Identification and Calculation of the Universal Asymptote for Drag Reduction by Polymers in Wall Bounded Turbulence

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Drag reduction by polymers in wall turbulence is bounded from above by a universal maximal drag reduction (MDR) velocity profile that is a log law, estimated experimentally by Virk as $V^+(y^+) \approx 11.7 \log y^+ - 17$. Here $V^+(y)$ and y^+ are the mean streamwise velocity and the distance from the wall in "wall" units. In this Letter we propose that this MDR profile is an edge solution of the Navier-Stokes equations (with an effective viscosity profile) beyond which no turbulent solutions exist. This insight rationalizes the universality of the MDR and provides a maximum principle which allows an *ab initio* calculation of the parameters in this law without any viscoelastic experimental input.

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Drag reduction in turbulent flows by a small concentration of polymers is an important phenomenon utilized in a number of technological applications including the Alaska pipeline. The efficacy of drag reduction by polymers is bounded by the so-called "maximum drag reduction asymptote" which was discovered and fully characterized experimentally more than 30 years ago [1]. The asymptote appears universal, and the parameters characterizing it were carefully measured, but to date not understood. In this Letter we offer a way to understand the nature of this asymptote and to calculate theoretically the parameters involved. This provides understanding of the intrinsic limit to how much the drag can be reduced by polymer agents.

The mean streamwise velocity profile in *Newtonian* turbulent flows in channel geometries satisfies the universal von-Kármán "log law of the wall" (cf. Fig. 1) which is written in wall units as

$$V^+(y^+) = \kappa_K^{-1} \ln y^+ + B, \qquad \text{for } y^+ \gtrsim 30. \tag{1}$$

Here x, y, and z are the streamwise, wall-normal, and spanwise directions, respectively [2]. Wall units are defined by denoting p' as the fixed pressure gradient $-\partial p/\partial x$, and L the midheight of the channel. The Reynolds number, Re, the normalized distance from the wall y^+, and the normalized mean velocity $V^+(y^+)$ (which is in the x direction with a dependence on y only) are defined by

$$\begin{aligned} \mathrm{Re} &\equiv L\sqrt{p'L}/\nu_0, \qquad y^+ \equiv y\,\mathrm{Re}/L, \\ V^+ &\equiv V/\sqrt{p'L}, \end{aligned} \tag{2}$$

where ν_0 is the kinematic viscosity. The law (1) is universal, independent of the nature of the Newtonian fluid; it had been a shortcoming of the theory of wall-bounded turbulence that the von-Kármán constant $\kappa_K \approx 0.436$ and the intercept $B \approx 6.13$ had been only known from experiments and simulations [2,3]. Some recent progress on this can be found in [4].

One of the most significant experimental findings [1] concerning turbulent drag reduction by polymers is that in channel and pipe geometries the velocity profile (with polymers added to the Newtonian fluid) is bounded between von-Kármán's log law and another log law which describes the maximal possible velocity profile [Maximum drag reduction (MDR)] [5–8],

$$V^+(y^+) = \kappa_V^{-1} \ln(e\kappa_V y^+) \quad \text{for } y^+ \gtrsim 10. \tag{3}$$

This law, which was discovered experimentally by Virk (and hence the notation κ_V), is also claimed to be universal, independent of the Newtonian fluid and the nature of the

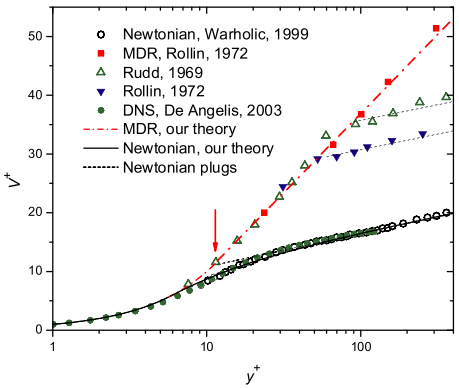

FIG. 1 (color online). Mean normalized velocity profiles as a function of the normalized distance from the wall. The data points from numerical simulations (green circles) [8] and the experimental points (open circles) refer to Newtonian flows. The red data points (squares) [1] represent the maximum drag reduction (MDR) asymptote. The dashed red curve represents the log law (4). The blue filled triangles [16] and green open triangles represent the crossover, for intermediate concentrations of the polymer, from the MDR asymptote to the Newtonian plug.

