

# Turbulent fluxes in stably stratified boundary layers

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## Abstract

We present here an extended version of an invited talk we gave at the international conference ‘Turbulent Mixing and Beyond’. The dynamical and statistical description of stably stratified turbulent boundary layers with the important example of the stable atmospheric boundary layer in mind is addressed. Traditional approaches to this problem, based on the profiles of mean quantities, velocity second-order correlations and dimensional estimates of the turbulent thermal flux, run into a well-known difficulty, predicting the suppression of turbulence at a small critical value of the Richardson number, in contradiction to observations.

Phenomenological attempts to overcome this problem suffer from various theoretical inconsistencies. Here, we present an approach taking into full account all the second-order statistics, which allows us to respect the conservation of total mechanical energy. The analysis culminates in an analytic solution of the profiles of all mean quantities and all second-order correlations, removing the unphysical predictions of previous theories. We propose that the approach taken here is sufficient to describe the lower parts of the atmospheric boundary layer, as long as the Richardson number does not exceed an order of unity. For much higher Richardson numbers, the physics may change qualitatively, requiring careful consideration of the potential Kelvin–Helmholtz waves and their interaction with the vortical turbulence.

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(Some figures in this article are in colour only in the electronic version.)

## Nomenclature

$\mathcal{A}$	Thermal flux production vector, (2.7e)	$Pr_T$	Turbulent Prandtl number, $\nu_T/\chi_T$
$\mathcal{B}$	Pressure–temperature–gradient-vector, (2.7f)	$Ri_{\text{flux}}$	Flux Richardson number, $\beta F_z/\tau_{xz} S_U$
$C_{ij}$	Energy conversion tensor, (2.7b)	$Ri_{\text{grad}}$	Gradient Richardson number, $\beta S_\Theta/S_U^2$
$D/Dt$	Substantial derivative, $\partial/\partial t + \mathbf{U} \cdot \nabla$	$\mathcal{R}_{ij}$	Pressure–rate-of-strain-tensor, (2.7c)
$D/Dt$	Mean substantial derivative, $\partial/\partial t + \mathbf{U} \cdot \nabla$	$S_U$	Mean velocity gradient, $dU/dz$
$E_K$	Turbulent kinetic energy per unit mass, $ \mathbf{u} ^2/2$	$S_\Theta$	Mean potential temperature gradient, $d\Theta/dz$
$E_\Theta$	‘Temperature energy’ per unit mass, $\theta^2/2$	$T$	Molecular temperature
$\mathbf{F}$	Turbulent thermal flux per unit mass, $\langle \mathbf{u}\theta \rangle$	$\mathbf{U}$	Velocity field
$F_*$	Thermal flux at zero elevation, $z = 0$	$U$	Mean velocity, $\langle \mathbf{U} \rangle$
$\mathbf{g}$	Gravity acceleration, $\mathbf{g} = -g\hat{\mathbf{z}}$	$\mathbf{u}$	Fluctuating velocity, $\mathbf{U} - U$
$L$	Monin–Obukhov length, $u_*^3/\beta F_*$	$u_*$	(Wall) friction velocity, $\sqrt{\tau_*}$
$\ell$	Outer scale of turbulence, external parameter	$\hat{\mathbf{x}}$	Horizontal (streamwise) unit vector
$\mathcal{P}_{ij}$	Rate of Reynolds stress production, (2.7a)	$\beta$	Buoyancy parameter, $g\hat{\beta}$
$p, \tilde{p}, p_*$	Total, fluctuating and zero-level pressures	$\hat{\beta}$	Thermal expansion coefficient
		$\gamma_{\text{RI}}$	Relaxation frequency of $\tau_{ij}$ , $i = j$
		$\tilde{\gamma}_{\text{RI}}$	Relaxation frequency of $\tau_{ij}$ , $i \neq j$

$\gamma_{RD}$	Relaxation frequency of $\mathbf{F}$
$\gamma_{uu}$	Relaxation frequency of $E_K$
$\gamma_{\theta\theta}$	Relaxation frequency of $E_\Theta$
$\varepsilon_{ij}$	Dissipation tensor of $\tau_{ij}$ , (2.6)
$\boldsymbol{\varepsilon}$	Dissipation vector of $\mathbf{F}$ , (2.6)
$\varepsilon$	Dissipation of $E_\Theta$ , (2.6)
$\Theta_d$	Deviation of potential temperature from BRS
$\Theta$	Mean potential temperature, $\langle \Theta_d \rangle$
$\theta$	Fluctuating potential temperature, $\Theta_d - \langle \Theta_d \rangle$
$\theta_*$	Potential temperature at zero elevation, $F_*/u_*$
$\lambda_*$	Viscous lengthscale, $\nu/u_*$
$\nu$	Kinematic viscosity
$\nu_T$	Turbulent viscosity
$\rho$	Density of the fluid
$\tau_{ij}$	Reynolds stress tensor, $\langle u_i u_j \rangle$
$\tau_*$	Mechanical momentum flux at zero elevation (at the ground)
$\chi$	Kinematic thermal conductivity
$\chi_T$	Turbulent thermal conductivity
BRS	Basic Reference State
$\hat{\mathbf{z}}$	Vertical (wall-normal) unit vector

## 1. Introduction

The lower levels of the atmosphere are usually strongly influenced by the Earth's surface. Known as the atmospheric boundary layer, this is the part of the atmosphere where the surface influences the temperature, moisture and velocity of the air above through the turbulent transfer of mass.

The stability of the atmospheric boundary layer depends on the profiles of the density and temperature as a function of the height above the ground. During normal summer days, the land mass warms up and the temperature is higher at lower elevations. If it were not for the decrease in density of the air as a function of the height, such a situation of heating from below would have been always highly unstable. In fact, the boundary layer is considered stable as long as the temperature decreases at the dry adiabatic lapse rate ( $T' \approx -9.8^\circ\text{C km}^{-1}$ ) throughout most of the boundary layer. With such a rate of cooling, one balances out the decrease in density. With a higher degree of cooling, one refers to the atmospheric boundary layer as unstably stratified, whereas with a lower degree of cooling, the situation is stably stratified. A stably stratified boundary layer occurs typically during clear, calm nights. In extreme cases, turbulence tends to cease, and radiational cooling from the surface results in a temperature that increases with height above the surface.

The tendency of the atmosphere to be turbulent depends not only on the rate of cooling but also on the mean shear in the vertical direction. The parameter commonly used to describe the tendency of the atmosphere to be turbulent is the 'gradient' Richardson number (Richardson 1920), defined as

$$Ri_{\text{grad}} \equiv \frac{\beta d\Theta(z)/dz}{[dU_x/dz]^2}, \quad (1.1)$$

where  $x$  is the streamwise direction,  $z$  is the height above the ground,  $\Theta(z)$  is the mean potential temperature profile (which differs from the mean temperature profile  $T(z)$  by accounting for the adiabatic cooling of the air during

its expansion:  $d\Theta(z)/dz = dT(z)/dz + |T'|$ ),  $\beta = \tilde{\beta}g$  is the buoyancy parameter in which  $\tilde{\beta}$  is the adiabatic thermal expansion coefficient (for an ideal gas,  $\tilde{\beta} = 1/T$ ), and  $g$  is the gravitational acceleration. The mean shear  $dU_x/dz$  is defined in terms of the mean velocity  $\mathbf{U}$ , which in the simplest case of flat geometry depends only on the vertical coordinate  $z$ . The parameter  $Ri_{\text{grad}}$  represents the ratio of the generation or suppression of turbulence by buoyant production of energy to the mechanical generation of energy by wind shear.

This paper is an extended presentation of an invited talk we have given at the international conference 'Turbulent Mixing and Beyond' devoted, in particular, to the problems of fluid dynamics, turbulence, geophysics and statistics, which are long-standing challenging tasks. Here, we consider the description of stably stratified turbulent boundary layers (TBL), taking as an example the case of stable thermal stratification. Since the 1950s of the twentieth century, traditional models of stratified TBL generalize models of unstratified TBL, based on the budget equations for the kinetic energy and mechanical momentum; see reviews of Umlauf and Burchard (2005) and Weng and Taylor (2003). The main difficulty is that the budget equations are not closed; they involve turbulent fluxes of mechanical moments  $\tau_{ij}$  (known as the 'Reynolds stress' tensor) and a thermal flux  $\mathbf{F}$  (for the case of thermal stratification):

$$\tau_{ij} \equiv \langle u_i u_j \rangle, \quad \mathbf{F} \equiv \langle \mathbf{u} \theta \rangle, \quad (1.2)$$

where  $\mathbf{u}$  and  $\theta$  stand for the turbulent fluctuating velocity and the potential temperature with zero mean. The nature of the averaging procedure behind the symbol  $\langle \cdot \cdot \cdot \rangle$  will be specified below.

Earlier estimates of the fluxes (1.2) are based on the concept of the down-gradient turbulent transport, in which, similarly to the case of molecular transport, a flux is taken proportional to the gradient of transported property times a corresponding (turbulent) transport coefficient:

$$\tau_{xz} = -\nu_T dU_x/dz, \quad \nu_T \approx C_\nu \ell_z \sqrt{\tau_{zz}}, \quad (1.3a)$$

$$F_z = -\chi_T d\Theta/dz, \quad \chi_T \approx C_\chi \ell_z \sqrt{\tau_{zz}}, \quad \text{etc.} \quad (1.3b)$$

Here the turbulent-eddy viscosity  $\nu_T$  and turbulent thermal conductivity  $\chi_T$  are estimated by dimensional reasoning via the vertical turbulent velocity  $\sqrt{\tau_{zz}}$  and a scale  $\ell_z$  (which in the simplest case is determined by the elevation  $z$ ). The dimensionless coefficients  $C_\nu$  and  $C_\chi$  are assumed to be of the order of unity.

This approach meets serious difficulties (Zeman 1981); in particular, it predicts full suppression of turbulence when the stratification exceeds a critical level, for which  $Ri_{\text{grad}} \approx 0.25$ . On the other hand, in observations of the atmospheric turbulent boundary layer, turbulence exists for much larger values than  $Ri_{\text{grad}} = 0.25$ : experimentally above  $Ri_{\text{grad}} = 10$  and even higher (see Galperin *et al* (2007) and references cited therein). In models for weather predictions, this problem is 'fixed' by introducing fit functions  $C_\nu(Ri_{\text{grad}})$  and  $C_\chi(Ri_{\text{grad}})$  instead of the constant  $C_\nu$  and  $C_\chi$  in the model parameterization (1.3). This technical 'solution' is not based on any physical derivation and just masks the shortcomings of the model. To really solve the problem, one has to

understand its physical origin, even though from a purely formal viewpoint it is indeed possible that a dimensionless coefficient like  $C_\chi$  can be any function of  $Ri_{\text{grad}}$ .

To expose the physical reason for the failure of the down-gradient approach, recall that in a stratified flow, in the presence of gravity, the turbulent kinetic energy is *not* an integral of motion. Only the total mechanical energy, i.e. the sum of the kinetic and the potential energy, is conserved in the inviscid limit. As it has already been shown by Richardson, the difficult point is that an important contribution to the potential energy comes not just from the mean density profile, but from the density fluctuations. Clearly, any reasonable model of the turbulent boundary layer must obey the conservation laws.

The physical requirement of conserving the total mechanical energy calls for an explicit consideration not only of the mean profiles, but also of *all* the relevant second-order, one-point, simultaneous correlation functions of *all* the fluctuating fields together with some closure procedure for the appearing third-order moments. First of all, in order to account for the important effect of stratification on the anisotropy, we must write explicit equations for the entire Reynolds stress tensor,  $\tau_{ij} = \langle u_i u_j \rangle$ . Next, in the case of the temperature stratified turbulent boundary layers, we follow tradition (see e.g. Zeman (1981), Hunt *et al* (1988), Schumann and Gerz (1995), Hanazaki and Hunt (2004), Keller and van Atta (2000), Elperin *et al* (2002), Cheng *et al* (2002), Luyten *et al* (2002) and Rehmann and Hwang (2005)) and account for the turbulent potential energy, which is proportional to the variance of the potential temperature deviation,  $\langle \theta^2 \rangle$ . And last but not least, we have to consider explicitly equations for the vertical fluxes,  $\tau_{xz}$  and  $F_z$ , which include the down-gradient terms proportional to the velocity and temperature gradients, and counter-gradient terms, proportional to  $F_x$  (in the equation for  $\tau_{xz}$ ) and to  $\langle \theta^2 \rangle$  (in the equation for  $F_z$ ).

