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Energy Spectra of Developed Turbulence in Helium Superfluids

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We suggest a "minimal model" for the 3D turbulent energy spectra in superfluids, based on their two-fluid description. We start from the Navier–Stokes equation for the normal fluid and from the coarse-grained hydrodynamic equation for the superfluid component (obtained from the Euler equation for the superfluid velocity after averaging over the vortex lines) and introduce a mutual friction coupling term, proportional to the counterflow velocity, the average superfluid vorticity and to the temperature dependent parameter $q = \alpha/(1+\alpha')$*, where* α *and* α' *denote the dimensionless parameters characterizing the mutual friction between quantized vortices and the normal component of the liquid. We then derive the energy balance equations, taking into account the cross-velocity correlations. We obtain all asymptotical solutions for normal and superfluid energy spectra for limiting cases of small/big normal to superfluid density ratio and coupling. We discuss the applicability of our model to superfluid He II and to* ^{3}*He-B.*

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1. INTRODUCTION

Quantum turbulence[1] containing a tangle of singly quantized vortex line—such as turbulence in superfluid ^{4}He (He-II) and in the superfluid B-phase of ^{3}He (^{3}He-B)—besides of being for half a century[2] a playground for low temperature physicists increasingly attracts attention of the fluid dynamics community. In this paper, we address the important question of the turbulent energy distribution between scales (3D energy spectra) of such turbulence. In view of very sparse experimental data—besides indirect indications of its form in He II[1,3] and ^{3}He–B[4] deduced from various decay measurements or computer simulations[5–7] we recall the only direct measurement of the velocity spectrum based on pressure fluctuations in He II by Maurer and Tabeling[8]—there is a clear call to tackle this important issue theoretically. We consider the simplest case of classically generated turbulence in quantum liquids such as He II or ^{3}He–B that can be thought of as isothermal, homogeneous and isotropic. We work within a framework of the two-fluid model, building on ideas first introduced by Volovik[9] and Vinen,[10] by further developing our previous work.[11,12] We stress that our approach does not directly apply to quantum turbulence generated in superfluid helium by the thermal counterflow,[2] which is anisotropic, being generated thermally, by the temperature gradient in the channel.

Above the pressure dependent transition temperature ($T_\lambda \approx 2.17\,$K for ^{4}He and $T_c \approx 1\,$mK for ^{3}He) both ^{4}He and ^{3}He are ordinary viscous fluids that can be described by the Navier–Stokes equations and their turbulent flow is fully classical. From hydrodynamical viwepoint normal ^{4}He and ^{3}He liquids differ from each other mainly because of their very different values of kinematic viscosity. While liquid ^{4}He above T_λ possesses the lowest kinematic viscosity, ν, of all known fluids,[13] of order $\nu_4 \approx 2\times 10^{-4}$ cm^2/s (50 times smaller then that of water at room temperature), liquid ^{3}He at millikelvin temperature is a Fermi liquid[14] ($\nu_3 \propto T^{-2}$) with kinematic viscosity exceeding that of air (which is about $0.15\,$cm^{2}s) comparable with olive oil, of order $\nu_3 \approx 1\,$cm^2/s. In principle, both these liquids may become turbulent.

The plan of the paper is as follows. After this short introductory Section we introduce the continuous minimal model for two-fluid turbulence with mutual friction in Section 2 and we use it to derive the turbulent energy spectra in Section 3. We discuss the applicability of this continuous model and specify the crucial role of Kelvin waves in Section 4. We conclude and outline the future work in Section 5.

2. MINIMAL MODEL FOR TWO-FLUID TURBULENCE WITH MUTUAL FRICTION

2.1. Basic Equations for the Two-Fluid Model

In this paper we adopt the simplest form of the two fluid model for superfluid ^{3}He and ^{4}He (see e.g. Eqs. (2.2) and (2.3) in the Donnelly's textbook[15]) which neglects both bulk viscosity and thermal conductivity. These are the Euler Equation for the superfluid velocity $\boldsymbol{u}_{\rm s}$ (with zero viscosity $\nu_s = 0$) and the Navier–Stokes Equation for the normal component $\boldsymbol{u}_n$ (with the kinematic viscosity $\nu_n \equiv \nu$):

$$\rho_s \left[\frac{\partial \boldsymbol{u}_s}{\partial t} + (\boldsymbol{u}_s \cdot \nabla)\boldsymbol{u}_s \right] - \nabla_{\rho s} = -\boldsymbol{F}_{ns}, \tag{1a}$$

$$\rho_n \left[\frac{\partial \boldsymbol{u}_n}{\partial t} + (\boldsymbol{u}_n \cdot \nabla)\boldsymbol{u}_n \right] - \nabla_{pn} - \rho_n \nu \Delta \boldsymbol{u}_i = \boldsymbol{F}_{ns}. \tag{1b}$$

Here ρ_n, ρ_s are the densities of the normal and superfluid components, ρ_n, ρ_s are corresponding pressures:

$$p_n = \frac{\rho_n}{\rho} \left[p + \rho_s |\boldsymbol{u}_s - \boldsymbol{u}_n|^2 \right], \tag{1c}$$

$$p_s = \frac{\rho_s}{\rho} \left[p + \rho_n |\boldsymbol{u}_s - \boldsymbol{u}_n|^2 \right], \tag{1d}$$

