

# Additive equivalence in turbulent drag reduction by flexible and rodlike polymers

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We address the “additive equivalence” discovered by Virk and co-workers: drag reduction affected by flexible and rigid rodlike polymers added to turbulent wall-bounded flows is limited from above by a very similar maximum drag reduction (MDR) asymptote. Considering the equations of motion of rodlike polymers in wall-bounded turbulent ensembles, we show that although the microscopic mechanism of attaining the MDR is very different, the macroscopic theory is isomorphic, rationalizing the interesting experimental observations.

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## I. INTRODUCTION

Turbulent flows in a channel are conveniently discussed for fixed pressure gradients  $p' \equiv -\partial p / \partial x$ , where  $x, y$ , and  $z$  are the lengthwise, wall-normal, and spanwise directions, respectively [1]. The length and width of the channel are usually taken much larger than the midchannel height  $L$ , making the latter a natural rescaling length for the introduction of dimensionless (similarity) variables. Thus the Reynolds number  $Re$ , the normalized distance from the wall  $y^+$ , and the normalized mean velocity  $V^+(y^+)$  (which is in the  $x$  direction with a dependence on  $y$  only) are defined by

$$Re \equiv L\sqrt{p'L}/\nu_0, \quad y^+ \equiv yRe/L, \quad V^+ \equiv V/\sqrt{p'L}, \quad (1)$$

where  $\nu_0$  is the kinematic viscosity. Drag reduction by polymers in a channel geometry is bounded by two asymptotes [2]. One is the von Kármán log law of the wall for Newtonian fluids,

$$V^+(y^+) = \kappa_K^{-1} \ln y^+ + B \quad \text{for } y^+ \gtrsim 30. \quad (2)$$

While the log law can be derived using several approaches, the von Kármán constant  $\kappa_K \approx 0.436$  and intercept  $B \approx 6.13$  are only known from experiments and simulations [1,3]. The second asymptote is the maximum drag reduction (MDR) where the velocity field assumes another log law of the form

$$V^+(y^+) = \frac{1}{\kappa_V} \ln(e \kappa_V y^+) \quad \text{for } y^+ \gtrsim 12. \quad (3)$$

This law, which had been discovered experimentally by Virk (and hence the notation  $\kappa_V$ ), was derived theoretically for *flexible* polymers in [4–7]. The actual velocity profile in the presence of polymers is bounded between these two asymptotes: for sufficiently high values of  $Re$  and concentration of the polymer, the velocity profile in a channel is expected to follow the law (3). For finite  $Re$ , finite concentration, and finite extension of the polymers, one expects crossovers back to a velocity profile parallel to the law (2), but with a larger mean velocity (i.e., with a large value of the intercept  $B$ ). The positions of the crossovers are not universal, and they are understood fairly well [7,8].

In this paper, we address the experimental finding that *rigid rodlike* polymers appear to exhibit a very similar MDR

(3) as flexible polymers [9]. Since the bare equations of motion of rodlike polymers differ quite significantly from those of flexible polymers, one needs to examine the issue carefully to understand this similarity, which was termed by Virk “additive equivalence.” The aim of this paper is to understand this additive equivalence on the basis of the equations of motion.

In Sec. II, we address the theory of drag reduction by rigid polymers. In Sec. II A, we consider the equations of motion of rigid polymers (or fibers) in the presence of strong wall-bounded turbulence. We explain the interesting differences between the interaction of flexible and rodlike polymers with turbulent fluctuations. In Sec. II B, we discuss the difference in statistics between flexible and rodlike polymers near thermodynamic equilibrium (where the velocity field is laminar, and fluctuations are thermal). Section II C is devoted to the statistics of rodlike polymers in turbulent flows with a strong shear. In Sec. II D, we address the important issue of how to evaluate the various cross-correlation functions between the polymer conformation tensor and the turbulent fluctuations. These objects have a seminal role in the theory of drag reduction by rodlike polymers. In the following Secs. II E and II F, we demonstrate that in spite of the very significant microscopic differences between flexible and rodlike polymers, the balance equations for momentum and energy have exactly the same form as the corresponding equations for flexible polymers. The “additive equivalence” follows from this observation. In Sec. III, we summarize the paper and discuss further the correspondence between drag reduction by flexible and rodlike polymers.

## II. THEORY OF DRAG REDUCTION BY RODLIKE POLYMERS

### A. Basic equation of motion

#### 1. Hydrodynamic equations for the polymeric solutions

The hydrodynamic equations for an incompressible fluid velocity  $\mathbf{U} \equiv \mathbf{U}(t, \mathbf{r})$  in the presence of rodlike polymers have the form

$$\frac{D\mathbf{U}}{Dt} = \nu_0 \Delta \mathbf{U} - \nabla p + \nabla \cdot \boldsymbol{\sigma}, \quad (4a)$$

$$0 = \nabla \cdot \mathbf{U}. \quad (4b)$$

Here the fluid density is set to unity ( $\varrho \equiv 1$ ),  $D/Dt$  is the substantial derivative

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla, \quad (5)$$

$p \equiv p(t, \mathbf{r})$  is the pressure field,  $\nu_0$  is the kinematic viscosity of the carrier fluid, and  $\boldsymbol{\sigma} \Rightarrow \sigma_{ab}$  is an extra stress tensor caused by the polymers.

