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Correlators of Velocity Differences and Energy Dissipation as an Element in the Subcritical Scenario for Non-Kolmogorov Scaling in Turbulence.

V. S. L'vov and I. Procaccia

Department of Physics of Complex Systems and Department of Chemical Physics
The Weizmann Institute of Science - Rehovot 76100, Israel

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Abstract. – We discuss the theoretical implications of the experimental results for the cross correlations between velocity differences and dissipative fields which are reported in the companion (preceding) letter (*Europhys. Lett.*, 28 (1994) 635). The first implication is that 3d hydrodynamic turbulence has no conformal symmetry. Secondly, the experiment confirms the non-conformal scaling behaviour of such correlations as predicted by the analytical theory of the present authors. The results of the measurements lend support to the subcritical scenario that was suggested recently as an explanation of the non-Kolmogorov scaling of the structure functions in large but finite Reynolds number turbulence.

The scaling properties of high–Reynolds-number (*Re*) turbulence are usually probed by examining the structure functions of velocity differences $S_n(R)$ and the correlation functions of the dissipation field $K_{\varepsilon\varepsilon}(R)$. The former are obtained from the longitudinal part of the velocity differences $\delta u(\boldsymbol{x}+\boldsymbol{R},\boldsymbol{x}) \equiv [\boldsymbol{u}(\boldsymbol{x}+\boldsymbol{R}) - \boldsymbol{u}(\boldsymbol{x})]\cdot \boldsymbol{R}/R$,

$$S_n(R) \equiv \langle [\delta u(\boldsymbol{x}+\boldsymbol{R},\boldsymbol{x})]^n \rangle \sim R^{n\zeta_n}\,, \tag{1}$$

where ζ_n are scaling exponents. The latter is determined by the field of dissipation $\varepsilon(\boldsymbol{x}) \equiv \nu[\nabla_\alpha u_\beta(\boldsymbol{x}) + \nabla_\beta u_\alpha(\boldsymbol{x})]^2/2$, and it reads

$$K_{\varepsilon\varepsilon}(R) \equiv \langle \hat{\varepsilon}(\boldsymbol{x}+\boldsymbol{R})\hat{\varepsilon}(\boldsymbol{x})\rangle \sim R^{-\mu}\,, \tag{2}$$

where $\hat{\varepsilon}(\boldsymbol{x}) = \varepsilon(\boldsymbol{x}) - \langle\varepsilon\rangle$ and μ is another scaling exponent which is known in some quarters as the «intermittency exponent». In eqs. (1) and (2) $\langle\ldots\rangle$ stands for an ensemble average.

In the preceding letter [1] Oncley and Praskovsky report on experimental measurements of cross-correlation functions between velocity differences and the dissipation field, and in particular the quantity

$$K_1(R) = \langle [\widehat{\delta u}(\boldsymbol{x},\boldsymbol{x}+\boldsymbol{R})]^2 \hat{\varepsilon}(\boldsymbol{x})\rangle \sim R^{\beta_1}\,, \tag{3}$$

where again the symbol $\hat{}$ stands for subtracting the mean value of the quantity. In addition to the inherent interest in such correlations, it was demonstrated recently [2,3] that quantities like $K_1(R)$ play a crucial role in the theoretical understanding of scaling phenomena in turbulence. We turn now to an explanation of this role before discussing the experimental results.

It is well known that within the celebrated Kolmogorov theory (KO41) one expects that $\zeta_n = 1/3$ for all n, and it was also stated [4] that KO41 required that $\mu = 0$. Experimentally it was found repeatedly that the values of ζ_n differed from the KO41 prediction, and also the exponent μ was determined [5,6] to lie in the range of 0.25-0.3. These experimental results led to a growing doubt on the validity of the KO41 theory, and to a large number of theoretical attempts to explain the deviations from KO41. None of these attempts had a satisfactory foundation from the point of view of fluid mechanics. Recently, however, a novel scenario for non-Kolmogorov scaling was proposed on the basis of the Navier-Stokes equations [2]. This scenario is based on several findings that we summarize now in brief:

i) Renormalized perturbation theory [7] based on the Navier-Stokes equations reached the conclusion that the KO41 solution is an order-by-order scale-invariant solution for turbulence. In fact, renormalized perturbation expansions revealed no divergences either in the infra-red or in the ultraviolet when the series for $S_n(R)$ was examined. There is no formalism known to us to renormalize the scaling exponents if the diagrammatic series is convergent term by term. Moreover, the fact that the KO41 solution is a scaling solution of the Navier-Stokes equations has been demonstrated [8] recently to be a non-perturbative result.