Unfortunately, the resulting second-order closure seems to be inconsistent with the variety of boundary-layer data, and many authors took the liberty to introduce additional fitting parameters and sometimes fitting functions to achieve better agreement with the data (see reviews of Umlauf and Burchard (2005), Weng and Taylor (2003), Zeman (1981), Mellor and Yamada (1974)) and references cited therein). Moreover, in the second-order closures the problem of critical Richardson number seems to persist (Canuto 2002, Cheng *et al* 2002).

Notice that in spite of obvious inconsistency of the first-order schemes, most of the practically used turbulent models are based on the concept of the down-gradient transport. One of the reasons is that, in the second-order schemes, instead of two down-gradient equations (1.3) one needs to take into account eight nonlinear coupled additional equations, i.e. four equations for the Reynolds stresses, three equations for the heat fluxes and one equation for the temperature variance. As a result, the second-order schemes have seemed to be rather cumbersome for comprehensive analytical treatment and have allowed to find only some relationships between correlation functions (see e.g. Cheng *et al* (2002)). Unfortunately, the numerical solutions to the complete set of the second-order schemes equations which involve too many fitting parameters are much less informative in clarification of the physical picture of the phenomenon than desired analytical ones.

In this paper, we suggest a relatively simple second-order closure model of turbulent boundary layer with stable temperature stratification that, on the one hand, accounts for the main relevant physics in the stratified TBL and, on the other hand, is simple enough to allow complete analytical treatment including the problem of critical  $Ri_{\text{grad}}$ . To reach this goal, we approximate the third-order correlations via the first- and second-order ones, accounting only for the most physically important terms. We will try to expose the approximations in a clear and logical way, providing the physical justification as we go along. The resulting second-order model consists of nine coupled equations for the mean velocity and temperature gradients, four components of the Reynolds stresses, two components of the temperature fluxes and the temperature variance. Thanks to the simplicity achieved by the model, we found an approximate analytical solution of these equations, expressing all nine correlations as functions of only one governing parameter,  $\ell(z)/L$ , where  $\ell(z)$  is the outer scale of turbulence (depending on the elevation  $z$  and also known as the ‘dissipation scale’) and  $L$  – is the Obukhov length.

We would like to also stress that, in our approach,  $\ell(z)/L$  is an external parameter of the problem. For small elevations  $z \ll L$ , it is well accepted that  $\ell(z)$  is proportional to  $z$ , while the  $\ell(z)$  dependence is still under debate for  $z$  comparable with or exceeding  $L$ . For  $z \gtrsim L$ , the assignment and discussion of the actual dependence of the outer scale of turbulence,  $\ell(z)$ , which is manifested in Nature, is outside the scope of this paper, and is a topic for future work. For the time being, we can analyze the consequences of our approach for the following versions of  $\ell(z)$  dependence at  $z \gg L$ :

- Function  $\ell(z)$  is saturated at some level of the order of  $L$ . For concreteness, we can take

$$1/\ell(z) = \sqrt{(d_1 z)^{-2} + (d_2 L)^{-2}}, \quad d_1 \sim d_2 \sim 1. \quad (1.4)$$

- $\ell(z)$  is again proportional to  $z$  for elevations much larger than  $L$ :  $\ell(z) = d_3 z$  but with the proportionality constant  $d_3 < d_1$ . If so, we can also study the case  $\ell(z) \gg L$  even though such a condition may not be realizable in Nature. In that case, our analysis of the limit  $\ell(z) \gg L$  has only a methodological character: it allows to derive an approximate analytic solution for all the objects of interest as functions of  $\ell(z)/L$  that is also valid for the outer scale of turbulence not exceeding a value of the order of  $L$ .

It should be noticed that traditional turbulent closures (including ours) cannot be applied for strongly stratified flows with  $Ri_{\text{grad}} \gtrsim 1$  (maybe even at  $Ri_{\text{grad}} \sim 1$ ). The main reason is that these closures are roughly justified for developed *vortical* turbulence, in which the eddy-turnover time is of the order of its lifetime; in other words, there are no well-defined ‘quasi-particles’ or waves. This is not the case for stable stratification with  $Ri_{\text{grad}} \gtrsim 1$ , in which the Brunt–Väisälä frequency,

$$N \equiv \sqrt{\beta d\Theta(z)/dz}, \quad (1.5)$$

is larger than the eddy-turnover frequency  $\gamma$ . This means that for  $Ri_{\text{grad}} \gtrsim 1$  there are weakly decaying Kelvin–Helmholtz

internal gravity waves (with characteristic frequency  $N$  and decay time above  $1/\gamma$ ), propagating for large distances, essentially affecting TBL, as pointed out by Zilitinkevich (2002). We concentrate in our paper on a self-consistent description of the lower part of the atmospheric TBL, in which turbulence has a vortical character and, consequently, large values of  $Ri_{\text{grad}}$  do not appear. We relate the large values of  $Ri_{\text{grad}}$  in the upper part of TBL with contributions of the internal gravity waves to the energy and the energy flux in TBL, to the momentum flux and to the production of (vortical) turbulent energy. Owing to their instability in a shear flow, the waves can break and create turbulent kinetic energy. All these effects are beyond the framework of our paper. Their description in the upper ‘potential-wave’ TBL and intermediate region with the combined ‘vortical-potential’ turbulent velocity field is on our agenda for future work.

To make this paper clearer for a wide audience, not necessarily experts in atmospheric TBL, we attempt to present the material in a self-contained manner, and have organized it as follows.

In section 2.1, we use the Oberbeck–Boussinesq approximation and apply the standard Reynolds decomposition (into mean values and turbulent zero-mean fluctuations of the velocity and temperature fields) to derive equations for the mean values and balance equations for all relevant second-order correlation functions. In section 2.2, we demonstrate that the resulting balance equations exactly preserve (in the non-dissipative limit) the total mechanical energy of the system, which consists of the kinetic energy of the mean flow, kinetic energy and potential energy of the turbulent subsystem.

In section 3, we describe the proposed closure procedure that results in a model of stably stratified TBL, which accounts explicitly for all relevant second-order correlations. The third-order correlations that appear in the theory are modeled in terms of second-order correlations in sections 3.1 and 3.2. Further simplifications are presented in sections 3.3 and 3.4 for stationary turbulent flows in a plane geometry outside the viscous and buffer layers. In section 3.5, we suggest a generalization of the standard ‘wall-normalization’ to obtain the model equations in a dimensionless form with only one governing parameter,  $\ell(z)/L$ .

Section 4.1 contains the approximate analytical solution of the model. It is shown that the analytical solution deviates from the numerical counterpart by less than a few per cent in the entire interval  $0 \leq (\ell/L) < \infty$ .

The last section, section 4, is devoted to a detailed description of our results: profiles of the mean velocity and potential temperature (section 4.1), profiles of the turbulent kinetic and ‘temperature’ energies, profiles of the anisotropy of partial kinetic energies (section 4.2), profiles of the turbulent transport parameters  $\nu_T$  and  $\chi_T$ , profiles of the gradient- and flux-Richardson numbers  $Ri_{\text{grad}}$  and  $Ri_{\text{flux}}$ , and the dependence of the turbulent Prandtl number  $Pr_T$  versus  $\ell/L$  and  $Ri_{\text{grad}}$  (section 4.3). Finally, in section 4.4, we consider the validity of the down-gradient transport concept (1.3) and explain why it is violated in the upper part of TBL. The problem of critical  $Ri_{\text{grad}}$  is also discussed.

## 2. Simplified dynamics in a stably temperature-stratified TBL and their conservation laws

The aim of this section is to consider the simplified dynamics of a stably temperature-stratified turbulent boundary layer, aiming finally at an explicit description of the height dependence of important quantities like the mean velocity, mean temperature, turbulent kinetic and potential energies, etc. In general, one expects very different profiles from those known in standard (unstratified) wall-bounded turbulence. We want to focus on these differences and propose that they occur already relatively close to the ground, allowing us to neglect (to the leading order) the dependence of the density on height and the Coriolis force. We thus begin by simplifying the hydrodynamic equations that are used in this section.

### 2.1. Simplified hydrodynamic equations and Reynolds decomposition

Firstly, we briefly outline the derivation of the governing equations in the Boussinesq approximation. The system of hydrodynamic equations describing a fluid in which the temperature is not uniform consists of the Navier–Stokes equations for the fluid velocity,  $\mathbf{U}(\mathbf{r}, t)$ , a continuity equation for the space- and time-dependent (total) density of the fluid,  $\rho(\mathbf{r}, t)$ , and of the heat balance equation for the (total) entropy per unit mass,  $S(\mathbf{r}, t)$  (Landau and Lifshitz 1987).

These equations are considered with boundary conditions that maintain the solution far from the equilibrium state, where  $\mathbf{U} = S = 0$ . These boundary conditions are  $\mathbf{U} = 0$  at zero elevation,  $\mathbf{U} = \text{const}$  at a high elevation of a few kilometers. This reflects the existence of a wind at high elevation, but we do not attempt to model the physical origin of this wind in any detail. The only important condition with regard to this wind is that it maintains a momentum flux toward the ground that is prescribed as a function of the elevation. Similarly, we assume that a stable temperature stratification is maintained such that the heat flux toward the ground is prescribed as well.

We neglect the viscous entropy production term assuming that the temperature gradients are large enough such that the thermal entropy production term dominates. For simplicity of the presentation, we restrict ourselves to relatively small elevations and disregard the Coriolis force (for more details, see Wyngaard (1992)). On the other hand, we assume that the temperature and density gradients in the entire turbulent boundary layer are sufficiently small to allow employment of local thermodynamic equilibrium. In other words, we assume the validity of the equation of state.

As a ‘basic reference state’ (BRS), denoted hereinafter by a subscript ‘b’, we use the isentropic model of the atmosphere, where the entropy is considered space homogeneous. Now, assuming smallness of deviations of the density and pressure from their BRS values and exploiting the equation of state, one obtains a simplified equation, which is already very close to the standard Navier–Stokes equation in the Boussinesq approximation. Introducing (generalized) potential temperature (see Hauf and Höller 1987), one obtains the well-known system of hydrodynamic equations in the

Boussinesq approximation (Oberbeck 1879 and Boussinesq 1903). Close to the ground, where one can neglect the dependence of the density on height, the system reads:

$$\begin{aligned} \frac{D\mathbf{U}}{Dt} &= -\frac{\nabla p}{\rho_b} - \beta \Theta_d + \nu \Delta \mathbf{U}, \\ \frac{D\Theta_d}{Dt} &= \chi \Delta \Theta_d. \end{aligned} \quad (2.1)$$

Here,  $D/Dt \equiv \partial/\partial t + \mathbf{U} \cdot \nabla$  is the convection time derivative,  $p$  the deviation of pressure from BRS,  $\rho_b$  the density in BRS,  $\beta = g\tilde{\beta}$  the buoyancy parameter ( $\beta = -\hat{z}\beta$ , where  $\beta = g\tilde{\beta}$ ,  $g$  is the gravity acceleration and  $\tilde{\beta}$  is the thermal expansion coefficient, which is equal to  $1/T$ , the reciprocal molecular temperature, for an ideal gas),  $\Theta_d$  the deviation of the potential temperature from the BRS value,  $\nu$  the kinematic viscosity and  $\chi$  the kinematic thermal conductivity.