The calculation of the tensor  $\boldsymbol{\sigma}$  for rigid rods is offered in the literature [10], subject to the assumptions that the rodlike polymers are massless and have no inertia. In other words, the rodlike polymers are assumed to be at all times in local rotational equilibrium with the velocity field. Thus the stress tensor does not have a contribution from the rotational fluctuations against the fluid, but rather only from the velocity variations along the rodlike object. Such variations lead to “skin friction,” and this is the only extra dissipative effect that is taken into account [11–13]. The result of these considerations is the following expression for the additional stress tensor:

$$\sigma_{ab} = 6\nu_p n_a n_b (n_i n_j S_{ij}), \quad \text{rodlike polymers}, \quad (6)$$

where  $\nu_p$  is the polymeric contribution to the viscosity at vanishingly small and time-independent shear;  $\nu_p$  increases linearly with the polymer concentration, making it an appropriate measure for the polymer’s concentration. The other quantities in Eq. (6) are the velocity gradient tensor

$$\mathbf{S} \equiv (\nabla \mathbf{U})^T \Rightarrow S_{ab} = \partial U_a / \partial x_b, \quad (7)$$

and  $\mathbf{n} \equiv \mathbf{n}(t, \mathbf{r})$  is a unit ( $\mathbf{n} \cdot \mathbf{n} \equiv 1$ ) director field that describes the polymer’s orientation. Notice that for flexible polymers, the equation for  $\sigma_{ab}$  is completely different from Eq. (6),

$$\sigma_{ab} \approx \nu_p \gamma_p n_a n_b, \quad \text{flexible polymers}. \quad (8)$$

Here  $\gamma_p$  is the polymeric relaxation frequency. The difference between Eqs. (6) and (8) for  $\sigma_{ab}$  for the rodlike and flexible polymers reflects their very different microscopic dynamics. For the flexible polymers, the main source of interaction with the turbulent fluctuations is the stretching of the polymers by the shear. This is how energy is taken from the turbulent field, introducing an additional channel of dissipation without necessarily increasing the local gradient. In the rigid case, the dissipation is only taken as the skin friction along the rodlike polymers. Having in mind all these differences, it becomes even more astonishing that the macroscopic equations for the mechanical momentum and kinetic energy balances are isomorphic for the rodlike and flexible polymers, as is demonstrated below.

## 2. Equation for the orientation field of dilute rods solution

The equation for  $\mathbf{n}(\mathbf{r}, t)$  has the form

$$\frac{D\mathbf{n}}{Dt} = (\boldsymbol{\delta} - \mathbf{nn}) \cdot \mathbf{S} \cdot \mathbf{n} \Rightarrow \frac{Dn_a}{Dt} = (\delta_{ai} - n_a n_i) S_{ij} n_j. \quad (9)$$

This equation conserves the unit length of the director  $\mathbf{n}$ . The theory is written more compactly in terms of the (normalized) conformation tensor

$$\mathcal{R} \equiv \mathbf{nn} \Rightarrow \mathcal{R}_{ab} \equiv n_a n_b. \quad (10)$$

The equation of motion of this object follows from Eq. (9),

$$\frac{D\mathcal{R}_{ab}}{Dt} = S_{ai} \mathcal{R}_{ib} + S_{bi} \mathcal{R}_{ia} - 2 \mathcal{R}_{ab} (S_{ij} \mathcal{R}_{ij}). \quad (11)$$

In terms of  $\mathcal{R}$ , we can rewrite  $\boldsymbol{\sigma}$ , Eq. (6), as

$$\boldsymbol{\sigma} = 6\nu_p \mathcal{R} \text{Tr}\{\mathcal{R} \cdot \mathbf{S}\} \Rightarrow \sigma_{ab} = 6\nu_p \mathcal{R}_{ab} (\mathcal{R}_{ij} S_{ij}). \quad (12)$$

## B. The statistics of rodlike polymers near thermodynamic equilibrium

In order to understand the interaction of rodlike polymers with wall-bounded turbulent fluctuations, we need to start along the well charted road of dynamics near equilibrium, and examine the solution of these equation for strong shears.

### 1. Equations of motion near thermodynamic equilibrium

Consider the shear  $\mathbf{S}(\mathbf{r}, t)$  in Eq. (9) as independent of space and time, and replace  $\mathbf{S}(\mathbf{r}, t) \Rightarrow \mathbf{S}$ . The near-equilibrium rodlike polymer (or fibers) experiences rotational disorder due to local thermal velocity fluctuations that can be considered as a Brownian motion in the space of angles. As usual, the effect of thermal fluctuations can be mimicked by adding to the right-hand side (RHS) of Eq. (9) a Langevin random force  $\mathbf{f}(t)$ . When the rigid polymer is symmetric, the hydrodynamics can depend only on the dyadic product  $\mathbf{nn}$ . By pre- and postmultiplying the equation above by  $n_b$ , and taking average over the Brownian fluctuations, we have the transport equation for the second moment  $\overline{\mathcal{R}_{ab}} = \overline{n_a n_b}$ , where an overbar indicates an average over the polymer configurations,

$$\frac{D\overline{\mathcal{R}_{ab}}}{Dt} = S_{ai} \overline{\mathcal{R}_{ib}} + S_{bi} \overline{\mathcal{R}_{ia}} - 2 S_{ij} \overline{\mathcal{R}_{ij} \mathcal{R}_{ab}} - 6 \gamma_B \left( \overline{\mathcal{R}_{ab}} - \frac{\delta_{ab}}{3} \right), \quad (13)$$

where  $\gamma_B$  is the Brownian rotational frequency, proportional to the temperature. The derivation of this equation in the literature employs the closure

$$\overline{\mathcal{R}_{ij} \mathcal{R}_{ab}} S_{ab} = \overline{\mathcal{R}_{ij}} \overline{\mathcal{R}_{ab}} S_{ab}, \quad (14)$$

which is rationalized in [10].

