

## Direct Numerical Simulations of the Kraichnan Model: Scaling Exponents and Fusion Rules

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We present results from direct numerical simulations of the Kraichnan model for passive scalar advection by a rapidly varying random scaling velocity field for intermediate values of the velocity scaling exponent. These results are compared with the scaling exponents predicted for this model by Kraichnan. Further, we test the recently proposed fusion rules which govern the scaling properties of multipoint correlations, and present results on the linearity of the conditional statistics of the Laplacian operator on the scalar field. [S0031-9007(97)04512-2]

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As one of the simplest realizations of a model with turbulent statistics with nontrivial scaling exponents, the Kraichnan model [1] of advection by a white-in-time scaling velocity field has attracted much recent attention [2–6]. The model is analytically tractable, in the sense that its statistical description may be reduced to a set of closed form differential equations for the  $n$ -order correlation functions. The model concerns the equation of motion for a passively advected scalar field  $T$  driven by a velocity field  $\mathbf{u}$ :

$$\frac{\partial}{\partial t} T(\mathbf{x}, t) + \mathbf{u}(\mathbf{x}, t) \cdot \nabla T(\mathbf{x}, t) = \kappa \nabla^2 T(\mathbf{x}, t) + f(\mathbf{x}, t), \quad (1)$$

where  $\kappa$  is the molecular diffusivity. The velocity field is taken to be a Gaussian, white-in-time, incompressible homogeneous scaling random field. Statistical stationarity is achieved through the forcing  $f$ , which is also taken to be delta correlated in time, statistically homogeneous and isotropic, and to exhibit only large-scale spatial components. The parameter of interest in this model is the scaling exponent  $\zeta_h$  characterizing the so-called eddy diffusivity tensor  $h_{ij}(\mathbf{R})$  which contains the relevant information about the random velocity field  $\mathbf{u}(\mathbf{r}, t)$ :

$$h_{ij}(\mathbf{R}) \equiv \int_0^\infty d\tau \langle [u_i(\mathbf{r} + \mathbf{R}, t + \tau) - u_i(\mathbf{r}, t + \tau)] \times [u_j(\mathbf{r} + \mathbf{R}, t) - u_j(\mathbf{r}, t)] \rangle. \quad (2)$$

The notation  $\langle \dots \rangle$  refers to ensemble averaging. Under the conditions that the velocity field exhibits fast temporal decorrelation, scaling, and incompressibility,  $h_{ij}(\mathbf{R})$  takes the  $d$ -dimensional form [1]

$$h_{ij}(\mathbf{R}) = h(R) \left[ \frac{\zeta_h + d - 1}{d - 1} \delta_{ij} - \frac{\zeta_h}{d - 1} \frac{R_i R_j}{R^2} \right], \quad (3)$$

$$h(R) = H\left(\frac{R}{\mathcal{L}}\right)^{\zeta_h}, \quad 0 < \zeta_h < 2. \quad (4)$$

In the last equation the scaling of  $h(R)$  is expressed normalized with respect to  $\mathcal{L}$ , the outer scale of the

velocity field. For physically realizable fields  $\zeta_h$  may vary between 0 and 2.

Our aim is to express the statistical properties of the scalar field in terms of the parameter  $\zeta_h$ . The statistics is characterized by the  $n$ -point correlators, defined as

$$\mathcal{F}_n(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n) \equiv \left\langle \prod_{i=1}^n T(\mathbf{r}_i) \right\rangle. \quad (5)$$

