exponent being 0.54) than the dissipative cutoff frequency $\omega_d \propto \mathrm{Ra}^{0.78}$; see Ref. [23] or [4] for the experimental ω_d. Therefore the experimental power spectra [4] do not necessarily confirm the interpretation of the I-II transition as the onset of a Kolmogorov scaling range developing between the Bolgiano spectrum for smaller ω and the viscous range. But this might become more evident if the possible restrictions due to the probe size will be removed by diminishing the probe size from presently 200 μm to a sufficiently smaller size; cf. [23]. If one measures ω_B in units of $\kappa L^{-2}\mathrm{Ra}^{1/2}$ as in Ref. [4], one gets

$$\omega_B/\kappa L^{-2}\mathrm{Ra}^{1/2} \propto \mathrm{Ra}^{1/28}.$$

In these units thus the crossover ω_B increases only (very) slowly with Ra. In contrast, experimentally the high-frequency reduction shows a decreasing onset, in agreement with the interpretation by probe restrictions; see [23]. But, as mentioned, this probe limitation might yet hide the ω_B transition.

Let us remark, finally, that l_B can of course also be estimated from Eq. (30) using information about the forcing mechanism. First,

$$P_{u\theta} \sim \beta g L \Delta^{-1} P_{\theta\theta}, \tag{38}$$

from (16) and $\langle z f_\theta \rangle_V \sim \lambda \langle f_\theta \rangle_\lambda$, where λ is the width of the forcing zone; also $P_{\theta\theta} = \langle \theta f_\theta \rangle_V \sim \Delta \langle f_\theta \rangle_\lambda \lambda/L$. Remember $\int \langle f_\theta(\mathbf{x}) \rangle d^3x \sim \langle f_\theta \rangle_{\lambda,\mathrm{top}} + \langle f_\theta \rangle_{\lambda,\mathrm{bot}} = 0$. Assuming that (38) represents also $P_{uu} + P_{u\theta}$ one obtains from (30)

$$(k_B L)^4 \sim \beta g L^{-1} \Delta^5 P_{\theta\theta}^{-2}. \tag{39}$$

Continuing the estimate of $P_{\theta\theta}$ by $\langle f_\theta \rangle_\lambda \sim \Omega\Delta$ with the large-scale frequency $\Omega \sim UL^{-1} \sim \sqrt{\beta g \Delta l_m} L^{-1}$ we find from (39) that $(k_B L)^4 \sim L^3 \lambda^{-2} l_m^{-1}$. The relevant width of the forcing zone should be the mixing layer, so $\lambda \simeq l_m$, and therefore

$$k_B L \sim (L/l_m)^{3/4} \propto \mathrm{Ra}^{3/28},$$

as before; cf. (37).

To briefly summarize, our main conclusion is that in the Rayleigh-Bénard experiment the crossover scale l_B from Bolgiano-Obukhov scaling on the larger scales to Kolmogorov-Obukhov scaling on the smaller scales (but both well within the nonlinear range) has to be expected well *within* the inertial range, $10\eta < l_B < L$. Even more precisely, l_B behaves like the mixing layer scale l_m and might coincide with it, leading to an interesting interpretation of the length scale l_m. Our results are based on an analysis of the equations of motion and on the experimentally observed Ra dependence of the stirrings and dissipations.

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