### 2. Solution for a simple shear flow

For a simple shear, the velocity gradient satisfies

$$S_{ab} = S \delta_{ax} \delta_{by}. \quad (15)$$

Equation (13) in the stationary, space-homogeneous case turns into

$$\text{De}_r(\delta_{ax}\bar{\mathcal{R}}_{yb} + \delta_{bx}\bar{\mathcal{R}}_{ya} - 2\bar{\mathcal{R}}_{ab}\bar{\mathcal{R}}_{xy}) = 2(3\bar{\mathcal{R}}_{ab} - \delta_{ab}). \quad (16)$$

Here we introduce the Deborah number for the rodlike polymers,

$$\text{De}_r \equiv S/\gamma_B. \quad (17)$$

Equation (16) was solved in the limit  $\text{De}_r \gg 1$  [10] with the final results

$$\bar{\mathcal{R}}_{xx} \approx 1 \gg \bar{\mathcal{R}}_{xy} \approx \frac{1}{2\text{De}_r^{1/3}} \gg \bar{\mathcal{R}}_{yy} \approx \frac{2^{1/3}}{\text{De}_r^{2/3}},$$

$$2\bar{\mathcal{R}}_{xy}^2 \approx \bar{\mathcal{R}}_{xx}\bar{\mathcal{R}}_{yy}. \quad (18)$$

It is interesting to compare the statistics of rodlike and flexible polymers in strong laminar shears. For example (cf. [6] and references therein),

$$\bar{\mathcal{R}}_{xy} \sim \text{De}_r^{1/3}\bar{\mathcal{R}}_{yy}, \quad \text{rodlike}, \quad (19)$$

$$\bar{\mathcal{R}}_{xy} \sim \text{De}_r\bar{\mathcal{R}}_{yy}, \quad \text{flexible}. \quad (20)$$

Here the Deborah number for flexible polymers  $\text{De}_f$  is defined with the flexible polymer relaxation frequency  $\gamma_p$ ,  $\text{De}_f = S/\gamma_p$ . The different dependence on Deborah number stems from the very different microscopic dynamics that leads to different expressions for the polymeric stress tensor  $\sigma_{ab}$  for the rodlike and flexible polymers.

### C. Statistics of rigid rods in turbulent flow with strong shear

In the presence of turbulence, the fluctuations are no longer thermal and the statistical description of the polymer orientation is accomplished with the mean values

$$\mathcal{R}_{ab} \equiv \langle \mathcal{R}_{ab} \rangle, \quad (21)$$

where the angular brackets denote an average over the turbulent fluctuations. For well developed turbulence, it is expedient to use the Reynolds decomposition in which the velocity field  $\mathbf{U}(\mathbf{r})$  is written as a sum of its average (over time) and a fluctuating part,

$$\mathbf{U}(\mathbf{r}, t) = \mathbf{V}(y) + \mathbf{u}(\mathbf{r}, t), \quad \mathbf{V}(y) \equiv \langle \mathbf{U}(\mathbf{r}, t) \rangle. \quad (22)$$

For a channel of large length and width, all the averages, and in particular  $\mathbf{V}(y) = V(y)\hat{x}$ , are functions of  $y$  only. Correspondingly, the shear is written as a sum of the mean shear  $S_{ab}$  and fluctuating shear  $s(\mathbf{r}, t)$  with zero mean,

$$\mathcal{S} = \langle \mathcal{S}(\mathbf{r}, t) \rangle, \quad \mathcal{S}(\mathbf{r}, t) = \mathcal{S} + s(\mathbf{r}, t), \quad \langle s(\mathbf{r}, t) \rangle = 0. \quad (23)$$

With these notations and after averaging over the statistics of turbulence, Eq. (11) takes the form

$$\left\langle \frac{D\mathcal{R}_{ab}}{Dt} \right\rangle = A_{ab} - B_{ab}, \quad (24)$$

where  $A_{ab}$  contains all the terms in which the mean shear appears explicitly, and  $B_{ab}$  contains only the fluctuating part of the shear,

$$A_{ab} \equiv S_{ai}\mathcal{R}_{ib} + S_{bi}\mathcal{R}_{ia} - 2S_{ij}\langle \mathcal{R}_{ab}\mathcal{R}_{ij} \rangle,$$

$$B_{ab} \equiv 2\langle \mathcal{R}_{ab}(s_{ij}\mathcal{R}_{ij}) \rangle - \langle s_{ai}\mathcal{R}_{ib} \rangle - \langle s_{bi}\mathcal{R}_{ia} \rangle. \quad (25)$$

In a steady state in channel geometry, the left-hand side (LHS) of Eq. (24) contains only one nonvanishing term, which is  $\langle \mathbf{u} \cdot \nabla \mathcal{R} \rangle$ . This term is responsible for the turbulent part of the physical flux of  $\mathcal{R}$ . In all our derivation below, we will assume that such terms are small compared to all ‘‘local’’ contributions to the balance equations for the momentum and energy. This assumption will have to be tested *a posteriori*.