One expects the correlators to be homogeneous functions of their arguments,  $\mathcal{F}_n(\lambda \mathbf{r}_1, \lambda \mathbf{r}_2, \dots, \lambda \mathbf{r}_n) \sim \lambda^{\zeta_n} \mathcal{F}_n(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_n)$ , and one hopes to determine the dependence of the scaling exponents  $\zeta_n$  on  $\zeta_h$ . In this model, the rapid temporal decorrelation of the velocity allows one to derive a set of closed equations for these correlation functions [1]:

$$\left[ -\kappa \sum_{\alpha} \nabla_{\alpha}^2 + \sum_{\alpha > \beta} h_{ij}(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}) \frac{\partial^2}{\partial r_{\alpha,i} \partial r_{\beta,j}} \right] \mathcal{F}_{2n} = \sum_{\alpha > \beta} \Phi_0(\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}) \mathcal{F}_{2n-2}, \quad (6)$$

where  $\mathcal{F}_{2n}$  is a function of the  $2n$  variables  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2n}$  and  $\mathcal{F}_{2n-2}$  is a function of the  $2n - 2$  variables  $\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{2n}$  except for  $\mathbf{r}_{\alpha}$  and  $\mathbf{r}_{\beta}$ .  $\Phi_0$  is the forcing correlation and may be eliminated using the two-point equation. Only the  $2n$ th order moments are considered as by isotropy odd moments vanish. For  $n = 1$  these equations are readily solvable, leading to the exact result

$$\zeta_2 = 2 - \zeta_h. \quad (7)$$

For  $n \geq 2$  the equations are difficult to solve analytically for arbitrary values of  $\zeta_h$ , and to date only certain limits have been treated. The limit of  $\kappa \rightarrow 0$ ,  $\zeta_h \rightarrow 0$  (in that order) has been examined perturbatively in [4]. This limit is not realizable in direct numerical simulations due to numerical instabilities caused by small diffusivities; moreover fields with scaling exponents approaching zero become increasingly spatially rough and are very difficult to produce and treat reliably numerically. In [5] the perturbative small parameter was  $\zeta_h/d$ , with  $d$  the spatial dimension, which requires either the difficult  $\zeta_h \rightarrow 0$  limit

or the numerically inaccessible case of large dimension. The regime of  $\zeta_h \rightarrow 2$  has also been treated perturbatively in [7]. The only theory which treats the intermediate span of physical fields requires a closure that is not rigorous [3,6], and it is with the prediction arising from this theory that we will be able to make a comparison. Further we test in detail some of the more general scaling predictions afforded by the fusion rules for fluid dynamics developed in [8] and the particular statistical assumptions with respect to conditional statistics utilized in the theory of [3,6] in obtaining predictions for the scaling exponents.

The crucial assumption arises in the context of the equation for the  $n$ th order structure functions, defined

$$S_n(R) = \langle [T(\mathbf{x} + \mathbf{R}) - T(\mathbf{x})]^n \rangle: \\ R^{1-d} \frac{\partial}{\partial R} R^{d-1} h(R) \frac{\partial}{\partial R} S_{2n}(R) = J_{2n}(R). \quad (8)$$

The function  $J_{2n}(R)$  derives from the dissipative term and is given by

$$J_{2n}(R) = \kappa \langle \nabla^2 T(\mathbf{x}) [\delta_R T(\mathbf{x})]^{2n-1} \rangle, \quad (9)$$

where  $\delta_R T(\mathbf{x}) \equiv T(\mathbf{x} + \mathbf{R}) - T(\mathbf{x})$ . One may determine directly that  $J_2(R) = 4\bar{\epsilon}$ , the mean dissipation (independent of  $R$ ).

In order to obtain the scaling exponents  $\zeta_n$  of the  $n$ th order structure functions, one needs to evaluate  $J_{2n}(R)$ . In light of (4) and the exact result (7) one sees that  $J_{2n}$  must have a scaling form which agrees with

$$J_{2n}(R) = n C_{2n} J_2 S_{2n}(R) / S_2(R) \quad \text{for } n > 1. \quad (10)$$

This result can be derived without reference to (8) using the fusion rules derived in [6]. In either way the coefficients  $C_{2n}$  are undetermined. Kraichnan proposed [3] that  $C_{2n} = 1$  for all  $n$ . In this case one obtains from (8) a quadratic equation determining the  $\zeta_n$ s:

$$\zeta_{2n} = \frac{1}{2} [\zeta_2 - d + \sqrt{(\zeta_2 + d)^2 + 4\zeta_2(n-1)}]. \quad (11)$$