Heres $\rho \equiv \rho_s + \rho_n$ and $\boldsymbol{F}_{ns}$ describes the mutual friction:

$$\boldsymbol{F}_{ns} = -\rho_s \{ \alpha'(\boldsymbol{u}_s - \boldsymbol{u}_n) \times \boldsymbol{\omega} + \alpha \hat{\boldsymbol{\omega}}_s \times [\boldsymbol{\omega}_s \times (\boldsymbol{u}_s - \boldsymbol{u}_n)] \}. \tag{1e}$$

$\boldsymbol{\omega}_s \equiv \nabla \times \boldsymbol{u}_s$ is the superfluid vorticity; $\hat{\boldsymbol{\omega}}_s \equiv \boldsymbol{\omega}_s \times \boldsymbol{\omega}_s$; α' and α are dimensionless parameters describing the mutual friction between superfluid and normal components of the liquid mediated by quantized vortices which transfer momenta from the superfluid to the normal subsystem and vice versa. For the flow with vortices locally aligned with each other these parameters enter the reactive and dissipative forces acting on a vortex line as it moves with respect to the normal component. Here we consider α' and α as phenomenological parameters, assuming the general case where quantized vortices are not aligned locally and thus the bare parameters are renormalized.

Following Ref. 11 we approximate the "mutual friction term" $\boldsymbol{F}_{ns}$, Eq. (1e), as follows:

$$\boldsymbol{F}_{ns} = Q\omega_0(\boldsymbol{u}_s - \boldsymbol{u}_n). \tag{1f}$$

Here $Q \simeq \alpha\rho_s$ and ω_o is some characteristic superfluid vorticity

$$\omega_0 \equiv \sqrt{\langle |\boldsymbol{\omega}_s| \rangle^2}. \tag{1g}$$

Approximation (1f) accounts for the fact that the vorticity in developed turbulence usually is dominated by the smallest eddies in the system of scale η with the largest characteristic wave-vector $k_\eta \sim 1/\eta$. These eddies have the smallest turnover time $\tau_{\min}$ that is of the order of their decorrelation time. On the contrary, the main contribution to the velocity in the equation for the dissipation of the k-eddies with intermediate wave-vectors k, $k \ll k_{\max}$, is dominated by the k'-eddies with $k' \sim k$. Because the turnover time of these eddies $\tau_{k'} \gg \tau_{\min}$, approximation (1f) adopts self-averaging of Eq. (1e) on time intervals of interest ($\tau_{\min} \ll \tau \ll \tau_{k'}$) and thus vorticity can be considered as almost uncorrelated with the velocity $\boldsymbol{u}$ which is a dynamical variable. Actually, (1f) is the mean field approximation that neglects the fluctuating part of vorticity and replaces it by by its mean value.

2.2. Model for the Cross-Correlation $\langle \boldsymbol{u}_{\mathrm{s}} \cdot \boldsymbol{u}_{\mathrm{n}} \rangle$

Our first goal in this paper is to formulate the energy balance equations for the one-dimensional energy spectra $E_{ii}(k)$ for the normal and the superfluid subsystems, related with the simultaneous, same-point correlations $\langle \boldsymbol{u}_i \cdot \boldsymbol{u}_j \rangle$ as follows:

$$\int dk E_{ij}(k) = E_{ij} = 2\langle \boldsymbol{u}_i \cdot \boldsymbol{u}_j \rangle. \tag{2}$$

2.2.1. Some Definitions and Known Relationships

To find the cross-correlation $\langle \boldsymbol{u}_i \cdot \boldsymbol{u}_j \rangle$ we need to recall some definitions and relationships, well known in statistical physics. The first one is the Fourier transform:

$$\boldsymbol{u}_i(\boldsymbol{k}, \omega, t) \equiv \int \frac{d\boldsymbol{k} d\omega}{(2\pi)^4} \boldsymbol{u}_i(\boldsymbol{r}, t) \exp[-i(\boldsymbol{k} \cdot \boldsymbol{r} - \omega t)], \tag{3}$$

and the definition of the two-point, different-time cross-correlation functions $F_{ij}(\boldsymbol{k}, \omega)$:

$$\begin{aligned} &\langle \boldsymbol{u}_i(\boldsymbol{k}, \omega) \cdot \boldsymbol{u}_j^*(\boldsymbol{k}', \omega') \rangle \\ &\equiv (2\pi)^4 \delta(\boldsymbol{k} - \boldsymbol{k}') \delta(\omega - \omega') F_{ij}(\boldsymbol{k}, \omega). \end{aligned} \tag{4a}$$

Next we need to define simultaneous two-point (cross) correlators, $F_{ij}(\boldsymbol{k})$, in $\boldsymbol{k}$-representation:

$$\langle \boldsymbol{u}_i(\boldsymbol{k}, t) \cdot \boldsymbol{u}_j^*(\boldsymbol{k}', t) \rangle \equiv (2\pi)^3 \delta(\boldsymbol{k} - \boldsymbol{k}') F_{ij}(\boldsymbol{k}). \tag{4b}$$

The frequency integral of $F_{ij}(\boldsymbol{k}, \omega)$ produces $F_{ij}(\boldsymbol{k})$:

$$2\pi F_{ij}(\boldsymbol{k}) = \int d\omega F_{ij}(\boldsymbol{k}, \omega). \tag{4c}$$