polymer additive, including flexible and rigid polymers [9]. The numerical value of the coefficient κ_v is presently known only from experiments, $\kappa_v^{-1} \approx 11.7$, giving a phenomenological MDR law in the form [1]

$$V^+(y^+) = 11.7 \ln y^+ - 17. \tag{4}$$

For sufficiently high values of Re and polymer concentration, the velocity profile in a channel is expected to follow the law (4). For finite Re, concentration and extension of the polymers one expects crossovers back to a velocity profile parallel to the law (1), but with a larger mean velocity (i.e., with a larger value of the intercept B); see Fig. 1. The position of the crossovers are not universal in the sense that they depend on the nature of the polymers and the flow conditions; a theory for the crossovers is provided in [7,10]. The aim of this Letter is to identify the nature of the MDR log law as an edge turbulent state in wall-bounded flows, leading to a derivation of the parameters appearing in Eq. (4) without any viscoelastic input.

Wall-bounded turbulence in Newtonian fluids is discussed [2,11] by considering the fluid velocity $\boldsymbol{U}(\boldsymbol{r})$ as a sum of its average (over time) and a fluctuating part:

$$\boldsymbol{U}(\boldsymbol{r}, t) = \boldsymbol{V}(y) + \boldsymbol{u}(\boldsymbol{r}, t), \qquad \boldsymbol{V}(y) \equiv \langle \boldsymbol{U}(\boldsymbol{r}, t) \rangle. \tag{5}$$

The objects that enter the Newtonian theory are the mean shear $S(y)$, the Reynolds stress $W(y)$, and the kinetic energy $K(y)$ [12]:

$$\begin{gathered} S(y) \equiv dV(y)/dy, \qquad W(y) \equiv -\langle u_x u_y \rangle, \\ K(y) = \langle |\boldsymbol{u}|^2 \rangle / 2. \end{gathered} \tag{6}$$

In the presence of dilute polymers added to the Newtonian fluid one needs to complement these statistical objects with the dimensionless "conformation tensor" $\boldsymbol{R}(\boldsymbol{r}, t)$ which stems from the ensemble average of the dyadic product of the end-to-end distance of the polymer chains, normalized by its equilibrium value ρ_0^2 [13,14]. The way that the conformation tensor appears in the additional stress tensor which appears in the viscoelastic equations of motion is model dependent; it is different for flexible and rigid polymers, and it also depends on the actual model of the polymers. Nevertheless it was shown theoretically [7,15] that both for rigid and flexible polymers one can write down eventually the balance equations for momentum and energy at distance y away from the wall as

$$\nu(y) S(y) + W(y) = p' L, \qquad y \ll L, \tag{7}$$

$$\left[\nu(y) \frac{a^2}{y^2} + \frac{b\sqrt{K}}{y} \right] K(y) = W(y) S(y), \tag{8}$$

$$\nu(y) \equiv \nu_0 + C \nu_p \langle R_{yy} \rangle. \tag{9}$$

In Eq. (7) the right-hand side is the rate at which momentum is generated by the pressure head, $W(y)$ is the momentum flux in physical space towards the wall, and $\nu(y)S(y)$ stands for the Newtonian viscous dissipation of momentum in addition to the polymer contribution to the dissipation of momentum. The effective viscosity $\nu(y)$ is given by Eq. (9) where ν_p is the viscosity due to the polymers in the limit of zero shear and C is a constant of the order of unity. Similarly, in Eq. (8) the first term on the left -hand side is the combined (Newtonian and polymer) contributions to the dissipation of turbulent kinetic energy due to viscous effect near the wall while the second term represents the dissipation of turbulent kinetic energy parametrized using inertial energy fluxes. The right-hand side of (9) is the (exact) energy production rate. The coefficients a and b are dimensionless constants. The balance equations need to be supplemented by a relation between $K(y)$ and $W(y)$. Experiments [11] show that

$$W(y) = c_v^2 K(y), \tag{10}$$

with c_v apparently y independent outside the viscous boundary layer. To derive the functional form of the MDR [5] one asserts that the terms containing $\nu(y)$ in the balance Eqs. (7) and (8) overwhelm the inertial terms, and then together with (10) one derives $S(y) \sim \text{const}/y$ which is the log law for the MDR. As previously stated, our present aim is to compute the slope of the MDR log law.