To develop equations for the mean quantities and correlation functions, one applies the Reynolds decomposition:  $\mathbf{U} = \mathbf{U} + \mathbf{u}$ ,  $\langle \mathbf{U} \rangle = \mathbf{U}$ ,  $\langle \mathbf{u} \rangle = 0$ ,  $\Theta_d = \Theta + \theta$ ,  $\langle \Theta_d \rangle = \Theta$ ,  $\langle \theta \rangle = 0$ ,  $p = \langle p \rangle + \tilde{p}$ ,  $\langle \tilde{p} \rangle = 0$ . Here, the average  $\langle \dots \rangle$  stands for an averaging over a horizontal plane at a constant elevation. This leaves the average quantities with a  $z, t$  dependence only. Substituting in equations (2.1), one gets equations of motion for the mean velocity and mean temperature profiles

$$\begin{aligned} \frac{DU_i}{Dt} + \nabla_j \tilde{\tau}_{ij} &= -\frac{\nabla_i \langle p \rangle}{\rho_b} - \beta_i \Theta, \\ \frac{D\Theta}{Dt} + \nabla \cdot \tilde{\mathbf{F}} &= 0. \end{aligned} \quad (2.2)$$

Here,  $D/Dt \equiv \partial/\partial t + \mathbf{U} \cdot \nabla$  is the mean convection derivative. The total (molecular and turbulent) momentum and thermal fluxes are

$$\begin{aligned} \tilde{\tau}_{ij} &\equiv -\nu \nabla_j U_i + \tau_{ij}, \\ \tilde{\mathbf{F}} &\equiv -\chi \nabla \Theta + \mathbf{F}, \end{aligned} \quad (2.3)$$

where  $\tau_{ij} = \langle u_i u_j \rangle$  is the Reynolds stress tensor describing the turbulent momentum flux, and  $\mathbf{F} = \langle \mathbf{u}\theta \rangle$  is the turbulent thermal flux. In order to derive equations for these correlation functions, one considers the equations of motion for the fluctuating velocity and temperature:

$$\begin{aligned} D\mathbf{u}/Dt &= -\mathbf{u} \cdot \nabla \mathbf{U} - \mathbf{u} \cdot \nabla \mathbf{u} + \langle \mathbf{u} \cdot \nabla \mathbf{u} \rangle \\ &\quad - (\nabla \tilde{p}/\rho_b) + \nu \Delta \mathbf{u} - \beta \theta, \end{aligned} \quad (2.4a)$$

$$D\theta/Dt = -\mathbf{u} \cdot \nabla \Theta - \mathbf{u} \cdot \nabla \theta + \chi \Delta \theta + \langle \mathbf{u} \cdot \nabla \theta \rangle. \quad (2.4b)$$

The whole set of the second-order correlation functions includes the Reynolds stress  $\tau_{ij}$ , the turbulent thermal flux  $\mathbf{F}$  and the ‘temperature energy’  $E_\theta \equiv \langle \theta^2 \rangle / 2$ , which is denoted and named in analogy with the turbulent kinetic energy density (per unit mass and unit volume),  $E_K = \langle |\mathbf{u}|^2 \rangle / 2 = \text{Tr}\{\tau_{ij}\} / 2$ . Using (2.4), one gets the following ‘balance equations’ (see Kurbatsky 2000 and Zilitinkevich et al 2007):

$$\frac{D\tau_{ij}}{Dt} + \varepsilon_{ij} + \frac{\partial}{\partial x_k} T_{ijk} = \mathcal{P}_{ij} - \mathcal{C}_{ij} + R_{ij}, \quad (2.5a)$$

$$\frac{DF_i}{Dt} + \varepsilon_i + \frac{\partial}{\partial x_j} T_{ij} = \mathcal{A}_i + \mathcal{B}_i, \quad (2.5b)$$

$$\frac{DE_\theta}{Dt} + \varepsilon + \nabla \cdot \mathbf{T} = -\mathbf{F} \cdot \nabla \Theta. \quad (2.5c)$$

Here, we denote the dissipations of the Reynolds-stress, heat-flux and the temperature energy by

$$\begin{aligned} \varepsilon_{ij} &\equiv 2\nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right\rangle, \\ \varepsilon_i &\equiv (\nu + \chi) \left\langle \frac{\partial \theta}{\partial x_k} \frac{\partial u_i}{\partial x_k} \right\rangle, \\ \varepsilon &\equiv \chi \langle |\nabla \theta|^2 \rangle, \end{aligned} \quad (2.6)$$

The last term in the left-hand side (lhs) of each of equations (2.5) describes *spatial* flux of the corresponding quantity. In models of wall-bounded unstratified turbulence, it is known that these terms are very small almost everywhere. We do not have sufficient experience with the stratified counterpart to be able to assert that the same is true here. Nevertheless, for simplicity we are going to neglect these terms. It is possible to show that accounting for these terms does not influence the results much. Note that keeping these terms turns the model into a set of differential equations that are very cumbersome to analyze. This is a serious uncontrolled step in our development, so we cross our fingers and proceed with caution. Since these terms are neglected, we do not provide here the explicit expressions for  $T_{ijk}$ ,  $T_{ij}$  and  $\mathbf{T}$ .

The first term in the right-hand side (rhs) of the balance equation (2.5a) for the Reynolds stresses is the ‘energy production tensor’  $\mathcal{P}_{ij}$ , describing the production of the turbulent kinetic energy from the kinetic energy of the mean flow, proportional to the gradient of the mean velocity:

$$\mathcal{P}_{ij} \equiv -\tau_{ik} \partial U_j / \partial x_k - \tau_{jk} \partial U_i / \partial x_k. \quad (2.7a)$$

The second term in the rhs of equation (2.5a),  $\mathcal{C}_{ij}$ , will be referred to hereinafter as the ‘energy conversion tensor’. It describes the conversion of the turbulent kinetic energy into potential energy. This term is proportional to the buoyancy parameter  $\beta$  and the turbulent thermal flux  $\mathbf{F}$ :

$$\mathcal{C}_{ij} \equiv -\beta (F_i \delta_{jz} + F_j \delta_{iz}). \quad (2.7b)$$

The next term in the rhs of equation (2.5a) is known as the ‘pressure-rate-of-strain tensor’:

$$\begin{aligned} \mathcal{R}_{ij} &\equiv \langle \tilde{p} s_{ij} / \rho_b \rangle, \\ s_{ij} &\equiv \partial u_i / \partial x_j + \partial u_j / \partial x_i. \end{aligned} \quad (2.7c)$$

In incompressible turbulence its trace vanishes; therefore  $\mathcal{R}_{ij}$  does not contribute to the balance of the kinetic energy. As we will show in section 3.1, this tensor can be presented as the sum of three contributions (Zeman 1981):

$$\mathcal{R}_{ij} = R_{ij}^{\text{RI}} + R_{ij}^{\text{IP}} + R_{ij}^{\text{IC}}, \quad (2.7d)$$

in which  $R_{ij}^{\text{RI}}$  is responsible for the nonlinear process of isotropization of turbulence and is traditionally called

the ‘return-to-isotropy’, and  $R_{ij}^{IP}$  is similar to the energy production tensor (2.7a) and is called ‘isotropization of production’. A new term, appearing in the stratified flow, is  $R_{ij}^{IC}$ ; it is similar to the energy conversion tensor (2.7b) and will be referred to as the ‘isotropization of conversion’.

Consider the balance of the turbulent thermal flux  $\mathbf{F}$ , equation (2.5b). The first term in the rhs,  $\mathcal{A}$ , describes the source of  $\mathbf{F}$  and, by analogy with the energy-production tensor,  $\mathcal{P}_{ij}$ , is called the ‘thermal-flux production vector’. Like  $\mathcal{P}_{ij}$ , equation (2.7b), it has the contribution,  $A_i^{SU}$ , proportional to the mean velocity gradient:

$$\begin{aligned} \mathcal{A}_i &\equiv A_i^{SU} + A_i^{S\Theta} + A_i^{E\theta}, & A_i^{SU} &\equiv -\mathbf{F} \cdot \nabla U_i, \\ A_i^{S\Theta} &\equiv -\tau_{ij} \partial \Theta / \partial x_j, & A_i^{E\theta} &\equiv 2\beta E_\theta \delta_{iz}, \end{aligned} \quad (2.7e)$$

and two additional contributions, related to the temperature gradient and to the ‘temperature energy’,  $E_\theta$ , and the buoyancy parameter. One sees that, contrary to the oversimplified assumption (1.3b), the thermal flux in such a turbulent medium cannot be considered as proportional to the temperature gradient. It has also a contribution proportional to the velocity gradient and even to the square of the temperature fluctuations. Moreover, the rhs of the flux-balance equation (2.5b) has an additional term, the ‘pressure–temperature-gradient vector’, which, similarly to the pressure–rate-of-strain tensor (2.7d), can be divided into three parts (Zeman 1981):

$$\mathcal{B} \equiv \langle \tilde{p} \nabla \theta / \rho_b \rangle = \mathbf{B}^{RD} + \mathbf{B}^{SU} + \mathbf{B}^{E\theta}. \quad (2.7f)$$

As we will show in section 3.1, the first contribution,  $B_i^{RD} \propto \langle uu \nabla_i \theta \rangle$ , is responsible for the nonlinear flux of  $\mathbf{F}$  in the space of scales toward smaller scales, similarly to the correlation  $\langle uuu \rangle$ , which is responsible for the flux of kinetic energy  $\langle u^2 \rangle / 2$  toward smaller scales. The correlation  $B_i^{RD} \propto \langle uu \nabla_i \theta \rangle$  may be understood as the nonlinear contribution to the dissipation of the thermal flux. Correspondingly, we will call it ‘renormalization of the thermal-flux dissipation’ and will supply it with a superscript ‘RD’. The next two terms in the decomposition (2.7f) are  $B_i^{SU} \propto S_U$  and  $B_i^{E\theta} \propto E_\theta$ . They describe the renormalization of the thermal-flux production terms  $A_i^{SU} \propto S_U$  and  $A_i^{E\theta} \propto E_\theta$ , accordingly.

## 2.2. Conservation of total mechanical energy in the exact balance equations

The total mechanical energy of temperature-stratified turbulent flows consists of three parts with densities (per unit mass):  $E = E_K + E_\kappa + E_P$ , where  $E_K = |\mathbf{U}|^2 / 2$  is the density of kinetic energy of the mean flow,  $E_\kappa = \tau_{ii} / 2$  is the density of turbulent kinetic energy and  $E_P = \beta E_\theta / S_\Theta$  is the density of potential energy, associated with turbulent density fluctuation  $\tilde{\rho} = \tilde{\beta} \theta \rho_b$ , caused by the (potential) temperature fluctuations  $\theta$ , and  $S_\Theta = d\Theta / dz$ .

The balance equation for  $E_K$  follows from equation (2.2):

$$\begin{aligned} (DE_K / Dt) + \nu (\nabla_j U_i)^2 + \nabla_j (U_i \tilde{\tau}_{ij}) \\ = [\text{source } E_K] + \tau_{ij} \nabla_j U_i, \end{aligned} \quad (2.8a)$$

with the help of identity:  $U_i \nabla_j \tau_{ij} \equiv \nabla_j (U_i \tau_{ij}) - \tau_{ij} \nabla_j U_i$  and definition (2.3). The terms in the lhs of this equation, proportional to  $\nu$  and  $\tilde{\tau}_{ij}$ , respectively, describe the dissipation and the spatial flux of  $E_K$ . The term [source  $E_K$ ] in the rhs of equation (2.8a) describes the external source of energy, originating from the boundary conditions described above, and  $\tau_{ij} \nabla_j U_i$  describes the kinetic energy out-flux from the mean flow to the turbulent subsystem.

The balance equation for the turbulent kinetic energy follows directly from equation (2.5a):

$$DE_K / Dt + ([\varepsilon_{ii} + \nabla_j T_{ij}] / 2) = -\tau_{ij} \nabla_j U_i + \beta F_z. \quad (2.8b)$$

In the lhs of equation (2.8b), one sees the dissipation and spatial flux terms. The first term in the rhs originates from the energy production,  $\frac{1}{2} \mathcal{P}_{ii}$ , defined by equation (2.7a). This term has an opposite sign to the last term in the rhs of equation (2.8a) and describes the production of the turbulent kinetic energy from the kinetic energy of the mean flow. The last term in the rhs of equation (2.8b) originates from the energy conversion tensor  $\frac{1}{2} C_{ii}$ , equation (2.7b), and describes the conversion of the turbulent kinetic energy into the potential one.