Further $\boldsymbol{k}$-integration gives same-point correlations, which has a sense of (twice) kinetic energy density per unite mass of the normal (for $i = j = n$) or the superfluid (for $i = j = s$) kinetic energy density per unite mass:

$$(2\pi)^{-3} \int d^3k F_{ij}(\boldsymbol{k}) = \langle \boldsymbol{u}_i \cdot \boldsymbol{u}_j \rangle \equiv 2E_{ij}. \tag{4d}$$

Our approach is formulated in terms of one-dimensional density of E_{ij} in k-space,

$$E_{ij}(k) = k^2 F_{ij}(k)/(2\pi)^2, \tag{4e}$$

defined such that

$$\int dk E_{ij}(k) = E_{ij}. \tag{4f}$$

Finally, we define a scalar version of the velocity Green's (response) function:

$$\left\langle \frac{\delta u_i(\boldsymbol{k}, \omega)}{\delta f_i(\boldsymbol{k}', \omega')} \right\rangle = (2\pi)^4 \delta(\boldsymbol{k} - \boldsymbol{k}') \delta(\omega - \omega') G_{ij}(\boldsymbol{k}, \omega) \tag{5}$$

To determine $\langle \boldsymbol{u}_i \cdot \boldsymbol{u}_j \rangle$ we adopt a few more or less justified assumptions, discussed below.

2.2.2. Fluctuation–Dissipation Approximation for the Diagonal Turbulent Correlation Function

In the thermodynamical equilibrium, the different-time (cross) correlation functions (in the ω-representation) are related with the Greens' function by the mean of the fluctuation-dissipation theorem, which in hydrodynamics reads:

$$F_{ii}(\boldsymbol{k}, \omega) = 2T \operatorname{Im}[G_{ii}(\boldsymbol{k}, \omega)], \tag{6}$$

where T is the temperature of the system. Having in mind that the thermodynamical equilibrium is the equipartition of the energy with $T/2$, being the energy per degree of freedom (which in our notation is $F_{ii}(k)/2$,) we can write in equilibrium:

$$F_{ii}(\boldsymbol{k}, \omega) = 2F_{ii}(\boldsymbol{k}) \operatorname{Im}[G_{ii}(\boldsymbol{k}, \omega)]. \tag{7}$$

Our conjecture is that this equation is approximately (on a semi-qualitative level) valid also in the flux-equilibrium state, i.e., in the fully developed turbulence.

2.2.3. One-pole Approximation for the Diagonal Greens Functions

Generally speaking, the Greens function in the developed turbulence is very involved function of $\boldsymbol{k}$ and ω. The only facts that we know for sure are:

1. $G_{ii}(\boldsymbol{k},\omega)$ is analytical in the lower half-plane.
2. $\int_{-\infty}^{\infty} G_{ii}(\boldsymbol{k},\omega)d\omega=\pi$, $\lim_{\omega\to\infty} G_{ii}(\boldsymbol{k},\omega)=1/\omega$
3. At given $\boldsymbol{k}$, $G_{ii}(\boldsymbol{k},\omega)$ has a characteristic width (in ω), of about the total damping frequency in the system, i.e., $\nu_i k^2+\gamma i(k)+q_i\omega_0$.

The simplest analytical from, which satisfies these requirements is the so-called one-pole approximation

$$G_{ii}(\boldsymbol{k},\omega)\approx\frac{1}{\omega-i[\nu_i k^2+\gamma i(k)+q_i\omega_0]}, \tag{8a}$$

widely used in theoretical physics in general and in the theory of turbulence in particular. In Eq. (8) $\gamma_i(k)$ is the eddy-decorrelation frequency, which is the same as the eddy-turnover frequency. The Kolmogorov 1941 (K41) approximation for this object reads:

$$\gamma_i(k)\approx C_1\varepsilon_i(k)^{l/3}k^{2/3}, \tag{8b}$$

where $\varepsilon_i(k)$ is the energy flux in the i-subsystem. The well known effective turbulent viscosity $\nu_{\mathrm{T}}(k)$ is related with $\gamma(k)$ in the following way: $\nu_{\mathrm{T}}(k)\equiv\gamma(k)/k^2\propto k^{-4/3}$.

2.2.4. Approximation of the Gaussian Statistics

In the limit $\rho_{\mathrm{n}}\gg\rho_{\mathrm{s}}$ the turbulent velocity $\boldsymbol{u}_{\mathrm{n}}$ (or $\boldsymbol{u}_{\mathrm{s}}$ in the opposite limiting case, $\rho_{\mathrm{s}}\gg\rho_{\mathrm{n}}$) can be considered as a given one, independent of $\boldsymbol{u}_{\mathrm{s}}$ (or $\boldsymbol{u}_{\mathrm{n}}$, for $\rho_{\mathrm{s}}\gg\rho_{\mathrm{n}}$). Having in mind that in practice we are dealing with moderate extend of the inertial interval, and that the statistics of turbulence in the energy contained interval is very close to the Gaussian one, we can approximate the statistics of turbulent field $\boldsymbol{u}_{\mathrm{n}}$ (or $\boldsymbol{u}_{\mathrm{s}}$, for $\rho_{\mathrm{s}}\gg\rho_{\mathrm{n}}$) as Gaussian.