As has been pointed out in [3] this assumption bears a strong relation to the conditional statistics of the Laplacian of the field. One may rewrite  $J_{2n}(R)$  in terms of the average of the Laplacian conditioned on the value of a difference of  $T$  across the length scale  $R$ ,  $\delta_R T(\mathbf{x})$ :

$$J_{2n}(R) = -2n\kappa \int d\delta_R T P(\delta_R T) [\delta_R T]^{2n-1} \\ \times \langle \nabla^2 T(\mathbf{x}) | \delta_R T(\mathbf{x}) \rangle. \quad (12)$$

One way to ensure that  $J_{2n}(R)$  has the scaling (10) is for the conditional average to satisfy

$$\langle \nabla^2 T(\mathbf{x}) | \delta_R T(\mathbf{x}) \rangle = C\bar{\epsilon} \delta_R T(\mathbf{x}) / \kappa S_2(R). \quad (13)$$

Hence a linear behavior of the conditional average of the Laplacian is intimately connected with the determination of the scaling exponents.

The model has been studied by direct numerical simulations in [9] with  $\zeta_h = 1.5$ . These simulations have been criticized for the method of generation of the velocity field; two fixed scaling fields were swept past each other in orthogonal directions at a constant rate. In doing so one may lose isotropy in a way that can influence the apparent numerical values of the measured exponents. In our simulations we have evolved a scalar field in two dimensions on a  $1024^2$  grid. The scaling velocity field was implemented by Fourier transforming a set of  $k$ -vector coefficients which were each chosen randomly from a Gaussian distribution scaled to a standard deviation proportional to  $k^{-1-\zeta_h/2}$ . The direction of the  $k$ th component  $\mathbf{u}_k$  was chosen such that  $\mathbf{k} \cdot \mathbf{u}_k = 0$ . To reduce computation we have used an isotropized version of the method employed in [9]; namely, we generate two fixed realizations and shift them with respect to one another in order to obtain rapid variation. At each time step the two fields are independently shifted by a step of random size and *direction*. The fields are renewed after around every 500 time steps to reduce any temporal correlation that this method might induce. We checked that the results are insensitive to a more frequent refreshment of these fields. The spatial discretization is second order, and the time evolution was performed using an explicit Euler scheme. The forcing was implemented by stimulating at every time step one of the nine smallest wave numbers with an amplitude chosen from a Gaussian distribution. Our initial conditions for the scalar field (for a given value of  $\zeta_h$ ) were Gaussian random with the 2nd order scaling exponent distinct from the expected result of  $2 - \zeta_h$ , and truncated in  $k$  space. Typically, saturation to statistical steady state required about thirty million time steps on the CRAY J90. We have converged results for three values of  $\zeta_h$ , i.e., 0.6, 1.0, and 1.2. The diffusivity in every run was chosen to obtain the longest possible inertial range while retaining stability in the small scales.

In Fig. 1 we present a typical realization of the scalar field for  $\zeta_h = 1.0$ . It shows significant development of small scale structures. In Fig. 2 we present the structure functions  $S_n(R)$  as a function of  $R$  for the three values of  $\zeta_h$ , computed using spatial averaging over single realizations after statistical stationarity was reached, and then time averaging over one hundred snapshots taken at intervals of ten thousand time steps. This figure shows that we have one and a half decades of scaling, or "inertial range."

Figure 3 displays the dependence of  $\zeta_n$  on  $n$  for the three values of  $\zeta_h$ . Also shown is the prediction of Kraichnan Eq. (11) for these values. It is evident that for the three parameter values tested we have close agreement. In Fig. 3 we display also the odd values for the exponents. These were calculated from the field by taking absolute values; strictly this is not covered by the theory but one sees here that they smoothly interpolate the law for the even orders. We remark that although the grid is relatively small the structure functions display



FIG. 1. Typical realization of the scalar field.

well-developed scaling ranges for orders as high as 12. The relatively good statistics resulted from averaging over many snapshots. We checked however that also the single-time realizations appear to be well self-averaged.