According to the last of equations (2.5), one gets the balance equation for the potential energy  $E_P$ ; multiplying equation (2.5c) for  $E_\theta$  by  $\beta / S_\Theta$ :

$$DE_P / Dt + (\beta [\varepsilon + \nabla_j T_j] / S_\Theta) = -\beta F_z. \quad (2.8c)$$

The rhs of this equation (coinciding up to a sign with the last term in the rhs of equation (2.8b)) is the source of potential energy (from the kinetic one).

In the sum of the three balance equations, the conversion terms (of the kinetic energy from the mean to turbulent flows and of the turbulent kinetic energy to the potential one) cancel and one gets the total mechanical energy balance:

$$DE / Dt + [\text{diss } E] + \nabla [\text{flux } E] = [\text{source } E_K]. \quad (2.9)$$

This equation exactly respects the conservation of total mechanical energy in the dissipationless limit, irrespective of the closure approximations. This is because the energy production and conversion terms are exact and do not require any closures, whereas the pressure–rate-of-strain tensor, which requires some closure, does not contribute to the total energy balance.

## 3. The closure procedure and the resulting model

In this section, we describe the proposed closure procedure that results in a model of stably stratified TBL. In developing this model, we strongly rely on the analogous well-developed modeling of standard (unstratified) TBL. The final justification of this approach can be done only in comparison with data from experiments and simulations. We will do below what we can to use the existing data, but we propose at this point that much more experimental and simulational work is necessary to solidify all the steps taken in this section.

### 3.1. The pressure–rate-of-strain tensor $\mathcal{R}_{ij}$ and the pressure–temperature-gradient vector $\mathcal{B}$

The correlation functions  $\mathcal{R}_{ij}$  and  $\mathcal{B}$ , defined by equations (2.7c) and (2.7f), include the fluctuating part of the pressure  $\tilde{p}$ . The Poisson equation for  $\tilde{p}$  follows from equation (2.4):  $\Delta\tilde{p} = \rho_b[-\nabla_i\nabla_j(u_i u_j - \langle u_i u_j \rangle) + U_i u_j + U_j u_i] + \beta\nabla_z\theta$ . The solution of this equation includes a harmonic part,  $\Delta\tilde{p} = 0$ , which is responsible for sound propagation and does not contribute to turbulent dynamics at small Mach numbers. Thus this contribution can be neglected. The inhomogeneous solution includes three parts:  $\tilde{p} = \rho_b[p_{uu} + p_{Uu} + p_\theta]$ , where

$$\begin{aligned} p_{uu} &= \Delta^{-1}\nabla_i\nabla_j(\langle u_i u_j \rangle - u_i u_j), \\ p_{Uu} &= \Delta^{-1}\nabla_i\nabla_j(U_i u_j + U_j u_i), \\ p_\theta &= \beta\Delta^{-1}\nabla_z\theta, \end{aligned} \quad (3.1)$$

and the inverse Laplace operator  $\Delta^{-1}$  is defined as usual in terms of an integral over Green's function.

Correspondingly the correlations  $\mathcal{R}_{ij}$  and  $\mathcal{B}$  consist of three terms, equations (2.7d) and (2.7f), in which

$$\begin{aligned} R_{ij}^{\text{RI}} &= \langle p_{uu} s_{ij} \rangle, & R_{ij}^{\text{IP}} &\equiv \langle p_{Uu} s_{ij} \rangle, & R_{ij}^{\text{IC}} &\equiv \langle p_\theta s_{ij} \rangle, \\ B_i^{\text{RD}} &= \langle p_{uu} \nabla\theta \rangle, & B_i^{\text{SU}} &\equiv \langle p_{Uu} \nabla\theta \rangle, & B_i^{E\theta} &\equiv \langle p_\theta \nabla\theta \rangle. \end{aligned} \quad (3.2)$$

All of those terms originating from  $p_{uu}$  are the most problematic because they introduce coupling to triple correlation functions:  $R_{ij}^{\text{RI}} \propto \langle u_i u_j u_k \rangle$  and  $B_i^{\text{RD}} \propto \langle u^2 \nabla\theta \rangle$ . Thus they require closure procedures whose justification can be tested only *a posteriori* against the data.

Having in mind the aim to simplify the model as much as possible, we adopt, for the diagonal part of the return-to-isotropy tensor, the simplest Rotta form (Rotta 1951)

$$R_{ii}^{\text{RI}} \simeq -\gamma_{\text{RI}}(\tau_{ii} - 2E_K/3), \quad (3.3a)$$

in which  $\gamma_{\text{RI}}$  is the relaxation frequency of diagonal components of the Reynolds-stress tensor toward its isotropic form,  $2E_K/3$ . The parameterization of  $\gamma_{\text{RI}}$  will be discussed later. The tensor  $R_{ij}^{\text{RI}}$  is traceless; therefore the frequency  $\gamma_{\text{RI}}$  must be the same for all the diagonal components of  $R_{ii}^{\text{RI}}$ . On the other hand, there are no reasons to assume that off-diagonal terms have the same relaxation frequency. Therefore, following L'vov *et al* (2006a) we assume that

$$R_{ij}^{\text{RI}} \simeq -\tilde{\gamma}_{\text{RI}}\tau_{ij}, \quad i \neq j, \quad (3.3b)$$

with, generally speaking,  $\tilde{\gamma}_{\text{RI}} \neq \gamma_{\text{RI}}$ . Moreover, at the intuitive level, we can expect that off-diagonal terms should decay faster than the diagonal ones, i.e.  $\tilde{\gamma}_{\text{RI}} > \gamma_{\text{RI}}$ . Indeed, our analysis of DNS results shows that  $\tilde{\gamma}_{\text{RI}}/\gamma_{\text{RI}} \simeq 1.46$  (L'vov *et al* 2006b).

The term  $B_i^{\text{RD}}$  also describes return-to-isotropy due to nonlinear turbulence self-interactions (Zeman 1981), and may be modeled as:

$$B_i^{\text{RD}} = -\gamma_{\text{RD}}F_i. \quad (3.3c)$$

This equation dictates the vectorial structure of  $B_i^{\text{RD}} \propto F_i$ , which will be confirmed below. The rest can be understood as

the definition of  $\gamma_{\text{RD}}$  as the relaxation frequency of the thermal flux. Its parameterization is the subject of further discussion in section 3.4.

The traceless 'isotropization-of-production' tensor,  $R_{ij}^{\text{IP}}$ , has a very similar structure to the production tensor,  $\mathcal{P}_{ij}$ , equation (2.7a), and thus is traditionally modeled in terms of  $\mathcal{P}_{ij}$  (Pope 2001):

$$R_{ij}^{\text{IP}} \simeq -C_{\text{IP}}(P_{ij} - \delta_{ij}\mathcal{P}/3), \quad \mathcal{P} \equiv \text{Tr}\{P_{ij}\}. \quad (3.3d)$$

The accepted value of the numerical constant  $C_{\text{IP}} = \frac{3}{5}$  (Pope 2001).

The traceless 'isotropization-of-conversion' tensor,  $R_{ij}^{\text{IC}}$ , does not exist in unstratified TBL. Its structure is very similar to the conversion tensor,  $C_{ij}$ , equation (2.7b). Therefore it is reasonable to model it in the same way in terms of  $C_{ij}$  (Zeman 1981):

$$R_{ij}^{\text{IC}} \simeq -C_{\text{IC}}(C_{ij} - \delta_{ij}\mathcal{C}/3), \quad \mathcal{C} \equiv \text{Tr}\{C_{ij}\}, \quad (3.3e)$$

with a new constant  $C_{\text{IC}}$ .

The renormalization of production terms  $B_i^{\text{SU}}$  and  $B_i^{E\theta}$  is very similar to the corresponding thermal flux production terms,  $A_i^{\text{SU}}$  and  $A_i^{E\theta}$ , defined by equation (2.7e). Therefore, in the spirit of equations (3.3d) and (3.3e), they are modeled as follows:

$$B_i^{\text{SU}} = (C_{\text{SU}} - 1)A_i^{\text{SU}} = (1 - C_{\text{SU}})(\mathbf{F} \cdot \nabla)U_i, \quad (3.3f)$$

$$B_i^{E\theta} = -(C_{E\theta} + 1)A_i^{E\theta} = -2\beta(C_{E\theta} + 1)E_\theta\delta_{iz}. \quad (3.3g)$$

Using this and (3.1), one finds the sign of  $C_{E\theta}$ :

$$\begin{aligned} -\beta(C_{E\theta} + 1)E_\theta &= \langle \tilde{p}_\theta \nabla_z \theta \rangle = \beta \langle (\nabla_z \theta) \Delta^{-1}(\nabla_z \theta) \rangle, \\ C_{E\theta} &= -\left(1 + \langle (\nabla_z \theta) \Delta^{-1}(\nabla_z \theta) \rangle / \langle \theta^2 \rangle\right) < 0. \end{aligned} \quad (3.4)$$

To estimate  $C_{E\theta}$ , we assume that on the gradient scales the temperature fluctuations are roughly isotropic, and therefore we can estimate  $\Delta = \nabla_x^2 + \nabla_y^2 + \nabla_z^2 \approx 3\nabla_z^2$ . Introducing this estimate and integrating by parts leads to  $C_{E\theta} \approx -2/3$ .

### 3.2. Reynolds-stress-, thermal-flux- and thermal-dissipation

Far away from the wall and for large Reynolds numbers, the dissipation tensors are dominated by the viscous scale motions, at which turbulence can be considered as isotropic. Therefore, the vector  $\epsilon$  should vanish, whereas the tensor  $\epsilon_{ij}$ , equation (2.6), should be diagonal:

$$\epsilon_i = 0, \quad \epsilon_{ij} = 2\gamma_{uu}E_K\delta_{ij}/3, \quad (3.5a)$$

where the numerical prefactor  $\frac{2}{3}$  is chosen such that  $\gamma_{uu}$  becomes the relaxation frequency of the turbulent kinetic energy. Under stationary conditions, the rate of turbulent kinetic energy dissipation is equal to the energy flux through scales, which can be estimated as  $\langle uuu \rangle/\ell$ , where  $\ell$  is the outer scale of turbulence. Therefore, the natural estimate of  $\gamma_{uu}$  involves the triple-velocity correlator,  $\gamma_{uu} \sim (\langle uuu \rangle/\ell\langle uu \rangle)$ , exactly in the same manner as the return-to-isotropy frequencies,  $\gamma_{\text{RI}}$  and  $\tilde{\gamma}_{\text{RI}}$  in equations (3.3a) and (3.3b). Similarly,

$$\epsilon = \gamma_{\theta\theta}E_\theta, \quad \gamma_{\theta\theta} \sim \langle \theta\theta u \rangle/\ell\langle \theta\theta \rangle. \quad (3.5b)$$

### 3.3. Stationary balance equations in plain geometry

In the plane geometry, the equations simplify further. The mean velocity is oriented in the (streamwise)  $\hat{\mathbf{x}}$ -direction and all mean values depend on the vertical (wall-normal) coordinate  $z$  only:  $\mathbf{U} = U(z)\hat{\mathbf{x}}$ ,  $\Theta = \Theta(z)$ ,  $\tau_{ij} = \tau_{ij}(z)$ ,  $\mathbf{F} = \mathbf{F}(z)$ ,  $E_\theta = E_\theta(z)$ . Therefore  $(\mathbf{U} \cdot \nabla)(\dots) = 0$ , and in the stationary case, when  $\partial/\partial t = 0$ , the mean convective derivative vanishes:  $D/Dt = 0$ . Moreover, owing to the  $y \rightarrow -y$  symmetry of the problem, the following correlations vanish:  $\tilde{\tau}_{xy} = \tilde{\tau}_{yz} = \tilde{F}_y = 0$ . The only nonzero components of the mean velocity and temperature gradients are:

$$S_U \equiv dU/dz, \quad S_\Theta \equiv d\Theta/dz. \quad (3.6)$$

**3.3.1. Equations for the mean velocity and temperature profiles.** Having in mind equations of section 3.3 and integrating equations (2.2) for  $U_x$  and  $\Theta$  over  $z$ , one gets equations for the total (turbulent and molecular) mechanical-momentum flux,  $\tilde{\tau}(z)$ , and thermal flux,  $\tilde{F}$ , toward the wall

$$\tilde{\tau}_{xz}(z) = -\nu S_U + \tau_{xz} \Rightarrow \tilde{\tau}_{xz}(0) \equiv -\tau_*, \quad (3.7a)$$

$$\tilde{F}_z(z) = -\chi S_\Theta + F_z \Rightarrow \tilde{F}_z(0) \equiv -F_*. \quad (3.7b)$$

The total flux of the  $x$ -component of the mechanical moment in the  $z$ -direction is  $\rho_b \tilde{\tau}_{xz}(z) \equiv \int dz(\partial\langle p \rangle/\partial x) + \text{const}$ . Generally speaking,  $\tilde{\tau}_{xz}(z)$  depends on  $z$ . For example, for the pressure-driven planar channel flow (of the half-weight  $\delta$ )  $\rho_b \tilde{\tau}_{xz}(z) = (\partial\langle p \rangle/\partial x)(\delta - z) < 0$ .