2.2.5. Here a Frog Jumps

As we suggested, in the limit $\rho_{\mathrm{n}}\gg\rho_{\mathrm{s}}$ we can consider the cross-velocity term $q_{\mathrm{s}}\omega_0\boldsymbol{u}_{\mathrm{n}}\equiv f_{\mathrm{s}}$ in the RHS of Eq. (1) for $\boldsymbol{u}_{\mathrm{s}}$ as a Gaussian random force f_{s} and compute the cross-correlation (for simplicity in the scalar version), $\langle u_{\mathrm{s}}f_{\mathrm{s}}\rangle$, using so called Gaussian integration by parts: $\langle u_{\mathrm{s}}f_{\mathrm{s}}^*\rangle$

$=\langle\delta u_{\rm s}/\delta f_{\rm s}\rangle\langle f_{\rm s}f_{\rm s}^{*}\rangle$. With $f_{\rm s}=q_{\rm s}\omega_0 u_{\rm n}$ this gives for the velocity cross-correlation:

$$F_{\rm sn}(\boldsymbol{k},w)=q_{\rm s}\omega_0 F_{\rm nn}(\boldsymbol{k},w)G_{\rm ss}(\boldsymbol{k},w). \tag{9}$$

Using our approximation (6) for $F_{\rm nn}(k,w)$, approximation (8a) for $G_{\rm nn}(k,w)$ and $G_{\rm ss}(k,w)$ relations (4c) and (4d), one gets after ω-integration:

$$E_{\rm sn}(k)\equiv q_{\rm s}\omega_0 E_{\rm n}(k)/\Delta_k, \quad \text{for} \quad \rho_{\rm n}\gg\rho_{\rm s}, \tag{10a}$$

where Δ_k is given below by (11b). Similarly, in the opposite limiting case:

$$E_{\rm sn}(k)\equiv q_{\rm n}\omega_0 E_{\rm s}(k)/\Delta_k, \quad \text{for} \quad \rho_{\rm n}\gg\rho_{\rm s}, \tag{10b}$$

For an arbitrary relation between $\rho_{\rm n}$ and $\rho_{\rm s}$ (including $\rho_{\rm s}\sim\rho_{\rm n}$) we suggest the interpolation formula

$$E_{\rm sn}(k) = \omega_0[q_{\rm n}E_{\rm s}(k)+q_{\rm s}E_{\rm n}(k)]/\Delta_k, \tag{11a}$$

$$\Delta_k \equiv \nu k^2+\gamma_{\rm n}(k)+\gamma_{\rm s}(k)+(q_{\rm s}+q_{\rm n})\omega_0, \tag{11b}$$

which obeys all needed limiting cases. Moreover, in the limiting case $\omega_0 q_i\gg\gamma_i$ it turns into a simpler form

$$E_{\rm sn}(k)=[\rho_{\rm s}E_{\rm s}(k)+\rho_{\rm n}E_{\rm n}(k)]/\rho. \tag{12a}$$

This equation has a physically motivated solution

$$E_{\rm sn}(k)=E_{\rm s}(k)=E_{\rm n}(k), \tag{12b}$$

that gives $\langle\boldsymbol{u}_{\rm n}(\boldsymbol{k},t)[\boldsymbol{u}_{\rm n}(\boldsymbol{k},t)-\boldsymbol{u}_{\rm s}(\boldsymbol{k},t)]\rangle=0$ and thus requires

$$\boldsymbol{u}_{\rm n}(\boldsymbol{r},t)=\boldsymbol{u}_{\rm s}(\boldsymbol{r},t), \tag{12c}$$

i.e., a fully coherent motion of the superfluid and the normal fluid velocities. It means that our interpolation (11) is physically justified and actually works better than one would expect, having in mind our rather crude approximations.

2.3. The Energy Balance Equations

The energy balance equations can be derived in a standard manner from the equations of motion, Eqs. (1)—see e.g. Ref. 11. The only new factor in this derivation is the cross-velocity correlations, for which we adopt Eqs. (11). This gives:

$$\begin{aligned} &\frac{\partial E_{\rm n}(k)}{\partial t}+\frac{\partial \varepsilon_{\rm n}(k)}{\partial k}+\nu k^2\\ &=q_{\rm n}\omega_0\left\{\tfrac{\omega_0}{\Delta_k}[q_{\rm s}E_{\rm n}(k)+q_{\rm n}E_{\rm s}(k)]-E_{\rm n}(k)\right\} \end{aligned} \tag{13a}$$

$$\begin{aligned} &\frac{\partial E_{\rm s}(k)}{\partial t}+\frac{\partial \varepsilon_{\rm s}(k)}{\partial k}\\ &=q_{\rm s}\omega_0\left\{\tfrac{\omega_0}{\Delta_k}[q_{\rm s}E_{\rm n}(k)+q_{\rm n}E_{\rm s}(k)]-E_{\rm s}(k)\right\}, \end{aligned} \tag{13b}$$

with Δ_k, given by Eq. (11b). The simplest way to model $\varepsilon_i(k)$, suggested in Ref. [16], is to relate $E_i(k)$ and $\varepsilon_i(k)$ in Eq. (13) in the spirit of the K41 dimensional reasoning:

$$E_i(k)=C\varepsilon_i(k)^{2/3}k^{-5/3}. \tag{14}$$

Here $C\simeq 1$ is the Kolmogorov dimensionless constant. In the absence of dissipation, Eq. (14) immediately produces the stationary solution $\varepsilon_k=\varepsilon$ with constant energy flux ε in the inertial interval of scales. Then Eq. (14) turns into the Kolmogorov-Obukhov 5/3–law for E_k:

$$E_k=C\varepsilon^{2/3}k^{-5/3}. \tag{15}$$

3. TURBULENT ENERGY SPECTRA IN THE MINIMAL MODEL

3.1. Small Normal Density

Let us first consider the case

$$\rho_{\rm s}\gg\rho_{\rm n}, \text{and thus}: q_{\rm s}\ll q_{\rm n}. \tag{16}$$