With this in mind, Eq. (24) reads simply  $A_{ab} = B_{ab}$ . This will allow us to estimate the crucial correlation functions that appear in the theory of drag reduction below. The LHS of this relation can be made explicit in channel geometry; using Eq. (15), we find

$$A_{ab} = S(\delta_{ax}\mathcal{R}_{yb} + \delta_{bx}\mathcal{R}_{ya} - 2\langle \mathcal{R}_{ab}\mathcal{R}_{xy} \rangle). \quad (26)$$

Using definition (10) and constraint  $|\mathbf{n}| = 1$ , this equation can be rewritten in components as

$$A_{xx} = 2S\langle \mathcal{R}_{xy}(\mathcal{R}_{yy} + \mathcal{R}_{zz}) \rangle, \quad (27a)$$

$$A_{yy} = -2S\langle \mathcal{R}_{xy}\mathcal{R}_{yy} \rangle, \quad (27b)$$

$$A_{zz} = -2S\langle \mathcal{R}_{xy}\mathcal{R}_{zz} \rangle, \quad (27c)$$

$$A_{xy} = S\langle \mathcal{R}_{yy}(1 - 2\mathcal{R}_{xx}) \rangle. \quad (27d)$$

In writing down expressions for  $B_{ab}$ , we will make explicit use of the expected solution for the conformation tensor in the case of large mean shear,  $S^2 \gg \langle s^2 \rangle$ . In such flows, we expect a strong alignment of the rodlike polymers along the streamwise direction  $x$ . The director components  $n_y$  and  $n_z$  are then much smaller than  $n_x \approx 1$ . For large shear, we can expand  $n_x$  according to

$$n_x = \sqrt{1 - n_y^2 - n_z^2} \approx 1 - \frac{1}{2}(n_y^2 + n_z^2). \quad (28)$$

We note that for  $n_x = 1$  (when the shear is actually infinite) the object  $B_{ab}$  vanishes since  $\langle s \rangle = 0$ . We therefore represent  $B_{ab}$  in order on  $n_y \sim n_z$ , keeping up to second order. We will show below that it is important to keep terms of second order since some of them have the same magnitude as terms which are formally of first order in the smallness. All terms of third order are smaller in magnitude than the terms that we keep. The first two orders read

$$B_{xx}^{(1)} = 2\langle s_{yx}\mathcal{R}_{xy} + s_{zx}\mathcal{R}_{xz} \rangle, \quad (29a)$$

$$B_{yy}^{(1)} = -2\langle s_{yx}\mathcal{R}_{xy} \rangle, \quad (29b)$$

$$B_{zz}^{(1)} = -2\langle s_{zx}\mathcal{R}_{xz} \rangle, \quad (29c)$$

$$B_{xy}^{(1)} = \langle \mathcal{R}_{xy}(s_{xx} - s_{yy}) - \mathcal{R}_{xz}s_{yz} \rangle, \quad (29d)$$

$$B_{xx}^{(2)} = 2\langle \mathcal{R}_{yy}(s_{yy} - s_{xx}) + \mathcal{R}_{zz}(s_{zz} - s_{xx}) + \mathcal{R}_{yz}(s_{yz} + s_{zy}) \rangle, \quad (30a)$$

$$B_{yy}^{(2)} = 2\langle \mathcal{R}_{yy}(s_{xx} - s_{yy}) - \mathcal{R}_{yz}s_{yz} \rangle, \quad (30b)$$

$$B_{zz}^{(2)} = 2\langle \mathcal{R}_{zz}(s_{xx} - s_{zz}) - \mathcal{R}_{yz}s_{zy} \rangle. \quad (30c)$$

$$B_{xy}^{(2)} = \langle \mathcal{R}_{yy}(s_{xy} + 3s_{yx}) + \mathcal{R}_{zz}s_{yz} + \mathcal{R}_{yz}(s_{zz} + 2s_{zx}) \rangle. \quad (30d)$$

Equations (27), (29), and (30) serve as a basis for further analysis.

## D. Closures and orders of magnitude

### 1. Statistical objects of interest

A theory of turbulent channel flows of Newtonian fluids can be constructed in terms of the mean shear  $S(y)$ , the Reynolds stress  $W(y)$ , and the kinetic energy  $K(y)$ ; these are defined, respectively, as

$$S(y) \equiv dV(y)/dy, \quad W(y) \equiv -\langle u_x u_y \rangle, \quad K(y) = \langle |\mathbf{u}|^2 \rangle / 2. \quad (31)$$

In the rodlike polymer case, the additional stress tensor  $\sigma_{ij}$  and its various correlation functions need to be considered as well. For that purpose, we turn now to the analysis of the necessary statistical objects.

First note that the expansion (28) allows us to express all products  $\mathcal{R}_{ab}\mathcal{R}_{cd} = n_a n_b n_c n_d$  in terms that are linear in  $\mathcal{R}$ , up to third-order terms in  $n_y \sim n_z$ . For example,

$$\begin{aligned} \mathcal{R}_{xx}^2 &\approx 1 - 2(\mathcal{R}_{yy} + \mathcal{R}_{zz}), & \mathcal{R}_{xy}^2 &\approx \mathcal{R}_{yy}, \\ \mathcal{R}_{xy}\mathcal{R}_{xz} &\approx \mathcal{R}_{yz}, & \mathcal{R}_{yy}\mathcal{R}_{ij} &\approx \mathcal{R}_{yy}\delta_{ix}\delta_{jx}, \quad \text{etc.} \end{aligned} \quad (32)$$

Actually we have used these estimates in the derivation of Eqs. (29a)–(29d) and (30a)–(30c).