Relatively close to the ground, where  $z \ll \delta$ , the  $z$  dependence of  $\tilde{\tau}_{xz}(z)$  can be neglected. In the absence of the mean horizontal pressure drop and spatially distributed heat sources,  $\tilde{\tau}$  and  $\tilde{F}$  are  $z$ -independent, and thus equal to their values at zero elevation, as indicated in equations (3.7) after the '⇒' sign. Notice that in our case of stable stratification, both vertical fluxes, the  $x$ -component of the mechanical momentum,  $\tilde{\tau}_{xz}$ , and the thermal flux,  $\tilde{F}_z$ , are directed toward the ground, i.e. negative. For the sake of convenience, we introduce in equations (3.7) notations for their (positive) zero-level absolute value:  $\tau_*$  and  $F_*$ .

Recall that in the plain geometry,  $U_z = 0$ . Nevertheless, one can write an equation for  $U_z$ :

$$d(\tau_{zz} + \langle p \rangle/\rho_b)/dz = \beta\Theta, \quad (3.8)$$

which describes a turbulent correction ( $\propto \tau_{zz}$ ) to the hydrostatic equilibrium. Actually, this equation determines the profile of  $\langle p \rangle$ , which does not appear in the system of balance equations (3.7).

**3.3.2. Equations for the pair (cross)-correlation functions.** Consider first the balance equation (2.5a) for the diagonal components of the Reynolds-stress tensor in algebraic model (which arises when we neglect the spatial fluxes):

$$\Gamma E_K + 3\gamma_{\text{RI}}\tau_{xx}/2 = -(3 - 2C_{\text{IP}})\tau_{xz}S_U - C_{\text{IC}}\beta F_z,$$

$$\Gamma E_K + 3\gamma_{\text{RI}}\tau_{yy}/2 = -C_{\text{IP}}\tau_{xz}S_U - C_{\text{IC}}\beta F_z, \quad (3.9)$$

$$\Gamma E_K + 3\gamma_{\text{RI}}\tau_{zz}/2 = -C_{\text{IP}}\tau_{xz}S_U + (3 + 2C_{\text{IC}})\beta F_z.$$

where  $\Gamma \equiv \gamma_{uu} - \gamma_{\text{RI}}$ . The lhs of these equations includes the dissipation and return-to-isotropy terms. In the rhs, we have the kinetic energy production and isotropization of production terms (both proportional to  $S_U$ ) together with the conversion and isotropization of conversion terms, which are proportional to the vertical thermal flux  $F_z$ . The horizontal component of the thermal flux,  $F_x$ , does not appear in these equations.

System (3.9) allows us to find anisotropy of the turbulent-velocity fluctuations and to get the balance equations for the turbulent kinetic energy with the energy production and conversion terms in the rhs:

$$3\tau_{xx} = 2\{[2(1 - C_{\text{IP}})\Gamma_{uu}/\gamma_{\text{RI}} + 1]E_K - (3 - 2C_{\text{IP}} + C_{\text{IC}})\beta F_z/\gamma_{\text{RI}}\}, \quad (3.10a)$$

$$3\tau_{yy} = 2\{[(C_{\text{IP}} - 1)\Gamma_{uu}/\gamma_{\text{RI}} + 1]E_K - (C_{\text{IP}} + C_{\text{IC}})\beta F_z/\gamma_{\text{RI}}\}, \quad (3.10b)$$

$$3\tau_{zz} = 2\{[(C_{\text{IP}} - 1)\Gamma_{uu}/\gamma_{\text{RI}} + 1]E_K - (C_{\text{IP}} - 2C_{\text{IC}} - 3)\beta F_z/\gamma_{\text{RI}}\}, \quad (3.10c)$$

$$\Gamma_{uu}E_K = -\tau_{xz}S_U + \beta F_z. \quad (3.10d)$$

Equation (3.10d) includes the only non-vanishing tangential (off-diagonal) Reynolds stress  $\tau_{xz}$  and has to be accompanied by an equation for this object:

$$\tilde{\gamma}_{\text{RI}}\tau_{xz} = (C_{\text{IP}} - 1)\tau_{zz}S_U + (1 + C_{\text{IC}})\beta F_x. \quad (3.10e)$$

This equation manifests that the tangential Reynolds stress  $\tau_{xz}$ , which determines the energy production (according to equation (3.10d)), influences, in turn, the value of the streamwise thermal flux  $F_x$ , which therefore affects the turbulent kinetic energy production.

As we have mentioned, in the plain geometry  $\tilde{F}_y = 0$ . Equation (2.5b) for the  $F_x$  and  $F_z$ , in this case, take the form:

$$\gamma_{\text{RD}}F_x = -(\tau_{xz}S_\Theta + C_{\text{SU}}F_zS_U), \quad (3.11a)$$

$$\gamma_{\text{RD}}F_z = -(\tau_{zz}S_\Theta + 2C_{\text{E}\Theta}\beta E_\theta), \quad (3.11b)$$

in which the rhs describes the thermal-flux production, corrected by the isotropization of production terms.

The last equation, equation (2.5c), for  $E_\theta$ , represents the balance between the dissipation (lhs) and production (rhs):

$$\gamma_{\theta\theta}E_\theta = -F_zS_\Theta. \quad (3.11c)$$

### 3.4. Simple closure of timescales and the balance equations in the turbulent region

At this point, we follow a tradition in modeling all the nonlinear inverse timescales by dimensional estimates (Kolmogorov 1941):

$$\gamma_{uu} = c_{uu}\sqrt{E_K}/\ell, \quad \gamma_{\text{RI}} = C_{\text{RI}}\gamma_{uu}, \quad (3.12)$$

$$\tilde{\gamma}_{\text{RI}} = \tilde{C}_{\text{RI}}\gamma_{\text{RI}}, \quad \gamma_{\theta\theta} = C_{\theta\theta}\gamma_{uu}, \quad \gamma_{\text{RD}} = C_{u\theta}\gamma_{uu}.$$



Recall that  $\ell$  is the ‘outer scale of turbulence’. This scale is equal to  $z$  for  $z < L$ , where  $L$  is the Obukhov length (definition is found below).

Detailed analysis of experimental, DNS and LES data (see L'vov *et al* (2006a, b) and references cited therein) shows that for unstratified flows,  $g = 0$ , the anisotropic boundary layers exhibit values of the Reynolds-stress tensor that can be well approximated by the values  $\tau_{xx} = E_K$ ,  $\tau_{yy} = \tau_{zz} = E_K/2$ . In our approach, this dictates the choice  $C_{RI} = 4(1 - C_{IP})$ . Also we can expect that  $\tau_{yy}$  is almost not affected by buoyancy. This gives simply  $C_{IC} = -C_{IP}$ . If so, equations (3.10) with the parameterization (3.12) can be identically rewritten as follows:

$$\begin{aligned} \tau_{xx} &= E_K - \frac{\beta F_z}{2\gamma_{uu}}, & \tau_{yy} &= \frac{E_K}{2}, & \tau_{zz} &= \frac{E_K}{2} + \frac{\beta F_z}{2\gamma_{uu}}, \\ \gamma_{uu} E_K &= \beta F_z - \tau_{xz} S_U, \\ \gamma_{uu} &= c_{uu} \sqrt{E_K/\ell}, \\ 4\tilde{C}_{RI} \gamma_{uu} \tau_{xz} &= \beta F_x - \tau_{zz} S_U. \end{aligned} \quad (3.13a)$$

For completeness we also repeated here the parameterization (3.12) of  $\gamma_{uu}$ . Finally, we present the version of the balance equations for the thermal flux (3.11a), (3.11b), and for the ‘temperature energy’, (3.11c), after all the simplified assumptions:

$$\begin{aligned} C_{\theta\theta} \gamma_{uu} E_\theta &= -F_z S_\Theta, \\ C_{u\theta} \gamma_{uu} F_x &= -(\tau_{xz} S_\Theta + C_{SU} F_z S_U), \\ C_{u\theta} \gamma_{uu} F_z &= -(\tau_{zz} S_\Theta + 2C_{E\Theta} \beta E_\theta). \end{aligned} \quad (3.13b)$$

### 3.5. Generalized wall normalization

The analysis of the balance equations (3.13) is drastically simplified if they are presented in a dimensionless form. Traditionally, the conventional ‘wall units’ are introduced via the wall friction velocity  $u_* \equiv \sqrt{\tau_*}$ , and the viscous lengthscale  $\lambda_* \equiv \nu/u_*$ . A wall unit for the temperature  $\theta_* \equiv F_*/u_*$  is defined via the thermal flux at the wall and friction velocity. Subsequently,  $\mathbf{r}^+ \equiv \mathbf{r}/\lambda_*$ ,  $t^+ \equiv t\lambda_*/u_*$ ,  $\mathbf{U}^+ \equiv \mathbf{U}/u_*$ ,  $p^+ \equiv p/\rho_b u_*^2$ ,  $\Theta^+ \equiv \Theta/\theta_*$ ,  $\theta^+ \equiv \theta/\theta_*$ , etc. Then the governing equations (2.1) take the form:

$$\begin{aligned} \mathcal{D}^+ \mathbf{U}^+ / \mathcal{D}t^+ + \nabla^+ p^+ &= \hat{\mathbf{z}} \Theta_d^+ / L^+ + \Delta^+ \mathbf{U}^+, \\ \mathcal{D}^+ \Theta_d^+ / \mathcal{D}t^+ &= \Delta^+ \Theta_d^+ / Pr. \end{aligned} \quad (3.14)$$

These equations include two dimensionless parameters: the conventional Prandtl number  $Pr = \nu/\kappa$ , and  $L^+$ —the Obukhov length  $L$  measured in wall units:  $L \equiv u_*^3/\beta F_*$ ,  $L^+ \equiv L/\lambda_*$ . We used here the modern definition of the Obukhov length, which differs from the old one by the absence of the von-Kármán constant  $\kappa$  in its denominator (Monin and Obukhov 1954).

Outside the viscous sublayer, where the kinematic viscosity and kinematic thermal conductivity can be ignored,  $L^+$  is the only dimensionless parameter in the problem, which

separates the region of weak stratification,  $z^+ < L^+$ , and the region of strong stratification, where  $z^+ > L^+$ .

Given the generalized wall normalization, we introduce objects with a superscript ‘+’ in the usual manner:

$$\begin{aligned} S_U^+ &\equiv t_* S_U, & S_\Theta^+ &\equiv \lambda_* S_\Theta/\theta_*, & \gamma^+ &\equiv t_* \gamma, \\ \tau_{ij}^+ &\equiv \tau_{ij}/u_*^2, & \mathbf{F}^+ &\equiv \mathbf{F}/u_* \theta_*, & E_\theta^+ &\equiv E_\theta/\theta_*^2. \end{aligned} \quad (3.15)$$

In the turbulent region, governed by  $L^+$  only, equations (3.7) simplify to  $\tau_{xz}^+ = -1$ ,  $F_z^+ = -1$ .