Clearly, in this case the massive superfluid component does not feel the tiny superfluid one and thus in the zeroth-order approximation [with respect of $(\rho_{\rm n}/\rho_{\rm s})\ll 1$]

$$E_{\rm s}(k)=C\varepsilon_{\rm s}^{2/3}k^{-5/3}, \quad \text{K41 spectrum}. \tag{17}$$

Indeed, in the limits $\rho_{\rm s}\to 0$ one has zero in the RHS of Eq. (13b). Then in the stationary cases $\partial\varepsilon_{\rm s}/\partial k=0$, i.e., $\varepsilon_{\rm s}$ becomes k-independent, and from Eq. (14) one immediately gets Eq. (17).

Much more interesting question is about a spectrum of the normal component of the small density, which is essentially affected by the massive superfluid component. In the limit (16) and for the stationary case Eqs. (13a) and (11b) for $E_n(k)$ take the form

$$\frac{\partial \varepsilon_n(k)}{\partial k} + \nu k^2 E_n(k) = q_n \omega_0 \left[\frac{\omega_0 q_n E_s(k)}{\Delta_k} - E_n(k) \right], \tag{18a}$$

$$\Delta_k \equiv \nu k^2 + \gamma_n(k) + \gamma_s(k) + q_n \omega_0, \tag{18b}$$

that will be analyzed for the two limiting cases.

3.1.1. a. Small $k \Rightarrow$ Full Coupling

For small k, γ_s (being $\propto k^{2/3}$) is small with respect of the (k-independent) $\omega_0 q_n$. If so, $\Delta_k \to q_n \omega_0$ and

$$\partial \varepsilon_n(k)/\partial k = q_n [E_n(k) - E_s(k)], \tag{19a}$$

with the K41 solution

$$E_s(k) = E_n(k) = C \varepsilon^{2/3} k^{-5/3}, \quad \varepsilon_s = \varepsilon_n = \varepsilon, \tag{19b}$$

having full coupling of the velocities, $\boldsymbol{u}_s = \boldsymbol{u}_n$.

3.1.2. Large $k \Rightarrow$ Decoupling with K41 Regime

K41 viscous micro-scale, $k_\eta \equiv 1/\eta$, defined in the standard manner:

$$\nu k_\eta^2 = \gamma_n(k_\eta), \ \Rightarrow k_\eta \simeq \varepsilon_n^{1/4} / \nu^{3/4}. \tag{20}$$

In absence of superfluid component, the spectrum of developed turbulence of the normal fluid would decay exponentially for $k \gg k_\eta$. We demonstrate that in the two-fluid system this does not occur due to an additional energy flux from the superfluid component to the normal one. Let $k > k_\eta$. Then one simplifies Eqs. (18) to an algebraic form with solution

$$E_n(k) = E_s(k) \frac{q_n^2 \omega_0^2}{(\nu k^2 + q_n \omega_0)^2}. \tag{21a}$$

One finds here a new scale:

$$\nu k_*^2 = q_n \omega_0. \tag{21b}$$

In the case of essential mutual friction, when $k_* \gg k_\eta$, (i.e. $q \gg \nu k_\eta^2$), in the subinterval

$$k_* \gg k \gg k_\eta, \tag{22a}$$

a spectrum of normal component (in the former viscous interval!) deviates from the K41 just slightly:

$$E_{\rm s}(k) - E_{\rm n}(k) \approx E_{\rm s}(k)\frac{\nu^2 k^4}{q_{\rm n}^2 \omega_0^2} \ll 1. \tag{22b}$$

It means that in the subinterval (22a) the mutual friction is still strong enough to keep the normal velocity almost coupled to the superfluid one:

$$|\boldsymbol{u}_{\rm s} - \boldsymbol{u}_{\rm n}| \approx u_{\rm s}\frac{\nu^2 k^4}{2 q_{\rm n}^2 \omega_0^2} \ll u_{\rm s}. \tag{22c}$$

However, for

$$k > k_* \tag{23a}$$

the strong coupling (22c) disappears, as it follows from Eq. (21a). In this region, according to Eq. (21a):

$$E_{\rm s} \gg E_{\rm n} \approx E_{\rm s}\frac{q_{\rm n}^2 \omega_0^2}{\nu^2 k^4} \alpha\, k^{-4-5/3}, \tag{23b}$$

i.e. $E_{\rm n} \ll E_{\rm s}$. However, in contrast with standard K41 spectrum, one has here a power-law rather than an exponential decay.

3.2. Small Superfluid Density

Consider now the opposite limiting case to Eq. (16)

$$\rho_{\rm s} \ll \rho_{\rm n}, \text{and thus}: q_{\rm s} \gg q_{\rm n}. \tag{24}$$

Clearly, then (in the zeroth-order approximation) in the inertial interval of scales, $k < k_\eta$, the normal component obeys the standard K41 spectrum

$$E_{\rm n}(k) = C \varepsilon_{\rm n}^{2/3} k^{-5/3}, \tag{25}$$

in the inertial interval of scales, $k < k_\eta$. The spectrum of the superfluid component is considered below in various limiting cases.

