Universal Scaling Exponents in Shell Models of Turbulence: Viscous Effects Are Finite-Sized Corrections to Scaling

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In a series of recent works it was proposed that shell models of turbulence exhibit inertial range scaling exponents that depend on the nature of the dissipative mechanism. If true, and if one could imply a similar phenomenon to Navier-Stokes turbulence, this finding would cast strong doubts on the universality of scaling in turbulence. In this Letter we propose that these "nonuniversalities" are just corrections to scaling that disappear when the Reynolds number goes to infinity. [S0031-9007(98)06693-9]

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The aim of this Letter is to question a growing consensus [1–4] concerning the nonuniversality of the scaling exponents that characterize correlation functions that appear in the context of numerical studies of shell models of turbulence. Shell models, and in particular the so-called Gledzer-Ohkitani-Yamada (GOY) model [5,6], became very popular due to their ease of simulation and the fact that they appear to exhibit anomalous scaling that is strongly reminiscent of the behavior of the scaling exponents of Navier-Stokes turbulence [7–9]. The GOY model is a simplified reduced wave number analog to the spectral Navier-Stokes equations. The wave numbers are represented as shells, each of which is defined by a (real) wave number $k_n \equiv k_0\lambda^n$ where λ is the "shell spacing." There are N degrees of freedom where N is the number of shells. The model specifies the dynamics of the "velocity" u_n which is considered a complex number, $n = 1,\ldots,N$. The GOY model reads

$$\left[\frac{d}{dt} + \nu k_n^{2\alpha}\right]u_n = f_n + i\left[k_{n+1}u_{n+1}u_{n+2} - \frac{k_n}{2}u_{n+1}u_{n-1} - \frac{k_{n-1}}{2}u_{n-1}u_{n-2}\right]^*. \quad (1)$$

In this equation $*$ stands for complex conjugation, ν is the "viscosity," and to motivate the discussion we already specified a one parameter family of models in which the exponent α determines the type of viscous dissipation. In the following we use $\lambda = 2$ and a forcing f_n restricted to the first two shells.

The scaling exponents can be defined using the "correlation functions" $\langle|u_n|^p\rangle$. It was shown, however, that the GOY model suffers from periodic oscillations superposed on the scaling laws obeyed by these functions [10]. It was proposed in [2] that it is advantageous therefore to focus on the scaling properties of correlation functions of the type

$$F_q(k_n) \equiv \left\langle\left|\mathrm{Im}\left(u_n u_{n+1} u_{n+2} + \frac{1}{4}u_{n-1}u_n u_{n+1}\right)\right|^{q/3}\right\rangle. \quad (2)$$

These functions scale with k_n, $F_q(k_n) \sim k_n^{-\zeta_q}$ as long as k_n is smaller than some dissipative cutoff scale k_d and sufficiently larger than k_1. The exponents ζ_q are the scaling exponents whose universality is questioned. Definition (2) has the advantage that ζ_3 is known exactly (if one assumes that inertial range scaling is universal) and it is $\zeta_3 = 1$ [8].

The issue to be discussed in this Letter is introduced in Fig. 1 which shows the results of numerical simulations of this family of models. The simulations were performed using the adaptive step-size backward differentiation algorithm (DDEBDF) from the SLATEC library [11,12]. We imposed a random forcing on the first two shells, $f_1 = 5 \times 10^{-3}$ and $f_2 = af_1$ where a typical value of a is 0.75. The time correlation of the forcing has been chosen to be of the magnitude of the natural forcing turnover time scale $\tau = 1/\sqrt{k_1 f} = 40$. We plot in log-log