As a further preparation for the theory, below we analyze various statistical objects in the turbulent environment and estimate their magnitudes. Based on experimental observations and DNS data, we assume that statistics of turbulent fluctuations do not deviate too much from isotropy. Explicitly,

$$\langle s_{xx}^2 \rangle \sim \langle s_{yy}^2 \rangle \sim \langle s_{zz}^2 \rangle \sim \langle s_{xy}^2 \rangle \sim \dots \quad (33)$$

Here and below, the notation  $\sim$  means “the same order of magnitude (i.e., correct to leading order up to coefficients of the order of unity).”

Second, consider correlation functions of turbulent fluctuation of the shear,  $s_{ij}$ , i.e.,  $\langle s_{ij}s_{kl} \rangle$ . At distance  $y$  from the wall the correlation functions are dominated by eddies of size  $y$ . Thus

$$\langle s_{ij}s_{kl} \rangle \sim K(y)/y^2 \quad \text{for all } ij, kl. \quad (34)$$

Third, consider the cross-correlation functions of the tensor  $\mathcal{R}$  with the turbulent shear  $s$ . For this goal, take the leading terms on the LHS and the RHS of  $xy$ , Eqs. (27d) and (29d),

$$|SR_{yy}| \approx |B_{xy}^{(1)}| \approx R_{xy} \frac{\sqrt{K}}{y}, \quad (35)$$

where on the RHS we have used the Cauchy-Schwartz inequality and Eq. (34). The notation  $\approx$  means “ $\sim$  in the sense defined above, or having a smaller order of magnitude.” Below (cf. Sec. II F), we show that this estimate is saturated. Therefore, expecting that  $R_{yy} \sim R_{xy}^2$ , we have

$$SR_{xy} \sim \frac{\sqrt{K}}{y}. \quad (36)$$

This estimate can be recast into an intuitive form, which is

$$R_{xy} = \langle n_x n_y \rangle \sim \frac{\langle \sqrt{s^2} \rangle}{S}. \quad (37)$$

This is in direct accord with the understanding that the degree of deviation from perfect alignment of the rodlike polymers ( $R_{xx}=1$ ) is proportional to the turbulent fluctuations relative to the mean shear.

Taking now the leading terms in  $yy$ , Eqs. (27b) and (29b), we have

$$S\langle \mathcal{R}_{yy}\mathcal{R}_{xy} \rangle \approx -\frac{1}{2}B_{yy}^{(1)} = \langle s_{yx}\mathcal{R}_{xy} \rangle. \quad (38)$$

Estimating  $\langle \mathcal{R}_{yy}\mathcal{R}_{xy} \rangle$  as  $\sim R_{yy}R_{xy}$ , and using Eq. (36), we have

$$\langle s_{yx}\mathcal{R}_{xy} \rangle \sim R_{yy} \frac{\sqrt{K}}{y}. \quad (39)$$

Notice then the Cauchy-Schwartz inequality for the same correlation,

$$\langle s_{yx}\mathcal{R}_{xy} \rangle \leq R_{xy} \frac{\sqrt{K}}{y}, \quad (40)$$

gives a much higher upper bound than the real estimate Eq. (39). This shows how important is the use of the equations of motion in estimating various correlation functions; sometimes the direct Cauchy-Schwartz estimate saturates, and sometimes it is a gross overestimate. The reason why different correlations have different orders of magnitude can be traced back to Eqs. (27a)–(27d), which indicate that the diagonal components of  $\mathbf{A}$  are cubic in the small parameter  $n_y \sim n_z$  while  $A_{xy}$  is quadratic.

Notice that the correlator  $\langle \mathcal{R}_{ij}s_{ij} \rangle$  has the contributions of the type of  $\langle s_{xy}\mathcal{R}_{xy} \rangle$  and  $\langle s_{yy}\mathcal{R}_{yy} \rangle$ . Both of them have the same estimate  $R_{yy}\sqrt{K}/y$ . Therefore, we can write

$$\langle \mathcal{R}_{ij}s_{ij} \rangle \sim R_{yy} \frac{\sqrt{K}}{y}. \quad (41)$$

## E. The momentum-balance equation

At this point, we can apply our estimates in the context of the balance equations for the mechanical momentum and the energy. We begin with the former, which is exact. It reads

$$\nu_0 S + W + \langle \sigma_{xy} \rangle = p'(L - y), \quad (42)$$

where  $p' \equiv -\partial p / \partial x$ . Near the wall,  $y \ll L$  and the RHS of this equation is approximated as  $p'L$ , a constant production of momentum due to the pressure gradient. On the LHS, we have the Reynolds stress which is the “turbulent” momentum flux to the wall, in addition to the viscous and the polymeric contributions to the momentum flux.

Using Eq. (6) and Reynolds decomposition (23), we compute

$$\langle \sigma_{xy} \rangle = 6\nu_p \langle \mathcal{R}_{xy} \mathcal{R}_{ij} s_{ij} \rangle = 6\nu_p [S \langle \mathcal{R}_{xy}^2 \rangle + \langle \mathcal{R}_{xy} \mathcal{R}_{ij} s_{ij} \rangle]. \quad (43)$$

With Eq. (32), the first term on the RHS of this equation can be estimated as follows:

$$6\nu_p S \langle \mathcal{R}_{xy}^2 \rangle = \tilde{c}_1 \nu_p R_{yy} S, \quad \tilde{c}_1 \simeq 6. \quad (44)$$

On the other hand, using the estimate (41) one sees that the second term in the RHS of Eq. (43) is of the same order as the first one.