The crucial new insight that will explain the universality of the MDR and furnish the basis for its calculation is that the MDR is a marginal flow state of wall-bounded turbulence: attempting to increase $S(y)$ beyond the MDR results in the collapse of the turbulent solutions in favor of a stable laminar solution $W = 0$. As such, the MDR is universal by definition, and the only question is whether a polymer (or other additive) can supply the particular effective viscosity $\nu(y)$ that drives Eqs. (7) and (8) to attain the marginal solution that maximizes the velocity profile. We predict that the same marginal state will exist in numerical solutions of the Navier-Stokes equations furnished with a y-dependent viscosity $\nu(y)$. There will be no turbulent solutions with velocity profiles higher than the MDR.

To see this explicitly, we first rewrite the balance equations in wall units. For constant viscosity [i.e., $\nu(y) \equiv \nu_0$], Eqs. (7)–(10) form a closed set of equations for $S^+ \equiv S\nu_0/(p'L)$ and $W^+ \equiv W/p'L$ in terms of two dimensionless constant $\delta^+ \equiv a\sqrt{K/W}$ (the thickness of the viscous boundary layer) and $\kappa_K \equiv b/c_V^3$ (the Von Kármán constant). Newtonian experiments and simulations agree well with a fit using $\delta^+ \sim 6$ and $\kappa_K \sim 0.436$ (see the black continuos line in Fig. 1 which shows the mean velocity profile using these very constants). Once the effective viscosity $\nu(y)$ is no longer constant we expect c_V to change and consequently the two dimensionless constants will change as well. We will denote the new constants as Δ and κ_C, respectively. Clearly one must require that for $\nu(y)/\nu_0 \to 1$, $\Delta \to \delta^+$ and $\kappa_C \to \kappa_K$. The balance equations are now written as

$$\nu^+(y^+)S^+(y^+) + W^+(y^+) = 1, \tag{11}$$

$$\nu^+(y^+)\frac{\Delta^2}{y^{+2}} + \frac{\sqrt{W^+}}{\kappa_C y^+} = S^+, \tag{12}$$

where $\nu^+(y^+) \equiv \nu(y^+)/\nu_0$. Substituting S^+ from Eq. (11) into Eq. (12) leads to a quadratic equation for $\sqrt{W^+}$. This equation has as a zero solution for W^+ (laminar solution) as long as $\nu^+(y^+)\Delta/y^+ = 1$. Turbulent solutions are possible only when $\nu^+(y^+)\Delta/y^+ < 1$. Thus at the edge of existence of turbulent solutions we find $\nu^+ \propto y^+$ for $y^+ \gg 1$. This is not surprising. As it was observed already in previous works, MDR solution is consistent with an effective viscosity which is asymptotically linear in y^+ [5,8]. It is therefore sufficient to seek the edge solution of the velocity profile with respect to linear viscosity profiles, and we rewrite Eqs. (11) and (12) with an effective viscosity that depends linearly on y^+ outside the boundary layer of thickness δ^+:

$$[1 + \alpha(y^+ - \delta^+)]S^+ + W^+ = 1, \tag{13}$$

$$[1 + \alpha(y^+ - \delta^+)]\frac{\Delta^2(\alpha)}{y^{+2}} + \frac{\sqrt{W^+}}{\kappa_C y^+} = S^+. \tag{14}$$