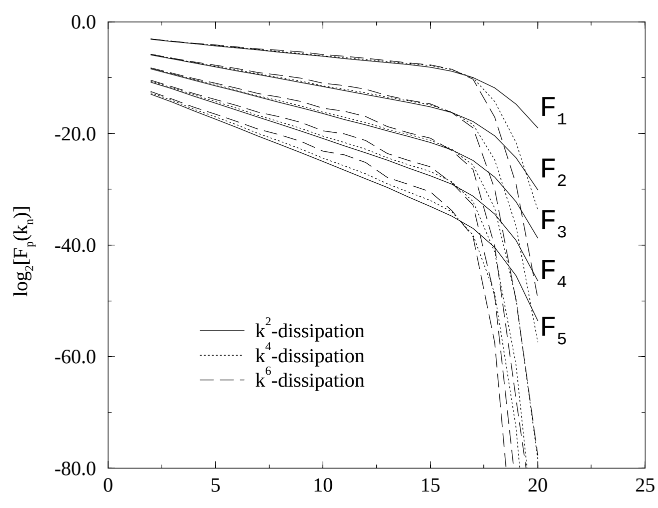

FIG. 1. The correlation functions F_1 to F_5 averaged over about 1500 forcing turnover scales for a model of 22 shells with a dissipative term proportional to k^2 ($\nu = 5 \times 10^{-7}$); k^4 ($\nu = 10^{-14}$); and k^6 ($\nu = 2 \times 10^{-22}$).

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coordinates the correlation functions F_q as a function of k_n, for different values of the parameter $\alpha = 1, 2, 3$ and $N = 22$. The dissipative cutoff is chosen (by fixing the parameter ν) to keep $k_d \approx k_0 \times 2^{17}$ for all values of α. In Table I we show the results obtained by simple linear fits between the shells 4 and 13. Results of this type were interpreted in previous works [1,4] as an indication that the inertial range scaling depends on α, casting therefore severe doubts on the universality of the scaling exponents ζ_q. It should be stressed that this is not a trivial issue; if indeed the scaling exponents depend on the dissipative mechanism this could either imply a major departure from the standard thinking about universality in turbulence, or an indication that shell models are fundamentally different from Navier-Stokes dynamics. The main point of this Letter is that none of this is necessary. We propose that these scaling plots should be interpreted as standard corrections to scaling that can be factored away by taking into acount the effect of the viscous dissipation. These effects become less important when the number of shells increases and when the length of the inertial range increases, as we show below in detail. The finite-sized effects discussed here are very reminiscent to corrections to scaling in critical phenomena [13]. To provide strong evidence that we are faced with finite-sized effects we integrated the equations of motion for a different number of shells. We chose three sets of data with $N = 22$, 28, and 34 shells. In all cases the integration was performed over 1500 forcing turnover time scales. The viscosities were chosen so that the width of the dissipative range is about 5–6 shells: for $\alpha = 1$ (normal dissipation), $\nu = 5 \times 10^{-7}$, 8×10^{-9} and 4×10^{-11} for $N = 22$, 28, and 34 respectively; for $\alpha = 2$, $\nu = 10^{-14}$, 2×10^{-20}, and 2×10^{-26}, and for $\alpha = 3$, $\nu = 2 \times 10^{-22}$, 10^{-30}, and 10^{-41}. The results of this simulation are a direct illustration of the finite-sized effect: for any value of α, we can replot our data for $N = 22$, 28, and 34, moving the abscissa to coalesce the near dissipative region. Consider, for example, $F_3(k_n)$ for which we have a theoretical expectation, i.e., $F_3(k_n) \sim k_n^{-1}$ in the inertial range, but for which the data in Table I indicate significant deviation from $\zeta_3 = 1$. In Fig. 2 we present double-logarithmic plots of $k_n F_3(k_n)$ vs k_n for $\alpha = 2$. We expect this function to be constant in the inertial range. Instead, we see a strong viscous effect, leading to deviation from constancy over ten shells even before deep dissipation sets in. Increasing the number of shells moves the problematic region to higher shells.

