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# Symmetries and Interaction Coefficients of Kelvin Waves

**Vladimir V. Lebedev · Victor S. L'vov**

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**Abstract** We considered symmetry restriction on the interaction coefficients of Kelvin waves and demonstrated that linear in small wave vector asymptotic, obtained analytically, is not forbidden, as one can expect by naive reasoning. Therefore now we have no reason to doubt in this asymptote, that results in the L'vov-Nazarenko energy spectrum of Kelvin waves.

**Keywords** Kelvin waves · Energy spectra · Symmetry restrictions

## 1 Introduction

Kelvin-wave (KW) energy cascade is believed to be a relevant ingredient of the quantum turbulence realized at low temperatures in superfluids [1]. The cascade transfers energy from the intervortex distance to smallest scales determined by the vortex core radius. A theory of the cascade can be constructed in spirit of weak turbulence theory [2] starting from the Hamiltonian representation of the equations describing the vortex dynamics [3]. Due to the one-dimensional nature of the KWs the conservation laws for energy and mechanical moments forbid energy exchange in the $2 \leftrightarrow 2$ wave scattering. Only higher-order processes, starting from $3 \leftrightarrow 3$ scattering may result in the energy exchange between KWs and thus are relevant for the cascade [4, 5]. Therefore a central question concerning the local character of the cascade is related to asymptotic behavior of the corresponding $3 \leftrightarrow 3$-interaction vertices of the KWs.

V.V. Lebedev
Landau Institute for Theoretical Physics RAS, Moscow, Kosygina 2, 119334, Russia
e-mail: lebede@itp.ac.ru

V.S. L'vov (✉)
Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
e-mail: victor.lvov@gmail.com

Recent discussion [4–7] on the local/nonlocal nature of the KW cascade raised an important question concerning asymptotical behavior of $3 \leftrightarrow 3$-KW interaction amplitude

$$\mathcal{W}_{1,2,3}^{4,5,6} \equiv \mathcal{W}(k_1, k_2, k_3 | k_4, k_5, k_6) \tag{1}$$

in the interaction Hamiltonian

$$\mathcal{H}_6 = \frac{1}{36} \sum_{k_j} \mathcal{W}_{1,2,3}^{4,5,6} w_1 w_2 w_3 w_4^* w_5^* w_6^* \Delta_{1,2,3}^{4,5,6}. \tag{2}$$

Here $w_j \equiv w(k_j)$ is an amplitude of KWs with wave vector $k$, a Fourier transform of a (complex) two-dimensional displacement vector $w(z) = x + iy$ of a vortex line from the straight line $x = y = 0$. The summation over six $k$-vectors $k_j = k_1 \dots k_6$ is restricted by the Kroneker symbol, $\Delta_{1,2,3}^{4,5,6}$, equal to one if $k_1 + k_2 + k_3 = k_4 + k_5 + k_6$, and to zero otherwise.

Explicit (and cumbersome) calculations, see Ref. [6], gave the value of $\mathcal{W}_{1,2,3}^{4,5,6}$ in the asymptotical limit, when one or few wave vectors are much smaller than others:

$$\mathcal{W}_{1,2,3}^{4,5,6} = -\frac{3}{4\pi} k_1 k_2 k_3 k_4 k_5 k_6. \tag{3}$$

It is important that (3) gives

$$\mathcal{W} \propto k_1 \quad \text{for } k_1 \ll k_2 \sim k_3 \sim \cdots k_6. \tag{4}$$

If so, then the interaction vertex of short-wave motions (with wave vectors $k_2 \sim k_3 \sim \cdots k_6$) with long-wave "$k_1$-motions" is proportional to $k_1 w_1$, or, in the physical space, is proportional to the spatial $z$-derivative of the displacement, $\partial_z w(z)$.

One may think [8] that this asymptotic behavior contradicts to physical intuition, according to which the interaction cannot depend on the local slope of long-wave disturbances, $\partial_z w(z)$, because of rotational invariance of the interaction Hamiltonian $\mathcal{H}_6$. Indeed, one can choose the coordinate system, oriented along the (local) direction of the long-wave disturbances, in which $\partial_z w(z) = 0$. Then the curvature of the long-wave disturbances (proportional to the second derivative $\partial_z^2 w$) is expected [8] to be relevant. If this is true, then instead of asymptote (4) one has

$$\mathcal{W} \propto k_1^2 \quad \text{for } k_1 \ll k_2 \sim k_3 \sim \cdots k_6. \tag{5}$$

So, the dilemma is: either the cumbersome calculations [6] are mistaken and linear asymptote (4) is wrong, or something is wrong with the simple symmetry analysis [8], leading to quadratic asymptote (5).