Where we endowed Δ with an explicit dependence on the slope of the effective viscosity $\nu^+(y)$, $\Delta = \Delta(\alpha)$. Since drag reduction must involve a decrease in W, we expect the ratio a^2K/W to depend on α, with the constraint that $\Delta(\alpha) \to \delta^+$ when $\alpha \to 0$. Although Δ, δ^+ and α are all dimensionless quantities, physically Δ and δ^+ represent (viscous) length scales (for the linear viscosity profile and for the Newtonian case, respectively) while α^{-1} is the scale associated to the slope of the linear viscosity profile. It follows that $\alpha\delta^+$ is dimensionless even in the original physical units. It is thus natural to present $\Delta(\alpha)$ in terms of a dimensionless scaling function $f(x)$,

$$\Delta(\alpha) = \delta^+ f(\alpha\delta^+). \tag{15}$$

Obviously, $f(0) = 1$. In the Appendix we show that the balance Eqs. (13) and (14) (with the prescribed form of the effective viscosity profile) have a nontrivial symmetry that leaves them invariant under rescaling of the wall units. This symmetry dictates the function $\Delta(\alpha)$ in the form

$$\Delta(\alpha) = \frac{\delta^+}{1 - \alpha\delta^+}. \tag{16}$$

Armed with this knowledge we can now find the maximal possible velocity far away from the wall, $y^+ \gg \delta^+$. There the balance equations simplify to

$$\alpha y^+ S^+ + W^+ = 1, \tag{17}$$

$$\alpha\Delta^2(\alpha) + \sqrt{W^+}/\kappa_C = y^+ S^+. \tag{18}$$

These equations have the y^+-independent solution for $\sqrt{W^+}$ and y^+S^+:

$$\sqrt{W^+} = -\frac{\alpha}{2\kappa_C} + \sqrt{\left(\frac{\alpha}{2\kappa_C}\right)^2 + 1 - \alpha^2\Delta^2(\alpha)},$$
$$y^+S^+ = \alpha\Delta^2(\alpha) + \sqrt{W^+}/\kappa_C. \tag{19}$$

By using Eq. (19) (see Fig. 2), we obtain that the edge solution ($W^+ \to 0$) corresponds to the supremum of y^+S^+, which happens precisely when $\alpha = 1/\Delta(\alpha)$. Using Eq. (16) we find the solution $\alpha = \alpha_m = 1/2\delta^+$. Then $y^+S^+ = \Delta(\alpha_m)$, giving $\kappa_v^{-1} = 2\delta^+$. Using the estimate $\delta^+ \approx 6$ we get the final prediction for the MDR. Using Eq. (3) with $\kappa_v^{-1} = 12$, we get

$$V^+(y^+) \approx 12\ln y^+ - 17.8. \tag{20}$$

This result is in close agreement with the empirical law (4) proposed by Virk. The value of the intercept on the right-hand side of Eq. (20) follows from Eq. (3) which is based on matching the viscous solution to the MDR log law in [5]. Note that the numbers appearing in Virk's law correspond to $\delta^+ = 5.85$, which is well within the error bar on the value of this Newtonian parameter. Note that we can easily predict where the asymptotic law turns into the viscous layer upon the approach to the wall. We can consider an infinitesimal W^+ and solve Eqs. (11) and (12) for S^+ and the viscosity profile. The result, as before, is $\nu^+(y) = \Delta(\alpha_m)y^+$. Since the effective viscosity cannot fall bellow the Newtonian limit $\nu^+ = 1$ we see that the MDR cannot go below $y^+ = \Delta(\alpha_m) = 2\delta^+$. We thus expect an extension of the viscous layer by a factor of 2, in very good agreement with the experimental data.

Note that the result $W^+ = 0$ should not be interpreted as $W = 0$. The difference between the two objects is the factor of Re^2, $W \propto \mathrm{Re}^2 W^+$. Since the MDR is reached asymptotically as $\mathrm{Re} \to \infty$, there is enough turbulence at

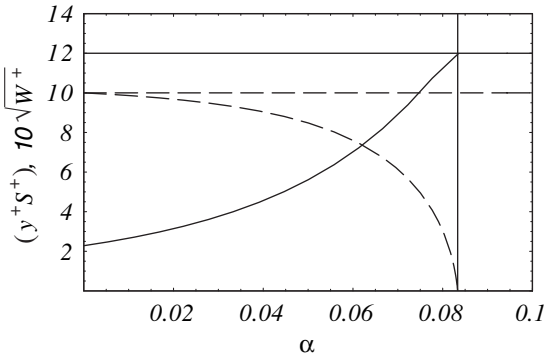

FIG. 2. The solution for $10\sqrt{W^+}$ (dashed line) and y^+S^+ (solid line) in the asymptotic region $y^+ \gg \delta^+$, as a function of α. The vertical solid line $\alpha = 1/2\delta^+ = 1/12$ which is the edge of turbulent solutions; since $\sqrt{W^+}$ changes sign here, to the right of this line there are only laminar states. The horizontal solid line indicates the highest attainable value of the slope of the MDR logarithmic law $1/\kappa_v = 12$.

this state to stretch the polymers to supply the needed effective viscosity. Indeed our discussion is in close correspondence with the experimental remark by Virk [1] that close to the MDR asymptote the flow appears laminar.