TABLE I. Simple scaling regression $k_n^{-\zeta_p}$ for regular and hyperviscosity; $\alpha = 1, 2, 3$.

k^p	N	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5
k^2	22	0.38	0.71	1.00	1.27	1.62
k^4	22	0.36	0.69	0.97	1.22	1.46
k^6	22	0.34	0.62	0.85	1.06	1.25

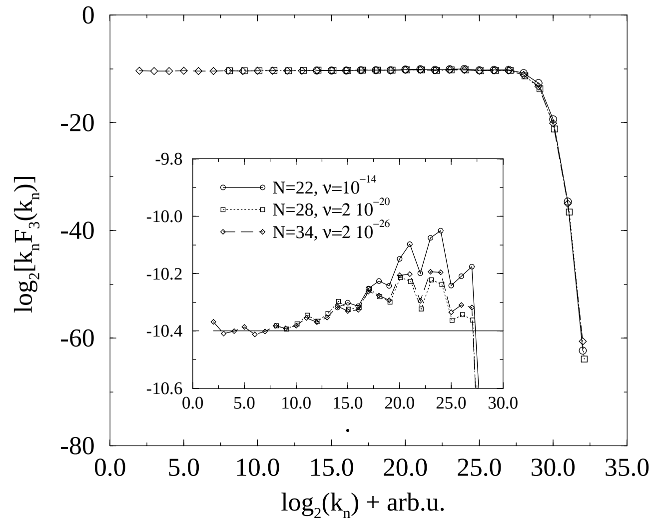

FIG. 2. Log-log plots of $k_n F_3(k_n)$ vs k_n in case of hyperviscosity of index $\alpha = 2$ with different numbers of shells and viscosities. The collapse has been obtained by shifting the abscissa. In the inset, we zoom on the y axis to exhibit the finite-sized effect occurring in the inertial range. The solid line shows the constant behavior expected theroretically. One observes clearly that the departure from this constant value occurs only in a region of about ten shells near to the viscous transition. When the inertial range is large enough, the predicted behavior is recovered.

We have shifted the abscissa in Fig. 2 to collapse all our data together. The data indeed collapse, with an N independent (fixed-sized) crossover region. This is a sure indication that we are dealing with a finite-sized effect and not the breaking of universality. Any deviation of the apparent ζ_3 from unity is going to disappear slowly when the size of the inertial range increases. Of course, the linear fit procedure is very sensitive to the existence of a bump of the type shown in Fig. 2 and the determination of the exponents in Table I is very unreliable.

As is commonly found in scaling systems that suffer strong finite-sized effects, it is advantageous to fit *the whole scaling function* rather than its power law part alone. In other words, to extract reliable inertial range scaling exponents, we propose to fit the correlation functions over their full domain, including the deep viscous regime. To achieve such a fit, we need to consider first the viscous range, where we may guess a generalized exponential decay:

$$u_n \sim \exp\left[-\left(\frac{k_n}{k_d}\right)^x\right], \qquad (3)$$

where k_d is an appropriate dissipative scale. In the stationary state we expect a balance between the nonlinear transfer and the dissipative damping for every shell. Because of the exponential decay, the most relevant nonlinear term for the nth shell is $k_{n-1}u_{n-1}u_{n-2}$, which has to balance with $k_n^{2\alpha} u_n$. Up to logarithmic corrections,

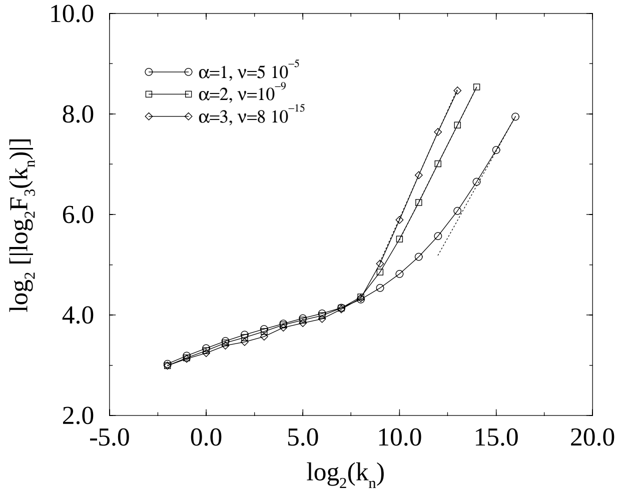

FIG. 3. Plots of $\log_2|\log_2 F_3(k_n)|$ vs $\log_2(k_n)$ for indices of hyperviscosity $\alpha = 1, 2, 3$, and viscosity $\nu = 5 \times 10^{-5}$, 10^{-9}, and 8×10^{-15}. The dotted lines represent the asymptotic stretched exponential behavior. The slopes are, respectively, 0.69, 0.76, and 0.86 to compare to the theoretical expectation $x = 0.694$.