Note that for  $\zeta_h = 1.0$  the agreement between the numerically computed value of  $\zeta_2$  and Eq. (7) is best. We believe that the reason for this is simply due to the difficulty of creating a velocity field with precise scaling on a finite grid. It is interesting that in fact the scaling in the passive scalar field appears cleaner than that which can be obtained by the Fourier transform method described above in grids of this size. If we check our apparent real space

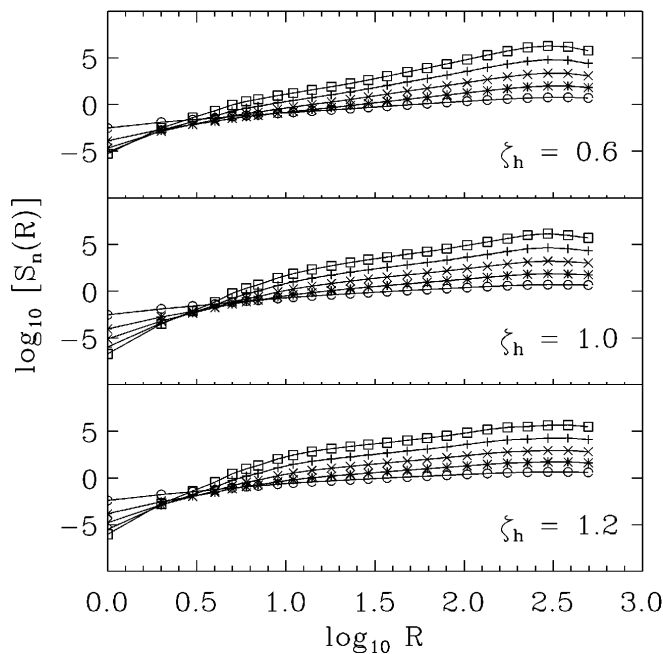


FIG. 2. Log-log plot of the structure functions  $S_n(R)$  as a function of  $R$  for  $n = 2, 4, 6, 8, 10$ .

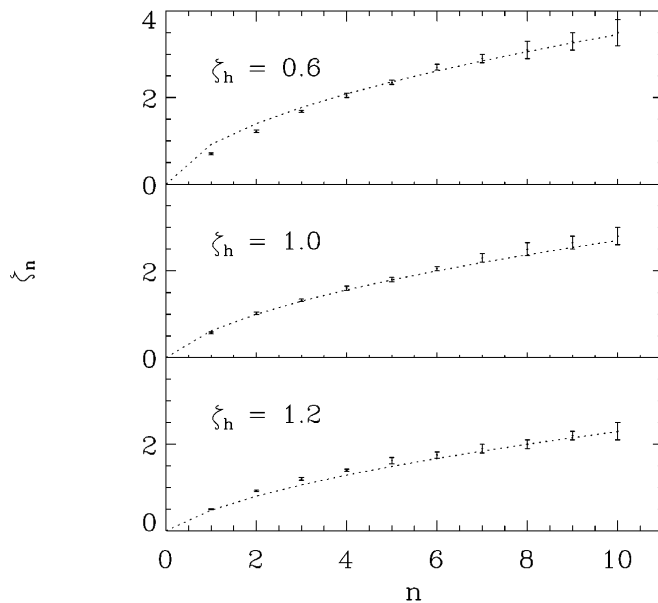


FIG. 3. The scaling exponents  $\zeta_n$  as a function of  $n = 2-10$  for three values of  $\zeta_h$ . The numerical data (error bars) are compared to the analytic prediction Eq. (11) (dotted line).

scaling exponents for the velocity field we find that the minimum error between the input  $\zeta_h$  in  $k$  space and the observed one occurs precisely at  $\zeta_h = 1$ . However the higher order scaling exponents do not seem to be as sensitive to this discrepancy.