In summary, one can probably improve further the model of the Newtonian wall-bounded flow, making it more elaborate and more precise. But the message of this Letter will stay unchanged; whatever is the model of choice, once endowed with effective viscosity $\nu(y)$ instead of ν_0, there would exist a profile of $\nu(y)$ that would result in a maximal possible velocity profile at the edge of existence of turbulent solutions. That profile is the prediction of the said model of choice for the MDR. In particular, we offer a prediction for simulations: direct numerical simulations of the Navier-Stokes equations in a channel, endowed with a linear viscosity profile [8], will not be able to support turbulent solutions when the slope of the viscosity profile exceeds the critical value that is in correspondence with the slope of the MDR. Notwithstanding, it is gratifying to discover that even a simple model of the balance of energy and momentum is sufficient, in light of the insight presented in this Letter, to predict *ab initio* the functional form *and* the parameters that determine the maximum drag reduction asymptote.

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Appendix.—Consider the following identity:

$$\begin{aligned}\nu^+(y^+) &= 1 + \alpha(y^+ - \delta^+) \\ &= [1 + \alpha(y^+ - \delta) + \alpha(\delta - \delta^+)] \\ &= g(\delta)\left[1 + \frac{\alpha}{g(\delta)}(y^+ - \delta)\right],\end{aligned} \tag{A1}$$

where

$$g(\delta) \equiv 1 + \alpha(\delta - \delta^+), \qquad \delta \geq \delta^+. \tag{A2}$$

Next introduce newly renormalized units using the effective viscosity $g(\delta)$, i.e.,

$$\begin{aligned} y^\ddagger &\equiv y^+ g(\delta), & \delta^\ddagger &\equiv \delta g(\delta), \\ S^\ddagger &\equiv S^+ g(\delta), & W^\ddagger &\equiv W^+. \end{aligned} \tag{A3}$$

In these variables the balance equations read

$$[1 + \alpha(y^\ddagger - \delta^\ddagger)]S^\ddagger + W^\ddagger = 1, \tag{A4}$$

$$[1 + \alpha(y^\ddagger - \delta^\ddagger)]\frac{\Delta^2(\alpha)}{y^{\ddagger 2}} + \frac{\sqrt{W^\ddagger}}{\kappa_K y^\ddagger} = S^\ddagger. \tag{A5}$$

These equations are isomorphic to (13) and (14) with δ^+ replaced by $\delta^\ddagger$. The ansatz (15) is then replaced by $\Delta(\alpha) = \delta^+ g(\delta)^{-1} f(\alpha\delta^\ddagger)$. This form is dictated by the following considerations: (i) $\Delta(\alpha) \to \delta^+$ when $\alpha \to 0$, (ii) all length scales in the rescaled units are divided by $g(\delta)$, and thus the prefactor in front of f becomes $\delta^+/g(\delta)$, and (iii) $\alpha\delta^+$ in Eq. (12) is now replaced in Eq. (A5) by $\alpha\delta^\ddagger$, leading to the new argument of f. Since the function $\Delta(\alpha)$ cannot change due to the change of variables, the function $\Delta(\alpha)$ should be identical to that given by Eq. (15):

$$\delta^+ f(\alpha\delta^+) = \frac{\delta^+}{g(\delta)} f(\alpha\delta^\ddagger). \tag{A6}$$

Using the explicit form of $g(\delta)$ Eq. (A2), and choosing (formally first) $\delta = \delta^\ddagger = 0$ we find that $f(\xi) = 1/(1 - \xi)$. It is easy to verify that this is indeed the solution of the above equation for any value of $\delta^\ddagger$, and therefore the unique form of Eq. (16).

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