The difference between the asymptote (4) and the asymptote (5) is of crucial importance for the physics of KW energy cascade:

- In the case of quadratic asymptote (5) the energy cascade should be dominated by local, step-by step energy transfer by interacting KWs with wave vectors of the same order of magnitude $k_1 \sim k_2 \sim k_3 \sim k_4 \sim k_5 \sim k_6$. This scenario leads to the

Kozik-Svistunov (KS) spectrum of KWs [4] with constant energy flux $\epsilon$:

$$E(k) \sim \frac{\Lambda \kappa^{7/5} \epsilon^{1/5}}{k^{7/5}}, \quad \text{KS energy spectrum.} \tag{6}$$

Here $\kappa$ is the circulation quantum, $\Lambda - \ln(\ell/a_0)$ with $\ell$ being mean intervortex distance, $a_0$-vortex core radius.

- In the case of the linear asymptote (4) the interactions between KWs in sixtets with very different wave vectors are much stronger and are dominated by the region, where two of three wave vectors from the sextet are of the order of inverse intervortex distance, $1/\ell$, and much smaller than the other four ones. For example, $k_1 \sim k_2 \sim k_3 \sim k_4 \gg k_5 \sim k_6 \sim \frac{1}{\ell}$. In this case the deviation of local direction of the vortex lines from its "global" direction [which can be characterized by the (dimensionless) mean-square of the "local" vortex direction $\Psi$ defined below in (7)] vortex lines of intervortex scales opens new "quartet" channel of an effective four-wave interaction with $k_1 + k_2 + k_3 = k_4$, that leads to the energy spectrum recently found by L'vov and Nazarenko (LN) [7]

$$E(k) \sim \frac{\Lambda \kappa \epsilon^{1/3}}{\Psi^{2/3} k^{-5/3}}, \quad \Psi \simeq \frac{8\pi}{\kappa \Lambda} \int_{1/\ell} E(k) dk, \quad \text{LN energy spectrum.} \tag{7}$$

Clearly, parameter $\Psi$ is dominated by the low-$k$ region of the wave vectors, $k\ell \sim 1$, where the KW energy spectrum is not universal and depends the form of the KW turbulence excitation. Therefore, dimensionless numerical factor $\Psi$ in the front of the LN energy spectrum (7) is not universal and depends on the particular KW energy distribution in the low $k$-region. However, the LN-scaling index $5/3$ is universal [and very different from $7/5$, the KS scaling index in (6)].

## 2 Rotational Symmetry and Line Length

To shed light on the contradiction between the asymptotics (4) and (5) let us consider a simple object: the length $L$ of a self-affine (without overhangs) line described by $x + iy = w(z)$ and fixed at the points $x = y = z = 0$ and $x = y = 0$, $z = L_0$. This length is given by the formally exact expression

$$L = \int_0^{L_0} \sqrt{1 + |\partial_z w(z)|^2} dz. \tag{8}$$

For small tilt, $|\partial_z w(z)| \ll 1$, the expression (8) can be expanded as follows:

$$L = L_0 + L_2 + L_4 + L_6 + \cdots, \tag{9}$$

where

$$L_2 = \frac{1}{2} \int_0^{L_0} |\partial_z w(z)|^2 \, dz = \frac{1}{2} \sum_k k^2 |w_k|^2, \tag{10}$$

describes individual contributions of waves with different wave vectors $k$ to the line length.

Higher terms are responsible for the cross-contributions to $L$ from waves with different $k$. For example:

$$L_4 = -\frac{1}{8}\int_0^{L_0} |\partial_z w(z)|^4\, dz = \sum_{k_i} T_{1,2}^{3,4} w_1 w_2 w_3^* w_4^* \Delta_{1,2}^{3,4}; \tag{11}$$

$$T_{1,2}^{3,4} = -\frac{1}{8} k_1 k_2 k_3 k_4, \tag{12}$$

$$L_6 = \frac{1}{16}\int_0^{L_0} |\partial_z w(z)|^6\, dz = \sum_{k_j} W_{1,2,3}^{4,5,6} w_1 w_2 w_3 w_4^* w_5^* w_6^* \Delta_{1,2,3}^{4,5,6}, \tag{13}$$

$$W_{1,2,3}^{4,5,6} = \frac{1}{16} k_1 k_2 k_3 k_4 k_5 k_6. \tag{14}$$

As one sees from (14), the vertices $T_{1,2}^{3,4}$ and $W_{1,2,3}^{4,5,6}$ have exactly the same linear in $k$ dependence as $\mathcal{W}_{1,2,3}^{4,5,6}$ in (3) and thus have the same problems with the "naive" physical intuition, reproduced above. Indeed, repeating the same reasoning, one may think [8] that effect of the long-wave disturbances cannot depend on its local slope, because "one can choose the coordinate system, oriented along the (local) direction of the long-wave disturbances, in which $\partial_z w(z) = 0$". If so, the vertices $T_{1,2}^{3,4}$ and $W_{1,2,3}^{4,5,6}$ have to be proportional to the square of the wave vector $k_1$ of the long-wave disturbances, and not to its first power, as in (14). However, the expressions (11)–(14) are definitely correct, being simple straightforward consequence of the expression (8).