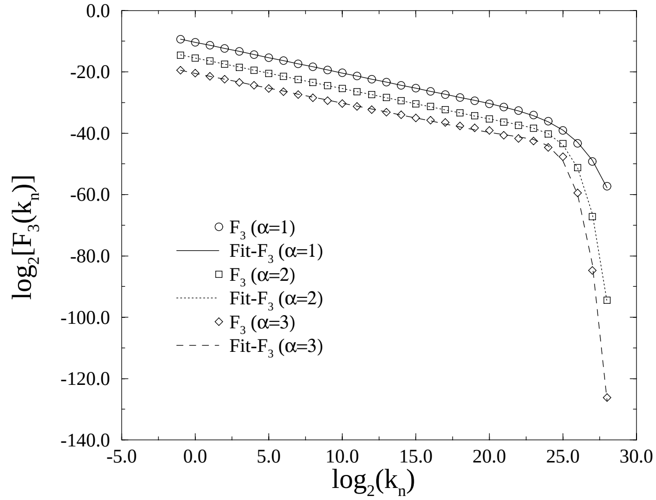

FIG. 4. Third order structure function $F_3(k_n)$ for $\alpha = 1$, 2, and 3. The symbols indicate the results of simulations with 34 shells; the lines are the fits performed between the 3rd and 32nd shells. The results have been shifted in the y coordinate for clarity.

the exponent x is then determined from

$$\exp\left[-\left(\frac{k_{n-1}}{k_d}\right)^x\right]\exp\left[-\left(\frac{k_{n-2}}{k_d}\right)^x\right] \sim \exp\left[-\left(\frac{k_n}{k_d}\right)^x\right], \tag{4}$$

or equivalently,

$$1 + \lambda^x - \lambda^{2x} = 0, \tag{5}$$

so that we have finally

$$x = \log_\lambda \frac{1 + \sqrt{5}}{2}. \tag{6}$$

Our simulations provide excellent support for the deep dissipative stretched exponential dependence, but the predicted value of x depends on the coefficient α; see Fig. 3. In the inertial range we expect a power law, and to estimate the behavior in the crossover region we note that the ratio of the viscous to nonlinear term can be written in the form

$$\frac{\nu k_n^{2\alpha} u_n}{k_n u_n^2} \sim \frac{\nu k_n^{2\alpha} u_n}{\nu k_d^{2\alpha} u_d} \frac{k_d u_d^2}{k_n u_n^2} \sim \left(\frac{k_n}{k_d}\right)^{2\alpha - 2/3}, \tag{7}$$

where u_d is the velocity at the viscous crossover shell with $k_n = k_d$. For simplicity, we assumed $u_n \sim k_n^{-1/3}$ for this estimate. Taking (7) as a first order correction of the inertial range power law by viscous effects we can propose a functional form for $F_q(k_n)$ over its full domain:

$$F_q(k_n) = A_q k_n^{-\zeta_q}\left(1 + \beta_q \frac{k_n}{k_{d,q}}\right)^{2\alpha - 2/3} \exp\left[-\left(\frac{k_n}{k_{d,q}}\right)^{x_q}\right]. \tag{8}$$

By construction this form fits the scaling law and the deep dissipative form for $k_n \ll k_d$ and $k_n \gg k_d$, respectively. For crossover values of k_n the formula offers two additional free parameters, i.e., β_q, $k_{d,q}$. This is the minimal number of fit parameters that is needed to interpolate between the inertial and the dissipative ranges.

