J Low Temp Phys (2010) 161: 606–610
DOI 10.1007/s10909-010-0240-1

Reply: On Role of Symmetries in Kelvin Wave Turbulence

V.V. Lebedev · V.S. L'vov · S.V. Nazarenko

Received: 1 July 2010 / Accepted: 30 September 2010 / Published online: 19 October 2010

Abstract In the Ref. (Lebedev and L'vov in J. Low Temp. Phys. 161, 2010, doi:10.1007/s10909-010-0215-2), this issue, two of us (VVL and VSL) considered symmetry restriction on the interaction coefficients of Kelvin waves and demonstrated that linear in small wave vector asymptotic, obtained analytically, is not forbidden, as Kosik and Svistunov (KS) expect by naive reasoning. Here we discuss this problem in additional details and show that theoretical objections by KS, presented in Ref. (Kozik and Svistunov in J. Low Temp. Phys. 161, 2010, doi:10.1007/s10909-010-0242-z), this issue, are irrelevant and their recent numerical simulation, presented in Ref. (Kozik and Svistunov in arXiv:1007.4927v1, 2010) is hardly convincing. There is neither proof of locality nor any refutation of the possibility of linear asymptotic of interaction vertices in the KS texts, Refs. (Kozik and Svistunov in J. Low Temp. Phys. 161, 2010, doi:10.1007/s10909-010-0242-z; arXiv:1006.0506v1, 2010). Therefore we can state again that we have no reason to doubt in this asymptote, that results in the L'vov–Nazarenko energy spectrum of Kelvin waves.

Keywords Kelvin waves · Interaction locality · Rotational symmetry · Numerical simulations

V.V. Lebedev
Landau Institute for Theoretical Physics RAS, Kosygina 2, Moscow 119334, Russia
e-mail: lebede@itp.ac.ru

V.S. L'vov (✉)
Department of Chemical Physics, The Weizmann Institute of Science, Rehovot 76100, Israel
e-mail: victor.lvov@gmail.com

S.V. Nazarenko
Warwick, UK

The argumentation of E.V. Kozik and B.V. Svistunov (KS) in their Comment [1] on Notes "*Symmetries and Interaction coefficients of Kelvin waves*" by two of us (VVL & VSL) [2] relies mainly on the KS text "*Geometric Symmetries in Superfluid Vortex Dynamics*" [3]. Therefore we start with commenting this text.

First part of Ref. [3] reproduces the KS theoretical scheme of the Kelvin wave (KW) cascade originally presented in Refs. [4, 5]. Next, they discuss separation of fast $u(z)$ and slow $\zeta(z)$ variables introducing a fast field u in the local reference system attached to the vortex position determined by the slow variable ζ. Then, in complete accordance with our derivation, presented earlier in [2], KS concluded that vertices describing interaction of slow and fast variables depend on the vortex curvature determined by the slow variable, i.e. that the vertices are proportional to the second power of the small wave vector. This is treated by KS as a proof of the cascade locality. However, as we stressed in [2], the interaction vertices in the kinetic equation for the KWs are not the same as obtained for the above variable w. The vertices in the kinetic equation are derived via a canonical transformation to the action variables (normal coordinates) which is obviously different from the above transformation to the local variables. And, as we demonstrated in Ref. [2], the symmetry does not forbid linear in k asymptotic behavior of such vertices, and, consequently, cannot rule out the L'vov-Nazarenko (LN) spectrum of KWs, in which nonlocal interaction of short KWs with the long ones are crucial for the energy transfer [6]. They make possible 4-wave processes, which lead to a power-law dependence $E(k) \propto k^{-5/3}$, in contrast to the KS local 6-wave scenario with $E(k) \propto k^{-17/5}$.

To restrict the discussion we would like to emphasize that both KS and LN approaches considered small amplitude KWs, propagating along straight vortex line in spite of the fact that the vortex lines in the real superfluid vortex tangle are never exactly straight. This approximation is justified, when studied short KWs for which the mean-free path λ is shorter then the typical curvature radius of vortex line ℓ. In this case a short piece of vortex of length λ can be approximate as linear. This kind of approximation is completely natural and common for many physical systems, e.g. in considering turbulence of the gravity waves on sea surface we treat the water surface locally as flat, even though at large scale the Ocean is spherical. In other words, in both KS and LN approaches there is "global coordinate frame" with a fixed z axis.

Theory of large amplitude KWs, or KWs with a wave-length compatible with ℓ is much more involved and still awaits for its development. In the framework of this future theory one may describe how the global topology of the vortex lines (e.g. close vortex loops) and the boundary conditions at its distant ends affect the KW dynamics.

Now some remarks concerning the specific KS assertions, made in [1]:

"In fact, the naiveness of the authors of Ref. [2] is in failing to appreciate that, being intermediate auxiliary concepts, the bare vertices themselves have no direct physical meaning and thus are not supposed to respect the tilt invariance on the individual basis."