3.2.1. Small Mutual Friction

Consider first the case when, in addition to inequalities (16), one has

$$k_{\dagger} \ll k_{\eta}, \tag{26a}$$

where the new characteristic scale $k_{\dagger}$ is defined by

$$q_{\mathrm{s}}\omega_0 = \gamma_{\mathrm{s}}(k_{\dagger}). \tag{26b}$$

For $k < k_{\dagger}$ the mutual friction dominate over the nonlinear interaction. We already know (see Eq. (12c)) that in this case $\boldsymbol{u}_{\mathrm{n}} = \boldsymbol{u}_{\mathrm{s}}$, i.e., there is a complete coupling of normal and superfluid velocities, both spectra obeying the 5/3-law (19b).

For $k > k_{\dagger}$ the nonlinear interaction dominates over the mutual friction:

$$\langle \boldsymbol{u}_{\mathrm{s}} \cdot \boldsymbol{u}_{\mathrm{n}} \rangle \ll \langle |\boldsymbol{u}_{\mathrm{s}}|^2 \rangle. \tag{27}$$

Nevertheless, both spectra obey 5/3-law with $\varepsilon_{\mathrm{s}} \simeq \varepsilon_{\mathrm{n}}$. Normal spectrum ends at $k \simeq k_{\eta}$. Superfluid component does not feel this cutoff and continues further until the limit of the classical description. Beyond this limit, the energy cascade is taken over by reconnections and Kelvin waves (see below).

3.2.2. Large Mutual Friction

In some sense, richer physics corresponds to the case with larger mutual friction, when

$$k_{\dagger} \gg k_{\eta}. \tag{28}$$

Then one has a coupled turbulent motion of the normal and superfluid components (with $\boldsymbol{u}_{\mathrm{n}}(\boldsymbol{r}, t) = \boldsymbol{u}_{\mathrm{s}}(\boldsymbol{r}, t)$) until the viscous cutoff (20).

For

$$k_{\eta} < k < k_{\dagger} \tag{29}$$

one has a case of strong mutual friction with the normal component at rest. This case has been considered in details in our Ref. 11 (and agrees with numerical result of Vinen[10]), giving the -3-spectrum

$$E_{\mathrm{s}}(k) \simeq q_{\mathrm{s}}^2 \omega_0^2 k^{-3}, \tag{30}$$

that originates from the balance of the nonlinear flux and the friction terms in Eq. (31) below.

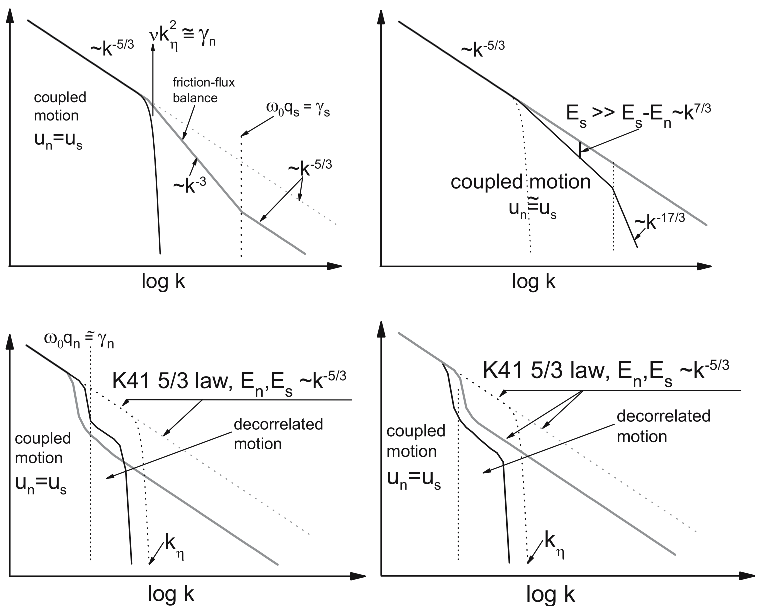

Fig. 1. The asymptotic forms of the coupled 3D energy spectra (schematic log-log plots of energy spectral density versus wave number) in the normal fluid (solid black lines) and in the superfluid (solid grey lines) resulting from the continuous two-fluid model. The dotted lines indicate the form of original conventional uncoupled K41 spectra. The top (bottom) figures correspond to limiting cases of large (low) mutual friction, for large (left) and small (right) normal fluid density. For further details, see text.

Indeed, in this case one puts in Eq. (13b) $E_{\rm n}=0$, $\Delta k \simeq \omega_0 q_{\rm s}$, and the RHS of Eq. (13b) can be approximated as

$$\text{RHS} \approx \omega_0 q_{\rm s} E_{\rm s} \left(\frac{q_{\rm n}}{q_{\rm s}} - 1 \right) \approx -\omega_0 q_{\rm s} E_{\rm s}.$$

In that case, instead of Eq. (13b), one arrives at

$$q_{\rm s}\omega_0 E_{\rm s}(k) = \frac{\partial \varepsilon_{\rm s}(k)}{\partial k} = C^{-3/2} \frac{\partial}{\partial k} [k^{5/2} E_{\rm s}^{3/2}(k)], \qquad (31)$$

which has the solution (30).