### 3.6. Rescaling symmetry and $\ddagger$ -representation

Outside the viscous region, where equations (3.13) were derived, the problem has only one characteristic length, i.e. the Obukhov scale  $L$ . Correspondingly, one expects that the only dimensionless parameter that governs the turbulent statistics in this region should be the ratio of the outer scale of turbulence,  $\ell$ , to the Obukhov lengthscale  $L$ , which we denote by  $\ell^\ddagger \equiv \ell/L = \ell^+/L^+$ . Indeed, introducing ‘ $\ddagger$ -objects’:

$$\ell^\ddagger \equiv \ell/L, \quad S_U^\ddagger \equiv S_U^+ \ell^+, \quad S_\Theta^\ddagger \equiv S_\Theta^+ \ell^+, \quad (3.16)$$

and using equations (3.15), one can rewrite the balance equations (3.13) as follows:

$$\tau_{xx}^+ = E_K^+ + \ell^\ddagger / 2c_{uu} \sqrt{E_K^+}, \quad \tau_{yy}^+ = E_K^+/2, \quad (3.17a)$$

$$2\tau_{zz}^+ = E_K^+ - \ell^\ddagger / c_{uu} \sqrt{E_K^+}, \quad (3.17b)$$

$$c_{uu} E_K^{3/2} = \ell^\ddagger F_z^+ - \tau_{xz}^+ S_U^\ddagger, \quad (3.17c)$$

$$4\tilde{C}_{RI} c_{uu} \sqrt{E_K^+} \tau_{xz}^+ = \ell^\ddagger F_x^+ - \tau_{zz}^+ S_U^\ddagger, \quad (3.17d)$$

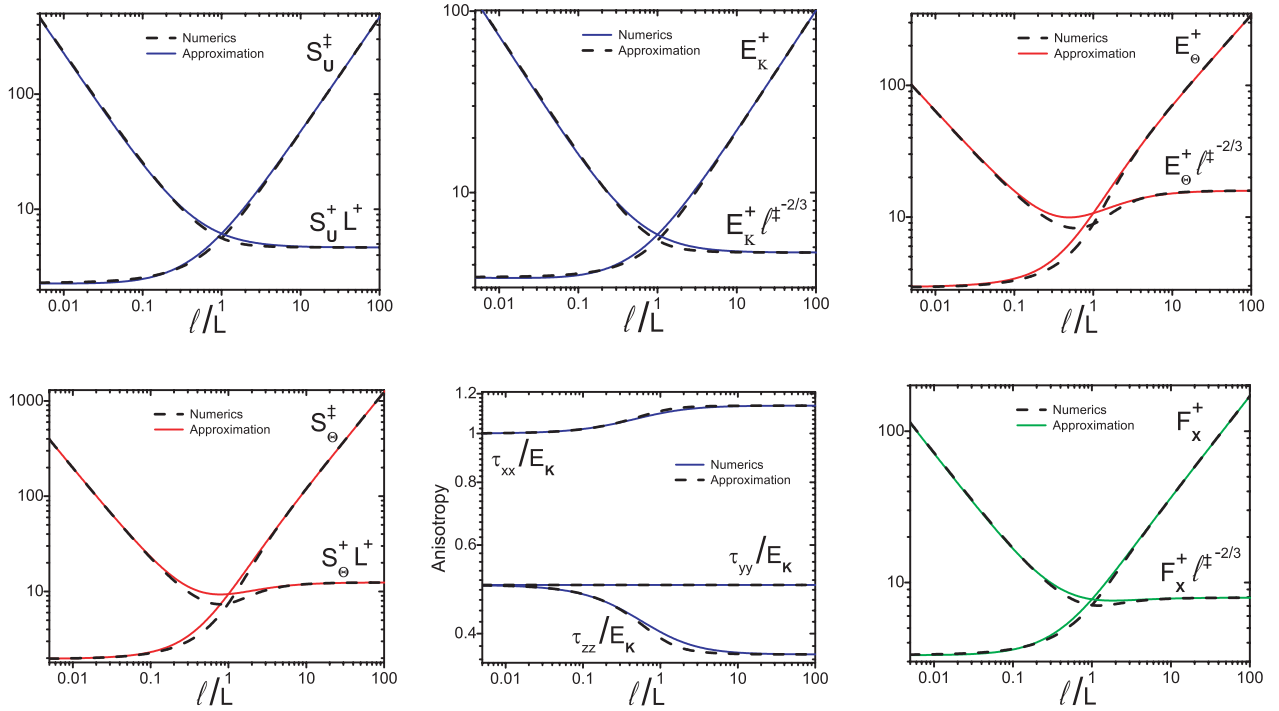
$$C_{\theta\theta} c_{uu} \sqrt{E_K^+} E_\theta^+ = -F_z^+ S_\Theta^\ddagger, \quad (3.17e)$$

$$C_{u\theta} c_{uu} \sqrt{E_K^+} F_x^+ = -\tau_{xz}^+ S_\Theta^\ddagger - C_{SU} F_z^+ S_U^\ddagger, \quad (3.17f)$$

$$C_{u\theta} c_{uu} \sqrt{E_K^+} F_z^+ = -\tau_{zz}^+ S_\Theta^\ddagger - 2C_{E\Theta} \ell^\ddagger E_\theta^+. \quad (3.17g)$$

These equations are the main result of this section 3. They may be considered as the ‘minimal model’ for stably stratified TBL, which respects the conservation of energy, describes anisotropy of turbulence and all relevant fluxes explicitly and, nevertheless, is simple enough to allow comprehensive analytical analysis, which results in an approximate analytical solution (with reasonable accuracy) for the mean velocity and temperature gradients  $S_U$  and  $S_\Theta$ , and all second-order (cross)-correlation functions.

As expected, the only parameter appearing in the minimal model (3.17) is  $\ell^\ddagger$ . The outer scale of turbulence,  $\ell$ , does not appear by itself but appears only via the definition of  $\ell^\ddagger$  (3.16). Therefore our goal now is to solve equations (3.17) in order to find five functions of only one argument  $\ell^\ddagger$ :  $S_U^\ddagger$ ,  $S_\Theta^\ddagger$ ,  $E_K^+$ ,  $E_\theta^+$  and  $F_x^+$ . After that, we can specify the dependence  $\ell^+(z^+)$  and then reconstruct the  $z^+$ -dependence of these five objects.



**Figure 1.** Log–log plots of the normalized velocity mean shears  $S_U^\dagger \equiv \ell^+ S_U^+$  and  $L^+ S_U^+$  (upper left panel), normalized mean-temperature gradients  $S_\Theta^\dagger \equiv \ell^+ S_\Theta^+$  and  $L^+ S_\Theta^+$  (lower left panel), the turbulent kinetic energy  $E_K^+$  and  $E_K^+/\ell^{\dagger 2/3}$  (upper middle panel), partial kinetic energies  $\tau_{ii}/E_K$  (lower middle panel), temperature energy  $E_\Theta^+$  and  $E_\Theta^+/\ell^{\dagger 2/3}$  (upper right panel) and horizontal thermal flux  $F_x^+$  and  $F_x^+/\ell^{\dagger 2/3}$  (lower right panel) versus  $\ell^\dagger = \ell/L = \ell^+/L^+$ . Red and blue solid lines: exact numerical solutions before and after normalization by the large  $\ell^\dagger$  asymptotics; black dashed and dot-dashed lines: approximate analytical solutions. The region  $\ell \gtrsim L$  may not be realized in Nature. In this case, it has only a methodological character.

## 4. Results and discussion

### 4.1. Analytical solution of the minimal-model balance equations (3.17)

This subsection is devoted to an analytical and numerical analysis of the minimal model (3.17). An example of numerical solution of equations (3.17) (with some reasonable choice of the phenomenological parameters) is shown in figure 1. Nevertheless, it would be much more instructive to have approximate analytical solutions for all correlations that will describe their  $\ell^\dagger$ -dependence with reasonable accuracy. The detailed cumbersome procedure of finding these solutions is skipped here, but a brief overview is as follows.

Equations (3.17) can be reformulated as a polynomial equation of ninth order for the only unknown  $\sqrt{E_K^+}$ . An analysis of its structure helps us to formulate an effective interpolation formula (4.1), discussed below. Hence, we found the solutions of equations (3.17) at neutral stratification,  $\ell^\dagger = 0$ , corrected up to the linear order in  $\ell^\dagger$ . Comparison of them with the existing DNS data resulted in an estimate for the constants  $\tilde{C}_{RI} \approx 1.46$  and  $c_{uu} \approx 0.36$ . Then, we considered the region  $\ell^\dagger \rightarrow \infty$ . Even though such a condition may not be realizable in Nature, from a methodological point of view, as we will see below, it enables us to obtain the desired analytical approximation. The  $\ell^\dagger \rightarrow \infty$  asymptotic solution with corrections, linear in the small parameter  $\ell^{\dagger -4/3}$ , were found. Now we are armed to suggest an interpolation formula

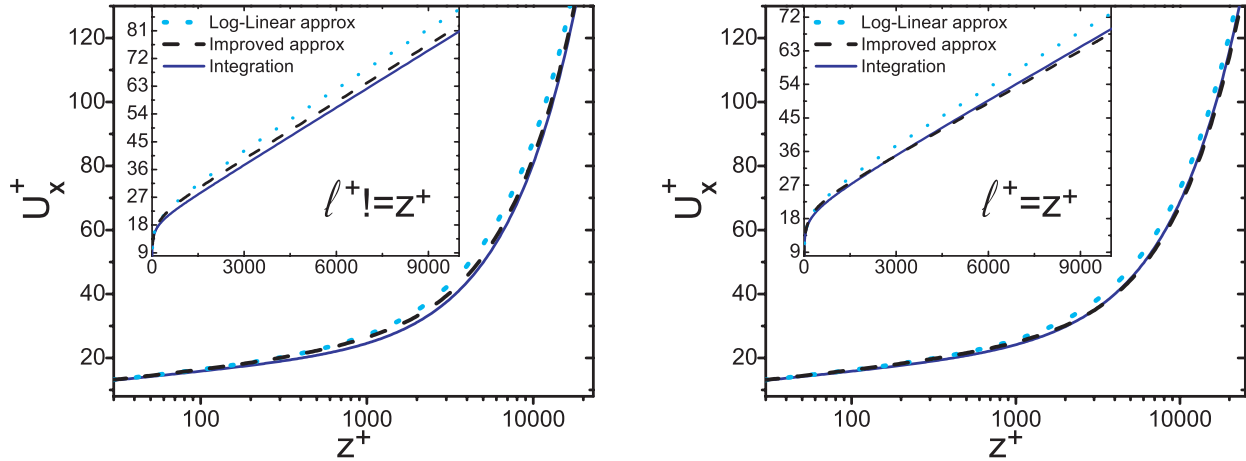
$$E_K^+ (\ell^\dagger)^{3/2} \simeq \frac{11\ell^\dagger}{3c_{uu}} + \frac{8\tilde{C}_{RI}}{\sqrt{(11\ell^\dagger/3c_{uu})^{2/3} + (8\tilde{C}_{RI})^{1/2}}}, \quad (4.1a)$$

which coincides with the exact solutions for  $\ell^\dagger = 0$  and for  $\ell^\dagger \rightarrow \infty$ , including the leading corrections to both asymptotics, linear in  $\ell^\dagger$  and  $\ell^{\dagger -4/3}$ . Moreover, in the region  $\ell^\dagger \sim 1$ , equation (4.1a) accounts for the structure of the exact polynomial. As a result, the interpolation formula (4.1a) is close to the numerical solution with deviations smaller than 3% in the entire region  $0 \leq \ell^\dagger < \infty$ , see the upper middle panel of figure 1. Together with equation (3.17c), it produces a solution for  $S_U^+$  that can be written as

$$S_U^+(\ell^+) \simeq (L_1^+)^{-1} + \left( \kappa \ell^+ \sqrt{1 + (\ell^+/L_2^+)^{2/3}} \right)^{-1}, \quad (4.1b)$$

where  $L_1^+ \equiv 3L^+/14$ ,  $L_2^+ \equiv 3L^+/11\kappa$  and  $\kappa$  is the von-Kármán constant. This formula gives the same accuracy  $\sim 3\%$ , see the upper left panel of figure 1. We demonstrate below that the proposed interpolation formulae describe the  $\ell^\dagger$ -dependence of the correlations with a very reasonable accuracy, about 10%, for any  $0 \leq \ell^\dagger < \infty$ , see the black dashed lines in figure 1.