To resolve this "contradiction" we will elaborate some consequences of the rotational symmetry of (8) for the line length. For this purpose we introduce "slow" line displacement $\xi(z) = x + iy$ [in the original, global $(x, y, z)$-reference system] and "fast" line displacement $u(z) = \tilde{x} + i\tilde{y}$ in the "local" $(\tilde{x}, \tilde{y}, \tilde{z})$-reference system with the origin following the slow displacement $\xi(z)$ and $\tilde{z}$ axis oriented along the local direction of the slow line $\xi(z)$. Then for small slow slopes, $\partial_z \xi(z) \ll 1$, the total line displacement in the global reference system can be approximated as follows:

$$w(z) = \xi(z) + u(z) + \mathrm{Re}[\partial_z \xi(z) u^*(z)]\partial_z u(z). \tag{15}$$

The last term here originates from the rotation of $\tilde{z}$-axis from the original direction of $z$-axis.

Now, in the $(\tilde{x}, \tilde{y}, \tilde{z})$-reference system, we can compute $\delta_{\xi u} L$, the cross-contribution to the line length caused by combine effect of the slow and fast displacements. Substituting $w(z)$ from (15) into (8) and integrating by parts, one gets in the linear in $\xi$ approximation:

$$\delta_{\xi u} L = \int dz\, \mathrm{Re}[\Phi^* \partial_z^2 \xi(z)], \quad \Phi \equiv \frac{u(z)}{\sqrt{1 + |\partial_z u(z)|^2}}. \tag{16}$$

In agreement with the symmetry reasoning the variation (16) is proportional to $\partial_z^2 \xi$ that is determined by the curvature of the line $w = \xi$.

However the line length (8) and its expansion (11)–(14) are written in terms of $w(z)$, (i.e. in the global reference system) while the variation (16) is presented via $\xi(z)$ in the local reference system. To rewrite the result (16) in terms of $w(z)$, one should use a transformation, inverse to (15). It can be done by iterations. The zero-order term is

$$\xi_0(z) = w(z) - u(z), \tag{17}$$

and the first iteration is

$$\xi_1(z) = \xi_0(z) - \mathrm{Re}[\partial_z \xi_0(z) u^*(z)]\partial_z u(z), \tag{18}$$

where $u(z)$ can be substituted by the fast part of $w(z)$. Substituting (18) into (16) one finds:

$$\delta L = -\int dz\, \mathrm{Re}\{\Phi^*[\partial_z \xi_0(z)\partial_z^2[u^*(z)\partial_z u(z)] \\ + \partial_z \xi_0(z)\partial_z^2[u(z)\partial_z u^*(z)]]\} + \cdots . \tag{19}$$

This equation explicitly contains the first derivative of the slow displacement $\xi_0(z)$. Therefore the rotational symmetry does not forbid linear in small $k_1$ terms in the expansion (9). It is confirmed by simple calculations leading to the expressions (11)–(14).

## 3 Long-Scale Behavior of Interaction Vertices

Here we show that the rotational symmetry also does not contradict to the linear in $k_1$ asymptote (4) of the vertex $\mathcal{W}_{1,2,3}^{4,5,6}$, (3). To see this we consider a single quantum self-affine vortex of length $L$, given by (8) and fixed, as before, at the boundary points. At zero temperature the vortex dynamics is determined by the formally exact Hamiltonian

$$\mathcal{H} = \frac{\kappa}{4\pi}\int \frac{dz_1\, dz_2}{R}\left\{1 + \mathrm{Re}\left[\frac{\partial w_1}{\partial z_1}\frac{\partial w_2^*}{\partial z_2}\right]\right\} \\ = -\frac{\kappa}{\pi}\int dz_1\, dz_2 \frac{\partial}{\partial z_1}\sqrt{R}\frac{\partial}{\partial z_2}\sqrt{R}, \tag{20}$$

with

$$R^2 = |w(z_1) - w(z_2)|^2 + (z_1 - z_2)^2 + a^2, \tag{21}$$

suggested by Sonin [9] (see also Ref. [3]). Here $\kappa$ is the circulation quantum and $a$ is the vortex core size (introduced for regularization).