Finally, we can present the momentum balance equation in the form

$$\nu_0 S + c_1 \nu_p R_{yy} S + W = p'L. \quad (45)$$

Another way of writing this result is in the form of an effective viscosity,

$$\nu(y) S + W = p'L, \quad (46)$$

where the effect of the rodlike polymers is included by the effective viscosity  $\nu(y)$ ,

$$\nu(y) \equiv \nu_0 + c_1 \nu_p R_{yy}. \quad (47)$$

### F. Turbulent energy balance equation

The effective viscosity that appeared in the momentum balance equation stems from the nondiagonal component of the polymeric stress tensor  $\langle \sigma_{xy} \rangle$ . On the other hand, the energy balance equation involves dissipation effects that stem from correlations between the polymeric stress tensor and the fluctuating turbulent shear field, see Eq. (53). We will show in this subsection that as far as the energy balance equation is concerned, these effect translate to the same form of the effective viscosity as the one appearing in the momentum balance equation.

In considering the balance of energy in a channel flow, it pays to separate the spatial directions, since we can learn separate bits of information from each such equation. Introduce the partial kinetic energy density

$$K_a(y) \equiv \frac{1}{2} \langle u_a^2 \rangle, \quad K(y) = K_x + K_y + K_z, \quad (48)$$

and consider the partial energy balance of  $K_a(y)$ ,

$$\frac{\partial K_a(y)}{\partial t} + R_a + \varepsilon_a^{\text{dis}} + \varepsilon_a^{\text{p}} = W(y) S(y) \delta_{ax}. \quad (49)$$

The total density of the kinetic energy at a given distance  $y$  from the wall is,

$$\frac{\partial K(y)}{\partial t} + \varepsilon^{\text{dis}} + \varepsilon^{\text{p}} = W(y) S(y),$$

$$\varepsilon^{\text{dis}} = \sum_a \varepsilon_a^{\text{dis}}, \quad \varepsilon^{\text{p}} = \sum_a \varepsilon_a^{\text{p}}, \quad \sum_a R_a = 0. \quad (50)$$

The various symbols in the last two equations are explained as follows: The RHS of these equations describes the energy flux from the mean flow to turbulent fluctuations due to the correlation between streamwise and cross-stream components of the turbulent velocity, known as the Reynolds stress  $W$ , see Eq. (31). Remarkably, in channel geometry this flux exists only in the equation for the streamwise velocity fluctuations,  $K_x$ .

The term  $R_a(y)$  is known as the “return to isotropy” [14], and it vanishes for isotropic turbulence in which  $K_x = K_y = K_z$ . Otherwise, it redistributes partial kinetic energy between different vectorial components and does not contribute to the total balance (50). A simple model for this term [14] is

$$R_a \sim \frac{\sqrt{K}}{y} \left( K_a - \frac{K}{3} \right). \quad (51)$$

As usual, the local “outer scale of turbulence” was estimated as the distance to the wall  $y$ . The order of magnitude estimate (33) is in accord with the role of this term in returning to local isotropy.

The term

$$\varepsilon_a^{\text{dis}} \simeq \nu_0 \sum_j \langle s_{ja}^2 \rangle, \quad \text{no sum over } a, \quad (52)$$

on the LHS of Eq. (49) is the rate of the viscous dissipation, proportional to the kinematic viscosity of the carrier fluid  $\nu_0$ . Lastly, the polymer contribution to the energy balance, denoted as  $\varepsilon_a^{\text{p}}$ , can be exactly computed as

$$\varepsilon_a^{\text{p}} = \langle \sigma_{aj} s_{aj} \rangle = 6\nu_p \langle s_{ai} \mathcal{R}_{ai} (S \mathcal{R}_{xy} + s_{jk} \mathcal{R}_{jk}) \rangle. \quad (53)$$

Notice that Eqs. (49) and (50) are written “in the local approximation,” in which the energy flux in the physical space is neglected. This is consistent with neglecting the term  $\langle \mathbf{u} \cdot \nabla \mathcal{R} \rangle$  in our discussion after Eq. (25). A justification of this approximation in the problem of drag reduction by polymers is found in [4,5].

Using the expansion (28), we can rewrite the equation for the dissipation rate  $\varepsilon^{\text{p}}$  as a series in the small parameter  $n_y \sim n_z$ ,

$$\begin{aligned} \varepsilon^{\text{p}} = & 6\nu_p \langle \{ s_{xx} + [\mathcal{R}_{xy}(s_{xy} + s_{yx}) + \mathcal{R}_{xz}(s_{xy} + s_{yx})]_1 \\ & + [s_{yy} \mathcal{R}_{yy} + s_{yz} \mathcal{R}_{yz} + \text{more}]_2 + \dots \} \\ & \times \{ S \mathcal{R}_{xy} + s_{xx} + [\dots]_1 + [\dots]_2 + \dots \} \rangle. \end{aligned} \quad (54)$$

In the square brackets  $[\dots]_1$  we displayed all the terms which are linear in the small parameter. In the square brackets  $[\dots]_2$  we show two quadratic terms, and there are more of them as indicated. The symbol  $\dots$  stands for “higher-order terms.” In the second curly brackets, the square brackets  $[\dots]_1$  and  $[\dots]_2$  are identical to the corresponding terms in the first curly brackets. The two leading terms in Eq. (54) are proportional to  $\langle s_{xx}^2 \rangle$  and  $S \langle \mathcal{R}_{xy} s_{xx} \rangle$ . Using Eqs. (29d), (34), and

(35), one sees that both leading terms have the same order of magnitude,  $\sim K(y)/y^2$ . In fact, we will argue that these two terms must cancel each other, up to terms of higher-order contributions  $\sim R_{yy}K(y)/y^2$ .