As an example of the excellent fit that this formula provides we show in Fig. 4 $F_3(k_n)$ for $\alpha = 1$, 2, and 3. This simulation employed 34 shells, and the fits were

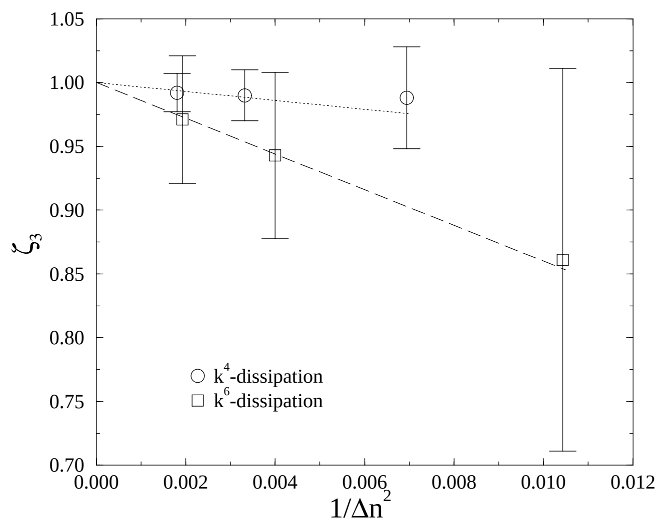

FIG. 5. Third order exponent ζ_3 computed via the fitting procedure described in the text for $\alpha = 2$ and 3 obtained for $N = 22$, 28, 34. The results have been plotted against $1/(\Delta n)^2$ where Δn is the estimated width of the inertial range. The straight lines (with respective slopes of -3.5 and -14) indicate that the calculated exponents converge linearly toward the predicted value $\zeta_3 = 1$.

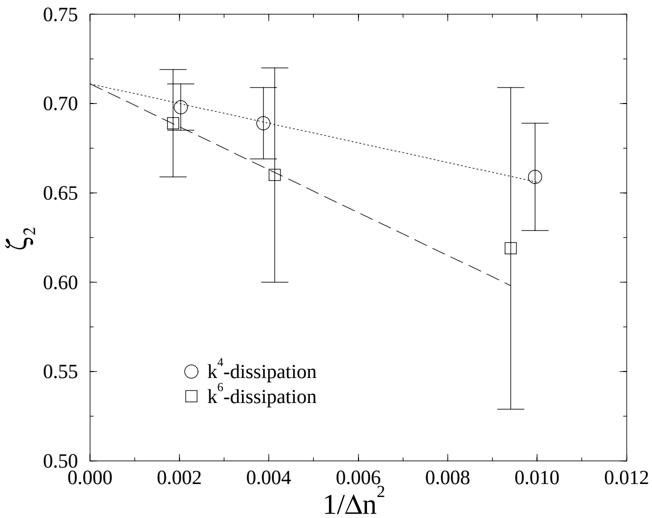

FIG. 6. Same figure as above for the second order exponent ζ_2. The illustrative lines have slopes -6 and -12. The final value $\zeta_2 = 0.711$ has been obtained in the normal dissipation case $\alpha = 1$.

performed between the 3rd and 32nd shells. Similarly good fits are found for all the data that we analyzed.

Accordingly, we recorded the apparent values of the scaling exponent ζ_q for increasing values of the N, and noticed that they converge to a given value. This convergence can be seen in Figs. 5 and 6 in which the scaling exponents ζ_2 and ζ_3 are plotted for $\alpha = 2, 3$ as a function of $1/(\Delta n)^2$ where $\Delta n = \log_2(k_d/k_2)$ is the estimated width of the inertial range, and k_d is obtained from the fit. It is obvious that the exponents converge toward a universal value that agrees with $\alpha = 1$. Note that a similar conclusion was obtained recently for shell models in which the eddy-viscosity was varied [14].

The conclusion of this Letter is that it is dangerous to measure scaling exponents in multiscaling situations without taking into account crossover and finite-sized effects. Even in the case of shell models which appear to have much longer inertial ranges than Navier-Stokes turbulence, simple log-log plots are misleading. We propose that in future determination of scaling exponents in numerical experiments similar care must be given to such issues.

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