ii) The renormalized perturbative series for correlations involving the dissipation field *do* exhibit ultraviolet logarithmic divergences [9]. Each dissipation field is responsible for the appearance of a series of «ladder» diagrams. These were resummed, resulting in the discovery that the scaling behaviour of the dissipation field involves an anomalous exponent Δ that is independent of the scaling exponents ζ_n. The correlation function $K_{\varepsilon\varepsilon}(R)$ can be expressed in terms of the exponent Δ, and at asymptotically large *Re* it scales like [2,9]

$$K_{\varepsilon\varepsilon}(R) \sim R^{-8/3+2\Delta}\,. \tag{4}$$

It was explained [2] that the numerical value of the anomalous exponent Δ is crucial for the scaling theory of turbulence. The value $\Delta = 4/3$ is a critical value for Δ. For $\Delta < 4/3$ the correlation (4) decays in space, and the renormalized theory for S_n converges term by term; the KO41 theory predicts correctly the values of the scaling exponents ζ_n for asymptotically large values of *Re*. For $\Delta \geqslant 4/3$ the KO41 predictions are no longer valid since the series for $S_n(R)$ ceases to converge order by order. The experimental evidence [5,6] is that the situation is just subcritical: the exponent μ, which is nothing but $8/3 - 2\Delta$, is in the range 0.25-0.3; thus $4/3 - \Delta$ is about 0.15 or less. This is sufficiently close to the destruction of KO41 to expect very important corrections to scaling. We refer [2] to the physics that leads to these corrections as «the subcritical scenario».

iii) The subcritical scenario works as follows [2,3]: the effect of the anomalous scaling of dissipative quantities on the scaling of the structure functions can be examined using the Navier-Stokes equations, from which one can derive the exact balance equation

$$D_n(R) = J_n(R)\,, \tag{5}$$

where

$$D_n(R) \equiv n\langle[\delta u(\boldsymbol{x}, \boldsymbol{x}')]^{n-1}[\boldsymbol{P}_l\boldsymbol{u}(\boldsymbol{x})\cdot\nabla\boldsymbol{u}(\boldsymbol{x}) - \boldsymbol{P}_l\boldsymbol{u}(\boldsymbol{x}')\cdot\nabla\boldsymbol{u}(\boldsymbol{x}')]\rangle\,, \tag{6}$$

$$J_n(R) = \nu n \langle [\delta u(\boldsymbol{x}, \boldsymbol{x}')]^{n-1} [\nabla^2 u_l(\boldsymbol{x}) - \nabla'^2 u_l(\boldsymbol{x}')] \rangle, \tag{7}$$

and $\boldsymbol{P}_l \equiv \{\boldsymbol{R}/R - [(\boldsymbol{R}/R)\cdot(1/\nabla^2)\nabla]\nabla\}$ and ν is the kinematic viscosity. It is easy to verify that $D_n(R)$ stems from the non-linear terms in the Navier-Stokes equations; it is independent of any viscous effects, and its evaluation leads to the estimate

$$D_n(R) \approx \frac{n(n-1)}{2} D_2(R) S_{n-2}(R) + n d_{n+1} \frac{S_{n+1}(R)}{R},$$

where d_n is an n-dependent coefficient. The term $J_n(R)$ arises from the dissipative terms, and one can show [2] that the major contribution to it, up to the coefficient $4n(n-1)/3$, is $B_n(R)$,

$$B_n(R) = \langle [\delta u(\boldsymbol{x}, \boldsymbol{x}')]^{n-2} \varepsilon(\boldsymbol{x}) \rangle \equiv B_n^{\mathrm{c}}(R) + B_n^{\mathrm{dec}}(R), \tag{8}$$

where

$$B_n^{\mathrm{dec}}(R) \equiv S_{n-2}(R) \langle \varepsilon \rangle, \qquad B_n^{\mathrm{c}}(R) \equiv \langle [\widehat{\delta u}(\boldsymbol{x}, \boldsymbol{x}')]^{n-2} \widehat{\varepsilon}(\boldsymbol{x}) \rangle. \tag{9}$$