For $k > k_\dagger$ the mutual friction is NOT important and the superfluid component recovers the 5/3-law, but with smaller energy flux.

4. ROLE OF KELVIN WAVES

The description we presented above ignores the fact that turbulence of the superfluid component consist of discrete tangled vortices with quantised circulation κ. This description is valid for the scales greater that the mean distance ℓ separating the quantised vortices. For example, in ^{4}He above 1.5 K the normal fluid Kolmogorov scale appears to be of the same order as ℓ and, therefore, in this case our model is not applicable for predicting the superfluid spectrum below the Kolmogorov scale[1]. Another example is turbulence at $T < 1$ K when the normal component is extremely weak and behaves like a Knudsen gas rather than a fluid. In this case, most of energy reaches ℓ along the cascade without dissipation and one should ask what happens to this energy below this scale. At scale ℓ, an essential role in turbulence evolution play vortex reconnections during which part of the energy is lost to phonon emission, and the rest of the energy is transferred to Kelvin waves, see e.g. paper of W.F. Vinen in this issue. The reconnections produce sharp cusps which quickly transform into a superposition of Kelvin waves whose nonlinear interaction leads to further turbulent cascades through scales. Note that these sharp cusps correspond to a broad distribution in wavenumber space and, therefore, both the direct energy cascade and the inverse cascade of waveaction can be important in subsequent evolution.[17] To describe the statistical nonlinear Kelvin waves one can use the weak turbulence approach, which results in a six-wave kinetic equation for the energy spectrum.[18] A differential equation model for Kelvin wave turbulence (kelvulence) which preserved the essential scalings and solutions of the original integral kinetic equation was derived in Ref. 17:

$$\dot{n} = \frac{C}{\kappa^{10}} \omega^{1/2} \frac{\partial^2}{\partial \omega^2} \left(n^6 \omega^{21/2} \frac{\partial^2}{\partial \omega^2} \frac{1}{n} \right), \tag{32}$$

where $n = E_s L/\omega$ is the waveaction spectrum,[a] L is the vortex length per unit volume—the vortex line density, κ is the circulation quantum, C is a dimensionless constant and $\omega = \omega(k) \cong \kappa k^2/(4\pi)$ is the Kelvin wave frequency.[b] This equation preserves the energy (per unit length of the vortex)

$$E = \frac{1}{2\sqrt{\kappa}} \int \omega^{1/2} n \, d\omega \tag{33}$$

[a] Factor L appears when one calculates energy per unit volume in terms of the energy per unit vortex length

[b] Here, we ignore logarithmic factors

and the waveaction (per unit length of the vortex)

$$N = \frac{1}{2\sqrt{\kappa}} \int \omega^{-1/2} n d\omega. \tag{34}$$

Equation (32) has both the direct cascade solution $n \sim k^{-17/5}$ and the inverse cascade solution $n \sim k^{-3}$. It also has the family of thermodynamic Rayleigh–Jeans solutions,

$$n = \frac{T}{\omega + \mu}. \tag{35}$$

where T and μ are constants having a meaning of temperature and the chemical potential, respectively.

The direct cascade of energy in kelvulence is eventually dissipated at high wavenumbers either via phonon radiation or via friction with the normal component. The phonon radiation is the dominant dissipation mechanism near absolute zero whereas the mutual friction becomes more important at higher temperatures, e.g., at $T > 0.4\,\mathrm{K}$ in ^{4}He, as estimated in Ref. 1. Using a dimensional argument and an assumption that the radiation is quadrupolar, we have the following expression for the sound dissipation term,[17]

$$(\dot{n})_{\rm rad} = \frac{\omega^{9/2} n^2}{\kappa^{1/2} c_s^4}, \tag{36}$$

where c_s is the speed of sound. This kind of sound radiation terminates the energy cascade at a finite frequency $\omega_{\rm rad} \sim (\epsilon^3 c_s^{20}/\kappa^{16})^{1/13}$, where ϵ is the total energy injection rate per unit vortex length.[17]

Now let us introduce the effect of mutual friction. It was argued in Ref. [1] that the characteristic time of the mutual friction dissipation at low temperatures, when the normal fluid is at rest, is $\tau_{\rm mf} \sim 1/(\alpha\kappa k^2)$ where α is the (temperature dependent[13,19]) mutual friction coefficient. Thus, we can postulate the following dissipation term in our differential model,

$$(\dot{n})_{\rm mf} = -\alpha\kappa k^2 n. \tag{37}$$

Interestingly, this term looks like a viscous dissipation with an effective viscosity coefficient $\alpha\kappa$. Its temperature dependence qualitatively agrees (within a numerical factor of order unity) with measurements of the effective kinematic viscosity in ^{4}He extracted from the towed grid experiments in the range between about 1.2 K–1.7 K.[20] For higher temperatures, coefficient α has to be replaced with a more complicated function of both friction parameters α and α'.[21] For a finite counterflow velocity $V_{\rm ns}$, one should replace (37) with[22]

$$(\dot{n})_{\rm mf} = \alpha \left[k V_{\rm ns} - \kappa k^2 \frac{\log(1/ka)}{4\pi} \right] n, \tag{38}$$

which describes Glaberson instability of Kelvin waves when their phase velocity is less than $V_{\rm ns}$. Finally, we are leaving for future consideration an interesting case of turbulent Glaberson amplification when $V_{\rm ns}$ is random due to turbulence in the normal fluid. Note that the friction dissipation in (38) differs from (37) by factor $\log(1/ka)/4\pi$ which, following Ref. 1, we will assume to be close to unity.