Unfortunately, a direct substitution of the interpolation formula (4.1) into the exact relation for  $S_\Theta^\dagger$  obtained from the system (3.17) works well only for small  $\ell^\dagger$ , in spite of the fact that the interpolation formula is rather accurate in the whole region. We therefore need to derive an independent



**Figure 2.** Computed with equation (1.4) (for  $d_1 = d_2 = 1$ ) plots of  $U^+$  (blue solid lines) versus  $\ln(z/L)$  and versus  $z/L$  (insets) for  $L^+ = 1000$ . In the left panel,  $\ell(z)$  is taken from equation (1.4), whereas in the right panel,  $\ell(z) = z$ . Log-linear approximation (4.3a) to all profiles is shown by dotted lines and its improved version (4.3b) is shown by dashed lines. ‘!’ stands for  $\neq$ . The region  $\ell \gtrsim L$  may not be realized in Nature. In this case, it has only a methodological character.

interpolation formula for  $S_{\Theta}^{\ddagger}$ . Using expansions for small  $\ell^{\ddagger} \ll 1$  and for large  $\ell^{\ddagger} \gg 1$ , we suggest

$$S_{\Theta}^+(\ell^+) \simeq S_{\Theta}^{+\infty} + \frac{S_{\Theta,0}^+ + 6(c_{uu}\alpha)^{4/3} S_{\Theta,1}^{+\infty}}{(1 + \alpha \ell^+/L^+)^{4/3}}, \quad (4.1c)$$

in which

$$S_{\Theta,0}^+ = 2^{1/4} c_{uu} C_{U\Theta} / \tilde{C}_{RI}^{1/4} \ell^+,$$

$$S_{\Theta,1}^{+\infty} = -C_{u\theta} (2\tilde{C}_{RI} - (11C_{u\theta} - 3C_{SU})/3 S_{\Theta}^{+\infty} L^+) / L^+,$$

$$S_{\Theta}^{+\infty} = -14(C_{SU} - 4C_{U\Theta}/3) / 3L^+,$$

and  $\alpha$  satisfies

$$S_{\Theta,1}^+ \ell^+ = S_{\Theta}^{+\infty} L^+ + 6S_{\Theta,1}^{+\infty} L^+ (c_{uu}\alpha)^{4/3} - 4\alpha S_{\Theta,0}^+,$$

with

$$S_{\Theta,1}^+ \ell^+ = -C_{u\theta} (3/4 \tilde{C}_{RI} - 22 + 3C_{SU}/C_{u\theta}) / 24 \tilde{C}_{RI}.$$

Equation (4.1c) is constructed such that the leading and subleading asymptotics for small and large  $\ell^{\ddagger}$  coincide with the first two terms in the exact expansions at ‘almost’ neutral stratification and extremely strong stratification. As a result, equation (4.1c) approximates the exact solution with errors smaller than 5% for  $\ell^{\ddagger} < 1$  and  $\ell^{\ddagger} > 50$  and with errors smaller than 10% for any  $\ell^{\ddagger}$ , see the lower left panel of figure 1.

Substituting the approximate equations (4.1) into the exact relations (2.17), one gets approximate solutions  $E_{\theta}^+$  and  $F_x^+$  with errors smaller than 10%, see the rightmost panels of figure 1.

#### 4.2. Mean velocity and temperature profiles

In principle, integrating the mean shear  $S_U^+$  and the mean temperature gradient  $S_{\Theta}^+$ , one can find the mean velocity and temperature profiles. Unfortunately, to do so we need to know  $S_U^+$  and  $S_{\Theta}^+$  as functions of the elevation  $z$ , while in

our approach they are found as functions of  $\ell/L$ . Remember that the external parameter  $\ell$  is the outer scale of turbulence that depends on the elevation  $z$ . The importance of an accounting for the proper physically motivated dependence of  $\ell$  on  $z$  for an example of channels and pipes has recently been shown by L'vov et al (2008). For the problem at hand, we can safely take  $\ell = z$  if  $z \ll L$ ; however, when  $z > L$ , the function  $\ell(z)$  is not found theoretically although it was discussed phenomenologically with the support of observational, experimental and numerical data. It is traditionally believed that for  $z \gtrsim L$  the scale  $\ell$  saturates at some level of order  $L$  (see e.g. equation (1.4)).

The resulting plots of  $U^+$  are shown in figure 2, left panel. Even taking  $\ell(z) = z$ , one gets a very similar velocity profile, see figure 2, right panel. With  $\ell(z) = z$  we found an analytical expression for the mean-velocity profile using the interpolation equation (4.1b) for  $S_U^+$ :

$$U^+(z) = \frac{1}{\kappa} \ln \left[ \frac{z/z_{u0}}{(1 + \sqrt{1 + (z/L_2)^{2/3}})^3} \right] + \frac{z}{L_1}. \quad (4.2)$$

Here  $z_{u0}$  is the roughness length.

The resulting mean velocity profiles have logarithmic asymptotic for  $z < L$  and a linear behavior for  $z > L$ , in agreement with meteorological observations. Usually, the observations are parameterized by a so-called log-linear approximation (Monin and Obukhov 1954):

$$U^+ = \kappa^{-1} \ln(z/z_{u0}) + z/L_1, \quad (4.3a)$$

which is plotted as dotted lines in figure 2. One sees some deviation in the region of intermediate  $z$ . The reason is that the real profile (see e.g. equation (4.2)) has a logarithmic term that saturates for  $z \gg L$ , whereas in the approximation (4.3a) this term continues to grow. To fix this, one can use equation (4.2) (with  $L_2 = L_1$  for simplicity), or even its simplified version

$$U^+ = \frac{1}{\kappa} \ln \frac{z}{z_{u0} \sqrt{1 + (z/L_1)^2}} + \frac{z}{L_1}. \quad (4.3b)$$

This approximation is plotted as a dashed line in figure 2 for comparison. One sees that the approximation (4.3b) works much better than the traditional one. Thus we suggest equation (4.3b) for parameterizing meteorological observations.

The temperature profiles in our approach look similar to the velocity ones: they have logarithmic asymptotic for  $\ell < L$  and linear behavior for  $\ell > L$ . Correspondingly, they can be fitted by a log-linear approximation, like (4.3a), or even better, by an improved version of it, like equation (4.3b). Clearly, the values of constants will be different:  $\kappa \Rightarrow \kappa_T$ ,  $L_1 \Rightarrow L_{1,T}$ , etc.

### 4.3. Profiles of second-order correlations

The computed profiles of the turbulent kinetic and temperature energies, horizontal thermal flux profile and the anisotropy profiles are shown in the middle and right panels of figure 1. The anisotropy profiles, lower middle panel, saturate at  $\ell/L \approx 2$ ; therefore they are not sensitive to the  $z$ -dependence of  $\ell(z)$ ; even quantitatively one can think of these profiles as if they were plotted as a function of  $z/L$ .

Another issue is the profiles of  $E_K^+$  (upper middle panel) and of  $E_\Theta$  and  $F_x^+$  (rightmost panels), which are  $\propto (\ell/L)^{2/3}$  for  $\ell \gg L$  (if realizable). With the interpolation formula (1.4) the profiles of the second-order correlations have to saturate at levels corresponding to  $\ell^\ddagger = 1$ . This sensitivity to the  $z$ -dependence of  $\ell(z)$  necessitates a comparison of the prediction with experimental data.

### 4.4. Turbulent transport, Richardson and Prandtl numbers

In our notations, the turbulent viscosity and thermal conductivity, turbulent Prandtl number, the gradient- and flux-Richardson numbers are

$$\nu_T \equiv -\frac{\tau_{xz}}{S_U} = \frac{1}{S_U^+} \equiv C_\nu \left(\ell^\ddagger\right) \frac{\tau_{zz}^+}{\gamma_{uu}^+}, \quad (4.4a)$$

$$\chi_T \equiv -\frac{F_z}{S_\Theta} = \frac{1}{S_\Theta^+} \equiv C_\chi \left(\ell^\ddagger\right) \frac{\tau_{zz}^+}{\gamma_{uu}^+}, \quad (4.4b)$$

$$Pr_T \equiv \frac{\nu_T}{\chi_T} = \frac{S_\Theta^+}{S_U^+} = \frac{S_\Theta^\ddagger}{S_U^\ddagger}, \quad (4.4c)$$

$$Ri_{\text{grad}} \equiv \frac{\beta S_\Theta}{S_U^2} = \frac{S_\Theta^+}{L^+ S_U^{+2}} = \frac{\ell^\ddagger S_\Theta^\ddagger}{S_U^{\ddagger 2}}, \quad (4.4d)$$

$$Ri_{\text{flux}} \equiv \frac{\beta F_z}{\tau_{xy} S_U} = \frac{1}{L^+ S_U^+} = \frac{\ell^\ddagger}{S_U^\ddagger}, \quad (4.4e)$$

$$Ri_{\text{grad}} = Ri_{\text{flux}} Pr_T. \quad (4.4f)$$

With equations (4.4a) and (4.4b), we introduce also two dimensionless functions  $C_\nu(\ell^\ddagger)$  and  $C_\chi(\ell^\ddagger)$  that are taken as  $\ell^\ddagger$ -independent constants in the down-gradient transport approximation (1.3) described in the Introduction section. We will show, however, that these functions have a strong dependence on  $\ell^\ddagger$ , going to zero in the limit  $\ell^\ddagger \rightarrow \infty$  as  $1/\ell^\ddagger^{4/3}$ . Therefore this approximation is not valid for large  $\ell^\ddagger$  even qualitatively.

4.3.1. Approximation of down-gradient transport and its violation in stably stratified TBL. As we have mentioned in the Introduction section, the concept of the down-gradient transport assumes that the momentum and thermal fluxes are proportional to the mean velocity and temperature gradients, see equations (1.3):

$$\tau_{xz} = -\nu_T S_U, \quad F_z = -\chi_T S_\Theta, \quad (4.5)$$

where  $\nu_T$  and  $\chi_T$  are effective turbulent viscosity and thermal conductivity, which can be estimated by dimensional reasoning. Equations (1.3), giving this estimate, include additional physical arguments that vertical transport parameters should be estimated via vertical turbulent velocity,  $\sqrt{\tau_{zz}}$ , and characteristic vertical scale of turbulence,  $\ell_z$ . The relations between the scales  $\ell_j$  in different  $j$ -directions in anisotropic turbulence can be found in the approximation of time-isotropy, according to which

$$\frac{\sqrt{\tau_{xx}}}{\ell_x} = \frac{\sqrt{\tau_{yy}}}{\ell_y} = \frac{\sqrt{\tau_{zz}}}{\ell_z} \equiv \gamma \Rightarrow \gamma_{uu}. \quad (4.6)$$

Here,  $\gamma$  is a characteristic isotropic frequency of turbulence that for concreteness can be taken as the kinetic energy relaxation frequency  $\gamma_{uu}$ . The approximation (4.6) is supported by experimental data, according to which in anisotropic turbulence the ratios  $\ell_i/\ell_j$  ( $i \neq j$ ) are larger than the ratios  $\ell_i \sqrt{\tau_{jj}}/\ell_j \sqrt{\tau_{ii}}$  that are close to unity. With this approximation,  $\nu_T$  and  $\chi_T$  can be estimated as follows:

$$\nu_T = C_\nu \tau_{zz}/\gamma_{uu}, \quad \chi_T = C_\chi \tau_{zz}/\gamma_{uu}, \quad (4.7)$$

where, according to the approximation of down-gradient transport, the dimensionless parameters  $C_\nu$  and  $C_\chi$  are taken as constants, independent of the level of stratification.