Let us repeat for the Hamiltonian (20) the same logic steps as for the length $L$. Now we are interested in an expansion of the Hamiltonian over the slow variable $\xi$ and define the fast variable $u$ in the reference system attached to the slow variable. The first term of the expansion of the Hamiltonian can be found by a bit more complicated calculations than the ones leading to (16). First of all, we find

$$\begin{aligned}\delta R^2 &= \mathrm{Re}\bigg\{(\xi_1' u_1^*)\frac{\partial}{\partial z_1}R^2 + (\xi_2' u_2^*)\frac{\partial}{\partial z_2}R^2 \\ &\quad + 2u_1^*[\xi_1 - \xi_2 - \xi_1'(z_1 - z_2)] + 2u_2^*[\xi_2 - \xi_1 - \xi_2'(z_2 - z_1)]\bigg\} \\ &\to \mathrm{Re}\bigg\{(\xi_1' u_1^*)\frac{\partial}{\partial z_1}R^2 + (\xi_2' u_2^*)\frac{\partial}{\partial z_2}R^2 + (z_1 - z_2)^2(u_1^*\xi_2'' + u_2^*\xi_1'')\bigg\}, \end{aligned} \tag{22}$$

where prime means derivative over $z$. The transformations of the expressions in square brackets are justified since $\xi$ is the soft variable. Next, the variation of the Hamiltonian (20) can be written as

$$\begin{aligned}\delta_{\xi u}\mathcal{H} &= -\frac{2\kappa}{\pi}\int dz_1\, dz_2 \frac{\partial}{\partial z_1}\sqrt{R}\frac{\partial}{\partial z_2}\delta\sqrt{R} \\ &\to -\frac{2\kappa}{\pi}\int dz_1\, dz_2 \frac{\partial}{\partial z_2}\left[\mathrm{Re}(\xi_2' u_2^*)\frac{\partial}{\partial z_1}\sqrt{R}\frac{\partial}{\partial z_2}\sqrt{R}\right] \\ &\quad - \frac{\kappa}{2\pi}\int dz_1\, dz_2 \frac{\partial}{\partial z_1}\sqrt{R}\frac{\partial}{\partial z_2}\bigg[R^{-3/2}(z_1 - z_2)^2 \\ &\quad \times \mathrm{Re}\left(u_1^*\frac{\partial^2 \xi_2}{\partial z_2^2} + u_2^*\frac{\partial^2 \xi_1}{\partial z_1^2}\right)\bigg]. \end{aligned} \tag{23}$$

The second line in (23) disappears after integration in part. Thus we conclude that the principal contribution to the first-order term of the Hamiltonian expansion is proportional to $\partial_z^2 \xi$, i.e. can be written in the form similar to (16):

$$\delta\mathcal{H} = \int dz\, \mathrm{Re}[\Psi^*(z)\partial_z^2\xi(z)]. \tag{24}$$

An explicit expression for $\Psi(z)$ can be found from (23). The expression (24) is in accordance with the symmetry expectations.

Now, as before, we should return to original variables substituting $\xi(z)$ from (18). The resulting expression for $\delta\mathcal{H}$ can be obtained from (19) by replacing $\Phi \to \Psi$. Therefore the expression for $\delta\mathcal{H}$ contains the first derivative of the slow variable $\xi_0$. Thus the interaction amplitude $\mathcal{W}(k_1, k_2, k_3|k_4, k_5, k_6)$ with the linear in the wave vector $k$ long-scale asymptote (3) is not forbidden by the rotational symmetry.

In our derivation we used the "non-slipping" boundary conditions assuming that the vortex endpoints are fixed. However, all our arguments are applied to the periodic boundary conditions that were used by Kozik and Svistunov [4, 5] as well.

## 4 Conclusion

We found that the linear in small wave vector asymptote (4) of the interaction vertices of Kelvin waves $\mathcal{W}(k_1, k_2, k_3|k_4, k_5, k_6)$ (3), obtained in [6] by direct analytical calculations, is not forbidden by the rotational symmetry because it "violates" due to the boundary conditions that fix the endpoints of the vortex and, consequently, introduce a preferred direction in space. The same happens in the case of the periodic boundary conditions that were used by Kozik and Svistunov [4, 5]. Consequently, alternative "quadratic" asymptotic (5), suggested in [8] without any analytical derivation, has no symmetry support and thus has no reason to be accepted. Therefore now we have no reason to doubt in linear asymptote (4) which results in the L'vov-Nazarenko energy spectrum (7).

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