The existence of $\widehat{\varepsilon}(\xi)$ in the correlation function $B_n^{\mathrm{c}}(R)$ leads to the appearance of the same anomalous exponent Δ that appears in $K_{\varepsilon\varepsilon}(R)$. The calculation in ref. [2] gave

$$B_n^{\mathrm{c}}(R) \approx C_n \frac{\nu S_n(R)}{R^2} \left(\frac{R}{\eta}\right)^{\Delta}, \tag{10}$$

where C_n is some unknown dimensionless constant, which is expected to be of $O(1)$ for $n > 2$. C_2 can be shown to vanish. Upon using (9) and (10) and the expression for D_n in the balance equation (5), we find that the structure functions for $n > 3$ are affected by the anomalous exponent as well. For example we find [2]

$$S_4(R) \sim (\langle \varepsilon \rangle R)^{4/3} \left(1 + \widetilde{C} \left(\frac{\eta}{R}\right)^{\mu/2}\right), \tag{11}$$

with $\widetilde{C}$ a constant proportional to C_3. Clearly, for R much larger than η, the correction term vanishes. However, we see now that due to the fact that $\mu/2$ is small (subcritical) we need a very large inertial range in order to reach the true KO41 scaling behaviour. For example, assuming that $\widetilde{C} = 6$ (and see below for the experimental results) and for $\mu = 0.3$ the correction term is dominant as long as $(\eta/R) > 10^{-5}$. The largest available Re in experiments [1] is about 10^8, which gives an inertial interval of less than 4 decades. The correction term remains more important than the KO41 term for the largest available values of Re. This leads to an «effective» scaling exponent for S_4 which is $4/3$-$\mu/2$. Iterating eq. (5) we can find corrections to the scaling exponents of all the structure functions $S_n(R)$ for $n > 3$, which appear of the order of the experimentally observed deviations. We stress, however, that these deviations are all subcritical and they disappear for sufficiently large Re.

From the theoretical point of view, the main uncertainty in the subcritical scenario is the value of the coefficient $\widetilde{C}$ in eq. (11), and of similar coefficients in the higher-order analogues of (11). The subcritical scenario relies on the fact that $\widetilde{C}$ is sufficiently large, and in particular non-zero. There is a real danger that $\widetilde{C}$ might be zero, since the diagrammatic series for $D_n(R)$ exhibits no ultraviolet divergences [10]. We use the balance equation to introduce anomalous scaling in $D_n(R)$, using the fact that $J_n(R)$ does have such resummable divergences. One possibility to explain why $D_n(R)$ shows no divergences is that $\widetilde{C}$ is precisely zero.

A pre-eminent reason that can lead to $\widetilde{C}$ being zero is the possible existence of conformal symmetry [11] (or another type of hidden symmetry that we are not aware of) in turbulence. The quantities $\delta u(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{R})$ and $\varepsilon(\boldsymbol{x})$ are associated with local fields that transform differently under rotation. $\delta u(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{R})$ is the integral of $\nabla_\alpha u_\beta(\boldsymbol{x})$ between the two points $\boldsymbol{x}$ and $\boldsymbol{x}+\boldsymbol{R}$, whereas $\varepsilon(\boldsymbol{x})$ is associated with $|\nabla \boldsymbol{u}|^2$. Roughly speaking, the consequence of the theory is that $\nabla_\alpha u_\beta(\boldsymbol{x})$ is dominated by the KO41 scaling $R^{-2/3}$, whereas $\varepsilon(\boldsymbol{x})$ is dominated by the anomalous scaling $R^{-\mu/2}$. The quantity $\widehat{\delta u}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{R})$ contributes to a correlation function in which there appears a factor of $R^{1/3}$ which is the KO41 behaviour. If we are interested in a quantity like $[\widehat{\delta u}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{R})]^2$, operator algebra [12] implies that it will have components on the two independent scaling fields $\widehat{\delta u}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{R})$ and $\widehat{\varepsilon}(\boldsymbol{x})$. In a correlation function it will contribute to the scaling behaviour contributions that come from each of the independent fields, *i.e.*