The quality of the prediction (11) can be independently tested by verifying that the coefficients  $C_n$  are close to unity, and that the conditional average (13) is indeed linear with the right  $R$ -dependent prefactor. To this end we computed from the simulation the quantities  $J_n(R)$  of Eq. (9).  $J_2$  was confirmed to be constant throughout the inertial range. In Fig. 4 we present  $J_n(R)$  as a function of

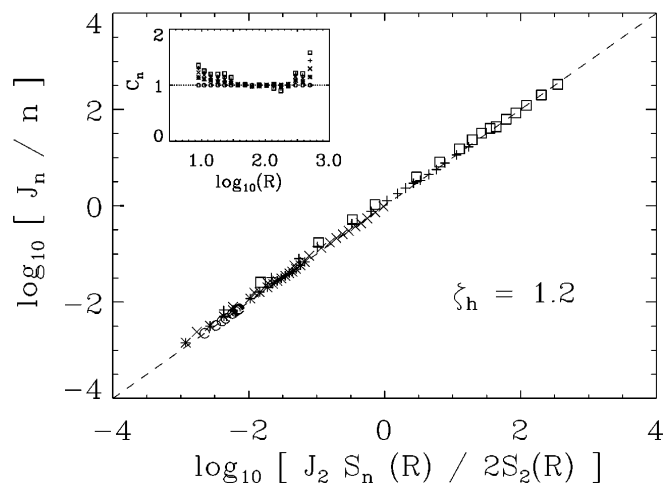


FIG. 4.  $J_n(R)$  as a function of the fusion rule prediction  $nJ_2 S_n(R)/2S_2(R)$  with  $C_n = 1$  for  $\zeta_h = 1.2$ . An independent measurement of  $C_n$  is exhibited in the inset. The other values of  $\zeta_h$  show equivalently good agreement.

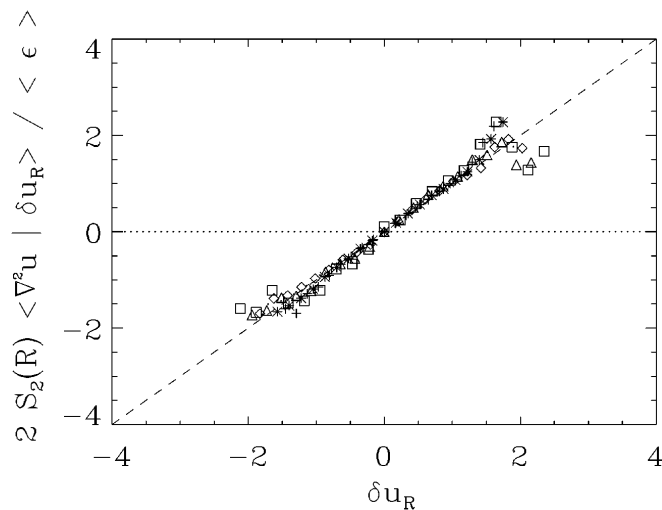


FIG. 5. Conditional averages normalized by the scaling of Eq. (13) calculated for the field with  $\zeta_h = 1.0$ , and from a single realization. Equally satisfactory results were obtained for the two other values of  $\zeta_h$ .

$nJ_2S_n(R)/2S_2(R)$  for  $n = 2, 4, 6, 8, 10$  and inertial range  $R$ . The dashed line is the line  $y = x$ , and we see that it passes through the data without any adjusted parameter. The coefficients  $C_n$  were obtained from the data for a range of values of  $R$  and  $n$ , and were found to be very close to unity in the inertial range; see inset in Fig. 4.

Finally we can check the postulated linearity of the conditional average (13). These quantities were calculated for a range of  $R$  values in the inertial range by averaging over several directions of  $\mathbf{R}$ . The results are displayed in Fig. 5. The range displayed has been truncated to that for which sufficiently stable statistics were obtained.

Our conclusions from these simulations are that the postulates that lead to the prediction (11) for the scaling exponents (i.e., linear conditional averages,  $C_{2n} = 1$ ) are

very well supported by the numerical data. As a result it is no surprise that the measured scaling exponents agree very closely with their predicted values. Because of the limitations of the computational techniques one cannot of course state that precise agreement is observed. It is our conviction however that the conditional average is very close to being linear; a persistent failure to prove the linearity mathematically may indicate that this property is not exact. It seems however very worthwhile to probe this question further to understand the close agreement between simulations and (11).

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