To see that such a cancellation must exist, we note that near the MDR we expect the polymer contribution to the dissipation to balance the production term  $WS$  [cf. Eq. (58) below]. We will show later that this production term is  $y$ -independent (where  $y$  is the distance from the wall). On the other hand, we will show that  $K(y)$  is linear in  $y$ , making  $K/y^2$  very large near the wall. Therefore,  $WS$  cannot be balanced by  $K/y^2$ . To avoid using the final results at this stage, and nevertheless to see that a cancellation must exist, we can focus just on the stationary balance Eq. (49) for the  $y$  component, in which the RHS is zero. Notice also that near the MDR, the polymer contribution to the energy balance dominates over the viscous dissipation and the nonlinear energy flux from large to small scales [4–6]. The latter was evaluated in [4] as  $K^{3/2}/y$ , which is exactly the evaluation of the return to isotropy term. The conclusion is that near the MDR,

$$\varepsilon_y^p = \langle \sigma_{yj} s_{yj} \rangle = 6 \nu_p \langle s_{yi} \mathcal{R}_{yi} (S \mathcal{R}_{xy} + s_{jk} \mathcal{R}_{jk}) \rangle \approx 0. \quad (55)$$

This expression can be again arranged in orders of magnitude, similarly to Eq. (54),

$$\varepsilon_y^p = 6 \nu_p \langle \{ [\mathcal{R}_{xy} s_{yx}]_{y1} + [s_{yy} \mathcal{R}_{yy} + s_{yz} \mathcal{R}_{yz}]_{y2} + \dots \} \times \{ S \mathcal{R}_{xy} + s_{xx} + [\dots]_1 + [\dots]_2 + \dots \} \rangle.$$

Here the second curly brackets is the same as the second curly brackets in Eq. (54). In Eq. (56), we find again the same two large contributions, i.e.,  $\mathcal{R}_{xy} s_{yx} \{ S \mathcal{R}_{xy} + s_{xx} \} + (\text{l.o.t.})$ , where the lower-order terms (l.o.t) have at least one small factor. Thus these terms must cancel each other, which means that  $S \mathcal{R}_{xy}$  cancels  $s_{xx}$  inside correlation functions. But these are exactly the terms that appear in the sum of the two large terms in Eq. (54), and we are therefore justified in neglecting them, allowing the other terms to share the burden of balancing  $WS$ . We note that this conclusion is justifying *a posteriori* the statement after Eq. (35) that the Cauchy-Schwartz inequality is saturated for the case considered.

An additional way to see that the cancellation must take place is to examine again the momentum balance equation (45). As discussed below, the MDR is obtained when the Reynolds stress term  $W$  is negligible compared to the polymer contribution  $c_1 \nu_p R_{yy} S$ . But when this happens it means that

$$WS \ll c_1 \nu_p R_{yy} S^2 \approx c_1 \nu_p K/y^2. \quad (56)$$

Evidently, this means that also in the energy balance equations  $WS$  would be overwhelmed by terms of the order of  $K/y^2$ , which therefore must cancel against each other.

Using our order of magnitude estimates for the remaining terms, we can therefore conclude that

$$\varepsilon^p(y) \approx c_2 \nu_p R_{yy}(y) K(y)/y^2, \quad (57)$$

where  $c_2$  is another parameter of the order of unity. Returning to the balance equation for the energy, we recall that we cannot calculate  $\varepsilon^{\text{dis}}(y)$  exactly, but we can estimate it rather well at a point  $y$  away from the wall. When viscous effects

are dominant, this term is estimated as  $\nu(a/y)^2 K(y)$  (the velocity is then rather smooth; the gradient exists and can be estimated by the typical velocity at  $y$  over the distance from the wall). Here  $a$  is a constant of the order of unity. When the Reynolds number is large, the viscous dissipation is the same as the turbulent energy flux down the scales, which can be estimated as  $K(y)/\tau(y)$ , where  $\tau(y)$  is the typical eddy turn over time at  $y$ . The latter is estimated as  $y/b\sqrt{K(y)}$ , where  $b$  is another constant of the order of unity. Together with Eq. (57), we can thus write the energy balance equation at point  $y$  as

$$a \nu_0 \frac{K(y)}{y^2} + b \frac{K^{3/2}(y)}{y} + c_2 \nu_p R_{yy}(y) \frac{K(y)}{y^2} = W(y) S(y). \quad (58)$$

We recognize the important result that the effective viscosity induced by the rodlike polymers in both the momentum and the energy balance equation is proportional to  $R_{yy}$ . *These balance equations are identical in form to those found for flexible polymers* [4]; this is an important step in understanding the “additive equivalence” discovered by Virk.