$$\langle [\widehat{\delta u}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{R})]^2 \ldots \rangle \sim [a_1 R^{2/3} + b_1 R^{-\mu/2}] \ldots, \tag{12}$$

where the dots ... stand for the contribution of another field.

Similarly, the field $\widehat{\varepsilon}(\boldsymbol{x})$ contributes to correlations according to

$$\langle \widehat{\varepsilon}(\boldsymbol{x}+\boldsymbol{R}) \ldots \rangle \sim [b_2 R^{-\mu/2} + a_2 R^{-4/3}] \ldots . \tag{13}$$

The second term in (12) stems from the fact that $[\widehat{\delta u}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{R})]^2$ has a scalar part that has a projection onto the anomalous scalar primary field $\widehat{\varepsilon}(\boldsymbol{x})$. The second term in (13) stems from the projection on the KO41 secondary scalar field which is constructed from the KO41 primary field, and it scales like $[\widehat{\delta u}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{R})]^2/R^2$. Physically this field originates from the contribution of eddies of scale R to the $K_{\varepsilon\varepsilon}(R)$ correlation.

Consider now a correlation like

$$B_4^c(R) = \langle [\widehat{\delta u}(\boldsymbol{x}, \boldsymbol{x}+\boldsymbol{R})]^2 \widehat{\varepsilon}(\boldsymbol{x}) \rangle . \tag{14}$$

If the problem possessed conformal symmetry, there could be no correlations between different primary fields. In that case $B_4^c(R)$ could have no terms proportional to $a_1 b_2$ or $a_2 b_1$. The only allowed contributions would be

$$B_4^c(R) \sim b_1 b_2 R^{-\mu} + a_1 a_2 R^{-2/3}, \qquad \text{conformal symmetry.} \tag{15}$$

If there is no conformal symmetry, the leading contribution would be from the cross terms,

$$B_4^c(R) \sim a_1 b_2 R^{2/3-\mu/2}, \qquad \text{no conformal symmetry.} \tag{16}$$

Thus, there is a large qualitative difference between these two predictions. If conformal symmetry exists the quantity $B_4^c(R)$ decays with R. If there is no conformal symmetry, the quantity increases with R. Looking at fig. 1 of the preceding, companion paper, one sees immediately that the prediction (15) is untenable, whereas (16) is possible. Thus the experiment indicates the important conclusion that *there is no conformal symmetry in* 3d *turbulence.*

Once we establish that there is no conformal symmetry (or any other hidden symmetry that causes the coefficients C_n to vanish), we can turn to a quantitative analysis of the exponents. To this end, examine eq. (10) for $n = 4$. After substituting (11), we find

$$B_4^c(R) = C_4 \langle \varepsilon \rangle^{5/3} R^{2/3} \left(\frac{\eta}{R} \right)^{\mu/2} \left[1 + \widetilde{C} \left(\frac{\eta}{R} \right)^{\mu/2} \right]. \tag{17}$$