To examine the effect of mutual friction on the energy cascade, let us, following Ref. 17, introduce a reduced second order model which ignores the waveaction conservation and the inverse cascade. With the friction term (37) included, we have

$$\dot{n} = -\frac{C_1}{\kappa^{10}\sqrt{\omega}} \frac{\partial}{\partial \omega} (n^5 \omega^{17/2}) - 4\pi\alpha\omega n. \tag{39}$$

where $C_1 > 0$ is a dimensionless constant. The general stationary solution of Eq. (39) is

$$n = \left[\left(\frac{2\epsilon}{C_1} \right)^{4/5} \kappa^{42/5} - \frac{4\pi\alpha\kappa^{10}}{C_1} \omega^{4/5} \right]^{1/4} \frac{1}{\omega^{17/10}}. \tag{40}$$

We see that at low frequencies this expression coincides with the non-dissipative energy cascade spectrum, $n \sim \omega^{-17/10}$, and we also see a sharp cut-off at

$$\omega_{\rm mf} = \frac{2C_1^{1/4}\epsilon}{(4\pi\alpha)^{5/4}\kappa^2}. \tag{41}$$

Comparing $\omega_{\rm rad}$ and $\omega_{\rm mf}$ one can see that the frictional dissipation becomes more important than the phonon radiation if

$$\epsilon < (4\pi\alpha)^{13/8} c_s^2 \kappa. \tag{42}$$

This expression differs from expression (87) of Ref. 1 (assuming their relation (74) between ϵ and l, they obtained crossover between the two regimes at $\alpha \simeq 7 \times 10^{-8}$ which corresponds to the temperature about 0.4 K). We attribute this difference to the fact that Ref. 1 assumed the sound radiation to be dipolar rather than quadrupolar as in Ref. 17.

Here we present our estimate of the He II temperature when condition (42) indicates a crossover between phonon radiation and mutual friction dissipation: injecting a power at the level of 1 W into 1l of liquid of density about 145 kg/m^3 (i.e., energy decay rate $\epsilon \approx 7\,\mathrm{m}^2/\mathrm{s}^3$) results to a typical vortex line density 10^{10}–10^{11} m^{-2}, giving $\epsilon \approx 10^{-10}\mathrm{m}^4/\mathrm{s}^3$. The condition (42) thus requires $4\pi\alpha$ of order 10^{-5} (i.e., about an order of magnitude higher than the estimate of Ref. 1), which roughly corresponds[23] to temperature of about 0.5 K.

5. CONCLUSIONS

In this paper, we presented a minimal model for turbulence of the coupled superfluid and normal components in superfluid helium. The model comprises a system of nonlinear partial differential equations for the energy spectra and its origin in the case of classical fluids can be traced back to Kovasznay 1947 paper.[16] The basic idea of such models is that the nonlinear terms, being of the simplest possible form, should preserve the original turbulence scalings and, in particular, predict correctly the Kolmogorov cascade. Having the Kolmogorov scalings built into the model and adding additional physical interactions, such as mutual friction and viscosity, one then obtains new nontrivial physical regimes charactrerised by non-Kolmogorov spectra. For superfluids, the first application of such model was done in Ref. 11 in the limiting case of the normal fluid at rest, and the new -3 spectrum was predicted.

In the present paper, we generalised this model to the case where both the normal and the superfluid components may be turbulent. The crucial theoretical step here is our estimation of the cross-correlation function between normal and superfluid velocities which determines their joint dynamics. This cross-correlations leads to appearance of a rich variety of interesting new regimes. In particular, we found for large superfluid to normal density ratio a regime in which the normal component is "dragged" by the superfluid component via the mutual friction with resulting Kolmogorov spectrum extending far below the Kolmogorov dissipation scale. Our model also confirms the picture suggested in Ref. 12 of a "knee" spectrum where the dissipative -3 scaling at medium wavenumbers exists in between of $-5/3$ Kolmogorov ranges at low and large k's. Our theory bridges classical turbulence with quantum turbulence and in a quantitative manner, it points out similarities and differences between the two, and we expect it to be useful and efficient for numerical simulations of more complicated experimental cases.

We also discuss the case when our continuous two-fluid description breaks down at scales below the mean distance between the quantised vortex filaments. At these scales, the turbulent cascades are believed to be carried through by random nonlinearly interacting Kelvin waves (kelvulence). A differential model for kelvulence, including the phonon radiation effect, was proposed in Ref. 17. In the present paper we extend this model to include the mutual friction effect and we obtain an analytical solution where the energy cascade is arrested at a finite wavenumber. Comparing this friction cutoff with the previously obtained radiation cutoff[17] we obtained an estimate for the crossover between the radiation and the friction dissipation mechanisms.

We leave the exact details of applicability to He II and to ^{3}He-B for future work, as well as an extension of our approach to (anisotropic) counterflow turbulence. Let us point out that the likely candidate of experimental verification of our theory might be ^{4}He–^{3}He superfluid mixtures where, due to presence of ^{3}He quasiparticles, mutual friction at low temperature is still expected to be significantly higher than in pure He II.

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