To complete the derivation, one adds to the balance equation the relation between  $K(y)$  and  $W(y)$ , which in the elastic layer are expected to be proportional to each other,

$$K = c_V^2 W. \quad (59)$$

It should be stressed that rigorously one can establish this relation only as an inequality with  $c_V \leq 1$ , and its use as an equality (which is common to the derivation of the Newtonian log law as well as to the derivation of the MDR in the flexible polymer case) rests on experimental and simulational confirmation. Near the MDR, the terms representing the effect of the polymers in Eqs. (45) and (58) are dominant, and one estimates from the momentum equation  $R_{yy}(y) \propto 1/S(y)$ . Using this in Eq. (58) together with Eq. (59), one ends up with the prediction that  $S(y) \propto 1/y$ , leading to a logarithmic law for the mean velocity. Repeating the derivation of [4] in wall units, one ends up with the MDR Eq. (3), with the identification

$$\kappa_V = c_V / c_N y_v^+ . \quad (60)$$

In this equation,  $c_N$  and  $y_v^+$  are constants that appear in the Newtonian theory, and cannot change from flexible to rodlike polymers. The existence of drag reduction is guaranteed, since  $\kappa_V$  was shown to be larger than its Newtonian counterpart  $\kappa_K$  [4]. The actual value of the slope at the MDR logarithmic law depends nonetheless on the numerical value of  $c_V$ . Thus the prediction of the theory is that *if  $c_V$  is about the same in rodlike and flexible polymers, than the slope of the MDR should be about the same.*

For completeness, we reiterate our previous result stating that the effective viscosity Eq. (47) is predicted to vary linearly in the distance from the wall when the system is at the MDR. This result follows immediately from  $R_{yy} \propto 1/S(y)$  and  $S(y) \propto 1/y$ .

### III. CONCLUSIONS

We have presented a scenario to rationalize the “additive equivalence” discovered by Virk and co-workers. The main conclusion of this paper is that although, on the face of it, the dynamics of flexible and rodlike polymers appear different, with flexible polymers being able to “stretch” and “store” energy (something that many researchers thought is central to drag reduction), the effective Reynolds balance equations for momentum and energy are isomorphic. Accordingly, the MDR is expected to be the same as long as  $c_V$  of Eq. (59) is the same.

On the other hand, one expects that the cross-over from the MDR to the Newtonian plug, which is nonuniversal even in flexible polymers [7,8], may show significant differences between rodlike and flexible polymers. Indeed, in friction coordinates, drag reduction by rodlike polymers appears as an upper bound on the drag reduction by flexible polymers [9]. According to the theory of [8], flexible polymers reach their maximal drag reduction when fully stretched, being then as effective as rodlike polymers. This is one way of rationalizing the findings of [9].

To make the difference between the flexible and rodlike polymers sharper, we note the different  $y$  dependence of  $K(y)$  and  $W(y)$  in the two cases. In the flexible case, one had a threshold condition for the onset of drag reduction in terms of the Deborah number, stating that the typical time scale for turbulent fluctuations,  $y/\sqrt{K(y)}$ , is of the order of the polymer relaxation time  $\tau_p$ . This immediately leads to the estimate  $K(y) \sim y^2$ , and the same for the Reynolds stress. In the present case, we have estimated  $SR_{xy} \sim \sqrt{K}/y$ , and with  $R_{xy}^2 \sim R_{yy} \sim y$  we get  $K(y) \sim y$ , and due to Eq. (59) we can write

$$K(y) \sim W(y) \sim y \quad \text{for rigid rodlike polymers,}$$

$$K(y) \sim W(y) \sim y^2 \quad \text{for flexible polymers.} \quad (61)$$

We note that this last statement is a sharp prediction of an important difference between the two drag-reducing universality classes, a difference that is not at all in contradiction with the “additive equivalence.” *A posteriori* we can also see why the terms of the order of  $K(y)/y^2$  in Eq. (54) must have canceled, being divergent as  $1/y$  against a  $y$ -independent energy input  $W(y)S(y)$ . We hope that this prediction would be put to experimental or simulational test.

An additional important difference between rigid and flexible polymers is that in the latter case an important condition for attaining the MDR was  $R_{xx} \gg R_{yy}$ . In the present case, we need to guarantee that  $c_1 \nu_p R_{yy} \gg \nu_0$  in order to enable the polymer terms to overwhelm the Newtonian terms in the balance equations. This condition means, however, that the concentration of the rigid polymer should be large enough before the MDR is obtained. In the flexible polymer case, one could reach the MDR conditions even for small concentrations as long as the  $Re$  is large enough and the Deborah number is large, leading to  $R_{xx} \gg R_{yy}$  [7]. This difference leads to the observed experimental behavior, where for flexible polymers the MDR is reached even for small concentrations and then a crossover back to the Newtonian plug is found, whereas in rigid polymers that MDR is obtained gradually as the concentration increases; see the figures in [9] for comparison.

Finally, it is interesting to note that our order of magnitude estimates of  $R_{ij}$  could be read directly from Eqs. (18) by replacing the thermal mean values  $\bar{R}_{ij}$  with turbulent means  $R_{ij} \equiv \langle \mathcal{R}_{ij} \rangle$  and simply identifying the Brownian frequency  $\gamma_B$  in the definition (17) of the Brownian Deborah number  $De_r$  with the characteristic turbulent frequency  $\gamma_{\text{turb}} \equiv \sqrt{K}R_{yy}/y$ . Put, for example, in the Eq. (18) for  $R_{yy}$ , we get

$$R_{yy} \approx 2^{1/3} K^{1/3} R_{yy}^{2/3} [S(y)y]^{2/3}. \quad (62)$$

Simplifying, this equation reads  $S^2 R_{yy} \sim K(y)/y^2$ , which is nothing but the square of Eq. (36). All the other orders of magnitude derived in Sec. II D follow as easily with this identification. We believe that this is another way to argue that our estimates are physically sensible and that we have captured the essence of drag reduction by rodlike polymers and the nature of the observation of the “additive equivalence.”

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