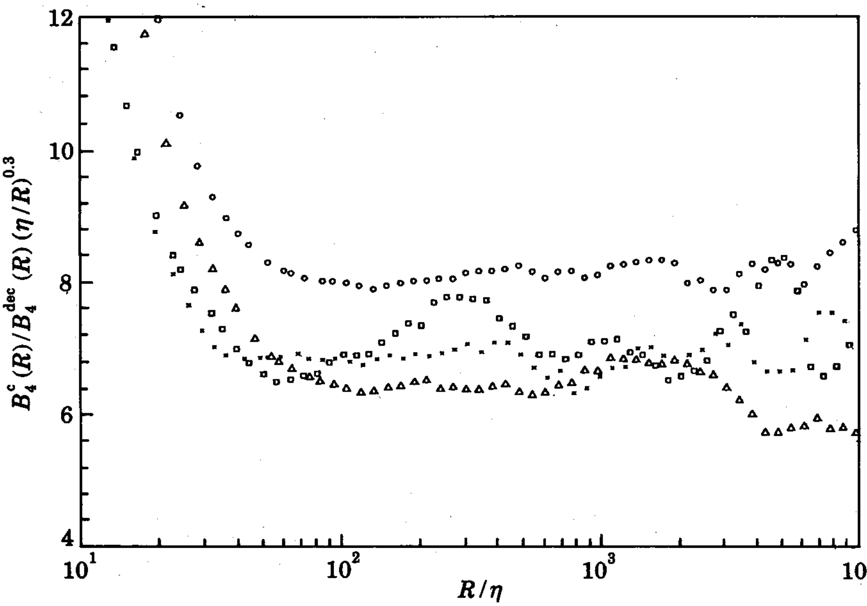

Fig. 1. – The correlator between the square of the velocity difference and the energy dissipation $B_4^c(R)$ (eq. (14)), normalized by $B_4^{dec}(R)$ (eq. (9)) and divided by $(\eta/R)^{0.3}$, as a function of (R/η), as measured by Praskovsky and Oncley [1]. × RC, $R_\lambda = 3200$; □ ASL-3, $R_\lambda = 6900$; △ ASL-5, $R_\lambda = 9200$; ○ ASL-6, $R_\lambda = 12\,700$.

Note that in the asymptotic limit $\eta/R \to 0$ eqs. (16) and (17) are equivalent. In intermediate asymptotics we expect the second term in (17) to be dominant. This leads to the prediction that

$$B_4^c(R) = bB_4^{dec}(R)(\eta/R)^\mu \sim R^{2/3-\mu}\,, \tag{18}$$

with b being a dimensionless constant. In the preceding letter [1] the exponent of $K_1(R) \equiv B_4^c(R)$ is called β_1, and is estimated from the local slopes of fig. 1, and shown in table I. The experimental conclusion is that $\beta_1 \approx 0.38 \pm 0.06$. Comparing with (18) we see that this would agree with a value of $\mu \approx 0.25 \pm 0.05$ which is in accordance with the direct measurement of this exponent via $K_{\varepsilon\varepsilon}(R)$.

Finally, we can estimate the value of the coefficient appearing in $B_4^c(R)$. In fig. 1 we show the data measured by Praskovsky and Oncley [1] for the quantity $B_4^c(R)/B_4^{dec}(R)(\eta/R)^{0.30}$ as a function of R. We see almost 2 decades of scales for which b is a constant between 6 and 8. From the theoretical point of view the finding that this coefficient is non-zero leaves us with the issue as to why the direct diagrammatic expansions for $S_n(R)$ and $D_n(R)$ seem to have no divergences leading directly to anomalous corrections to scaling. One way to explain this follows from the fact that the balance equation is not an order-by-order relation between D_n and J_n. Thus, since in J_n there are divergences term by term, we can understand it in only two ways: i) The sum of terms in D_n is divergent even though every term is convergent (which is possible for an asymptotic series without a small parameter), ii) there exists yet another primary field that we did not identify yet which causes divergences term by term. Understanding this issue remains an important challenge for the theory of hydrodynamic turbulence.

In summary, we have considered a cross-correlation function of a new type, and found that its experimentally measured [1] scaling behaviour conforms with eq. (17) which underlies the subcritical scenario. The quantity is consistent with a non-zero value of $\tilde{C}$, which is an important ingredient in the subcritical scenario for the apparent scaling exponents of the structure functions. These results can be considered as strong support for the subcritical scenario. Notwithstanding, additional experimental and theoretical studies must be accomplished before we can state that we understand the scaling properties of the small-scale structure of turbulence in satisfactory detail.

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