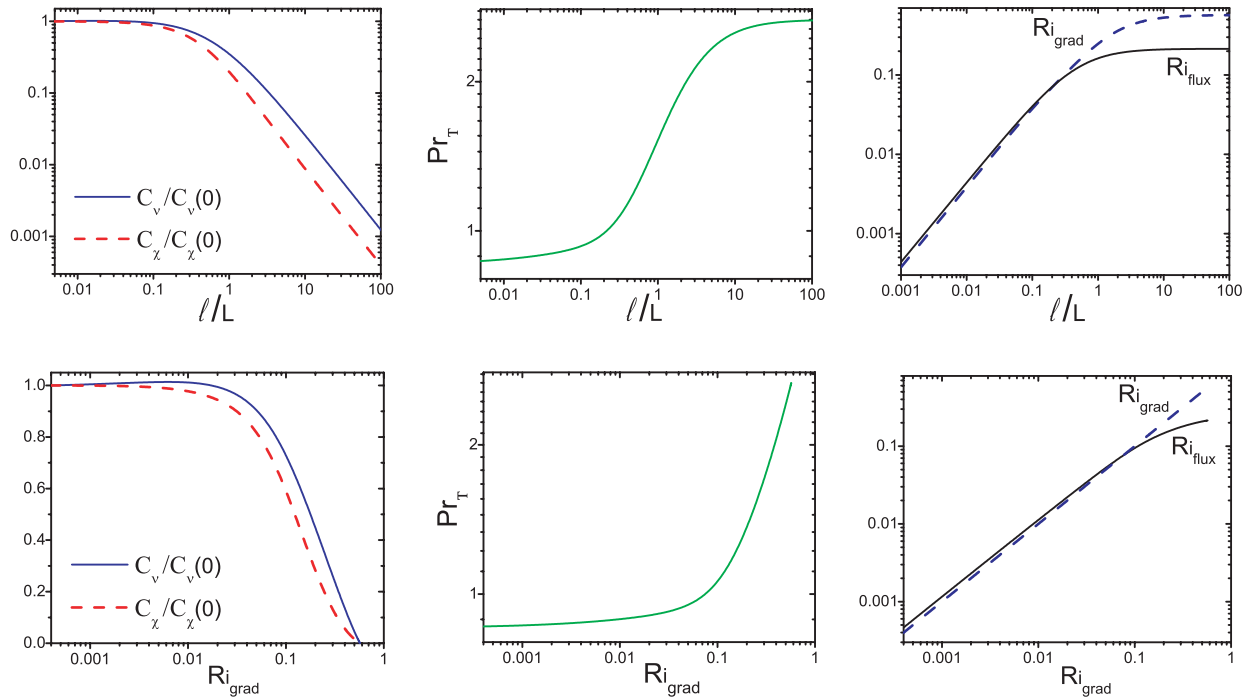
In order to check how the approximations (4.5) and (4.7) work in the stratified TBL for both fluxes, one can consider equations (4.5) as definitions of  $\nu_T$  and  $\chi_T$  and equations (4.7) as definitions of  $C_\nu$  and  $C_\chi$ . This gives

$$C_\nu \equiv -\frac{\tau_{xz}}{\tau_{zz}} \frac{\gamma_{uu}}{S_U} = \frac{\gamma_{uu}^+}{\tau_{zz}^+ S_U^+}, \quad (4.8a)$$

$$C_\chi \equiv -\frac{F_x}{\tau_{zz}} \frac{\gamma_{uu}}{S_\Theta} = \frac{\gamma_{uu}^+}{\tau_{zz}^+ S_\Theta^+}. \quad (4.8b)$$

Recall that, in this paper, the down-gradient approximation is not used at all. Instead, we are using exact balance equations for all relevant second-order correlations, including  $\tau_{xz}$  and  $F_x$ . Substituting our results in the rhs of the definitions (4.8), we can find how  $C_\nu$  and  $C_\chi$  depend on  $\ell^\ddagger = \ell/L$  that determines the level of stratification in our approach.

The resulting plots of the ratios  $C_\nu(\ell^\ddagger)/C_\nu(0)$  and  $C_\chi(\ell^\ddagger)/C_\chi(0)$  are shown in the leftmost panel of figure 3. One sees that  $C_\nu(\ell^\ddagger)$  and  $C_\chi(\ell^\ddagger)$  can be considered approximately as constants only for  $\ell \leq 0.2 L$ . For larger  $\ell/L$ , both  $C_\nu(\ell^\ddagger)$  and  $C_\chi(\ell^\ddagger)$  rapidly decrease, more or less in the same manner, diminishing by an order of magnitude already



**Figure 3.** Log–log plots of ‘down-gradient coefficient-functions’  $C_v$  (solid blue lines) and  $C_\chi$  (red dashed lines): left panels; turbulent Prandtl number  $Pr_T$  (green lines in middle panels) and  $Ri_{flux}$  (black solid lines),  $Ri_{grad}$  (black dashed lines): on right panels as a function of  $\ell^\ddagger = \ell/L$  (upper panels) and versus  $Ri_{grad}$  (lower panels). Notice that the presented dependencies have a qualitative character, and the choice of constants  $C_{\dots}$  depends on the actual functional form  $\ell(z)$ . For simplicity, we took  $\ell(z) = z$ . The region  $\ell \gtrsim L$  may not be realized in Nature. In this case, it has only a methodological character.

for  $\ell \approx 2L$ . For larger  $\ell/L$ , one can use the asymptotic solution according to which

$$\begin{aligned} S_U^+ &\simeq \frac{1}{L^+}, \\ \gamma_{uu} &\simeq \frac{\sqrt{E_K^+}}{\ell^+} \simeq \frac{\ell^\ddagger^{1/3}}{\ell^+}, \quad \tau_{zz} \simeq \ell^\ddagger^{2/3}. \end{aligned} \quad (4.9)$$

This means that both functions vanish as  $1/\ell^\ddagger^{4/3}$ :

$$\begin{aligned} C_v(\ell^\ddagger) &\simeq 0.01 \left(\frac{L}{\ell}\right)^{4/3}, \\ C_\chi(\ell^\ddagger) &\simeq 0.003 \left(\frac{L}{\ell}\right)^{4/3}, \end{aligned} \quad (4.10)$$

where numerical prefactors account for the accepted values of the dimensionless fit parameters.

The physical reason for the strong dependence of  $C_v$  and  $C_\chi$  on stratification is as follows: in the rhs of equation (3.10e) for the momentum flux and equation (3.11b) for the vertical heat flux, there are two terms. The first ones, proportional to  $\tau_{zz}$  and velocity (or temperature) gradients, correspond to the approximation (4.5), giving (in our notations)  $C_v = \text{const}$  and  $C_\chi = \text{const}$ , in agreement with the down-gradient transport concept. However, there are second contributions to the vertical momentum flux  $\propto F_\chi$  and to the vertical heat flux, which is proportional to  $\beta E_\theta$ . In our approach, both contributions are negative, giving rise to the *counter-gradient fluxes*. What follows from our approach is that these counter-gradient fluxes cancel (to the leading order) the down-gradient contributions in the limit  $\ell^\ddagger \rightarrow \infty$ . As a result, in this limit the effective turbulent diffusion

and thermal conductivity vanish, making the down-gradient approximation for them (with constant  $C_v$  and  $C_\chi$ ) irrelevant even qualitatively for  $\ell \gtrsim L$ .

In our picture of stable temperature-stratified TBL, the turbulence exists at any elevations, where one can neglect the Coriolis force. Moreover, the turbulent kinetic and temperature energies increase as  $(\ell/L)^{2/3}$  for  $\ell > L$ , see figure 1. At the same time, the mean velocity and potential temperature change the  $(\ell/L)$ -dependence from logarithmic to linear, see figure 2 and the (modified) log-linear interpolation formula (4.3b). Correspondingly, the shear of the mean velocity and the mean temperature gradient saturate at some elevation (and at some  $\ell/L$ ), and  $Ri_{grad}$  saturates as well. These predictions are in agreement with the large eddy simulation by Zilitinkevich and Esau (2006), where  $Ri_{grad}$  can be considered as saturating around 0.4 for  $z/L \approx 100$ .

Notice that the turbulent closures of the kind used above cannot be applied to strongly stratified flows with  $Ri_{grad} \gtrsim 1$  (maybe even at  $Ri_{grad} \sim 1$ ). There are two reasons for that. The first one was mentioned in the Introduction section. Namely, for  $Ri_{grad} \gtrsim 1$  the Brunt–Väisälä frequency  $N \equiv \sqrt{\beta S_\Theta}$ ,  $N^+ = \sqrt{S_\Theta^+/L^+}$ , is larger than the eddy-turnover frequency and therefore there are weakly decaying Kelvin–Helmholtz internal gravity waves, which, generally speaking, have to be accounted for in the momentum and energy balance equations.

The second reason, that makes the results very sensitive to the contribution of internal waves, follows from the fact that vortical turbulent fluxes vanish (at fixed velocity and temperature gradients). Therefore even relatively small

contributions of different kinds to the momentum and thermal fluxes may be important.

The final conclusion is that the TBL modeling at a large level of stratification requires accounting for the turbulence of the internal waves together with the vortical turbulence. It is very desirable that new observations and laboratory and numerical experiments with a control of internal wave activity will be conducted in future.

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## Appendix A. On the closure problem of triple correlations via second-order correlations

Let us look more carefully at the approximation (3.12), which is

$$\begin{aligned} \gamma_{uu} &= c_{uu} \sqrt{E_K} / \ell, & \gamma_{RI} &= C_{RI} \gamma_{uu}, \\ \tilde{\gamma}_{RI} &= \tilde{C}_{RI} \gamma_{RI}, & \gamma_{\theta\theta} &= C_{\theta\theta} \gamma_{uu}, & \gamma_{RD} &= C_{u\theta} \gamma_{uu}. \end{aligned} \quad (\text{A.1})$$

The dimensional reasoning that leads to this approximation is questionable for problems having a dimensionless parameter  $\ell^\ddagger$ . Generally speaking, all 'constants'  $c_{\dots}$  and  $C_{\dots}$  in equation (A.1) can be any function of  $\ell^\ddagger$ . Currently, we believe that a possible  $\ell^\ddagger$ -dependence of these functions is relatively weak and does not affect the qualitative picture of the phenomenon.

Moreover, even the assumption (3.5a) that the dissipation of the thermal flux  $\epsilon_i$  is proportional to the thermal flux and the assumption (3.5b) that the dissipation of  $E_\theta$ ,  $\epsilon \propto E_\theta$ , are also questionable. Formally speaking, one cannot guarantee that the triple cross-correlator  $\langle \theta uu \rangle^+$  that estimates  $\epsilon^+$  can be (roughly speaking) decomposed as  $\langle u\theta \rangle \sqrt{\langle uu \rangle}$ , i.e. really proportional to  $F = \langle u\theta \rangle$  as it is stated in equation (3.5a). Theoretically, one cannot exclude the decomposition  $\langle \theta uu \rangle \sim \langle uu \rangle \sqrt{\langle \theta\theta \rangle}$ , i.e. a contribution to  $\epsilon \propto E_K$ . Similarly, the dissipation  $\epsilon$  in the balance (2.5c) of  $E_\theta$ , which is determined by the correlator (3.5b), is  $\propto \langle \theta\theta u \rangle$ , as it follows from the decomposition  $\langle \theta\theta u \rangle \sim \langle \theta\theta \rangle \sqrt{\langle uu \rangle}$  and is stated in equation (3.5b). This correlator allows, for example, the decomposition  $\langle \theta\theta u \rangle \sim \langle \theta u \rangle \sqrt{\langle \theta\theta \rangle}$ , i.e. the contribution to  $\epsilon \propto F$ . This discussion demonstrates that the situation with the dissipation rates is not so simple as one may think and thus requires careful theoretical analysis, which is on our agenda for future work. Our preliminary analysis of this problem shows that all fitting constants are indeed functions of  $\ell^\ddagger$ . Fortunately, they vary within finite limits in the entire interval  $0 \leq \ell^\ddagger < \infty$ . Therefore we propose

that the approximations used in this paper preserve the qualitative picture of the phenomenon. Once again, the traditional down-gradient approximation does not work even qualitatively, because the corresponding 'constants'  $C_\nu$  and  $C_\chi$  vanish in the limit  $\ell^\ddagger \rightarrow \infty$ .

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