– This strange conclusion have probably appeared because VVL & VSL in Ref. [2] have considered the length functional L which happens to be a building block of the effective six-wave interaction Hamiltonian. However, this part of Ref. [2] did not mean to treat the *dynamical* problem of KW behavior, but considered instead a simpler *self-contained geometrical* example of finding total length L of an arbitrary (not necessarily vortex) line in a 3D space. Indeed, L is the simplest rotationally invariant object and its properties can be easily established by the direct calculations, which allows to shed light onto general features of the rotationally invariant quantities. However, as it shown in Ref. [2], the KS symmetry arguments applied to this system predict that the k-space expansion coefficients in L could not have linear asymptotics in k, which is definitely wrong, contradicting to the trivial two-line calculations (Taylor expansion and Fourier transform), presented, in particular, in the KS papers [4] as well.

We think that this contradiction between the KS symmetry "arguments" [1, 3] and KS calculations [4] is enough to close the discussions.[1] However for those who want to follow its more involved aspects we will continue to present our argumentation:

It is true, that the functional L (besides being a real geometrical object) is meaningful also in the context of the KW dynamics. Indeed, in this problem L represents the leading order Hamiltonian (with no additional contributions involved), known as the Local Induction Approximation (LIA). Of course, LIA is an integrable model and it would not lead the energy transfers between KWs with different k-vectors. However, the energy exchange is not the only physical effect that can be measured and discussed. For example, there is another nontrivial and observable physical effect: the nonlinear shift of KW with wave vector k, caused by another KW with wave vector k'.

This effect is described by the leading order four-KW "bare" vertex even in the LIA approximation. Again, the KS symmetry "arguments" [1, 3] gives wrong result for this physical effect. Moreover, this example shows that sometimes "the bare vertices themselves" do have direct physical meaning.

"Analogously, in a more general case of the full Biot-Savart model, the invariance of dynamics with respect to the shift and tilt of the vortex line prescribes that, after combining the corresponding vertices $T_{1,2}^{3,4}$ and $W_{1,2,3}^{4,5,6}$ into $V_{1,2,3}^{4,5,6}$, the terms $\propto k_1k_2k_3k_4k_5k_6$ at $k_1 \ll k_{2,\dots,6}$ must cancel exactly leaving the first non-vanishing contribution to $V_{1,2,3}^{4,5,6}$ proportional to k_1^2, which was demonstrated in our Ref. [3]." – Nothing of the kind was proved by KS, neither in the present nor in the previous paper. KS only showed absence of the linear k-asymptotics in the interaction vertex for the *local amplitudes* $u(z) = \tilde{x} + i\tilde{y}$ in the local $(\tilde{x}, \tilde{y}, \tilde{z})$-reference system with the origin following the slow displacement and $\tilde{z}$ axis oriented along the local direction of the slow line. This is not true for the basic Hamiltonian, written in terms of the global deviations $w(z) = x + iy$ in the original *global* reference (x, y, z) in which the kinetic equation is derived. Let us stress again, all "bare" vertexes in the basic Hamiltonian are physically meaning quantities, being responsible on observable effects, as we show on simple example of nonlinear frequency shift in the LIA Hamiltonian.

Most recently, KS decided not to reply directly to our counter-arguments, and tried to move the discussion from the field of theoretical physics to numerics, presenting

[1] One may ask, where KS were mistaken, in Refs. [1, 3] that present symmetry "arguments" or in Ref. [4] that shows well known result of trivial direct calculations?

their new simulation of KWs as an evidence of their correctness [7]. However, this simulation is hardly convincing for the following reasons.

– *Inconsistent numerical method used.* To accelerate the numerical code, KS have used a numerical method in which the short-scale dynamics was ignored in the large-scale evolution. Although it is not completely clear what it means for the interactions in the k-space, it is rather obvious that in essence such procedure alters the nonlocal interactions, possibly enforcing the interaction locality, which is precisely something one must avoid when testing the role of the nonlocal interactions.
– *Result of KS Ref.* [7] *is different from the KS Ref.* [8]. By direct examinations of the numerical spectrum plotted in Ref. [8], one can see that the LN spectrum ($-5/3$ slope) fits better their data—over 1.5 decades compared to just over half-decade of the KS fit ($-7/5$ slope). Why is it different in the new KS simulations [7]? One of the possible explanations is the bottleneck phenomenon related to either completely absent or a very abrupt dissipation of the energy cascade at high wavenumbers. KS in [7] used an abrupt cutoff of the kind which is notoriously known in hydrodynamic turbulence simulations to lead to bottlenecks. For the simulation KS [8], no mention about using of high frequency dissipation was given at all. Another reason for discrepancy between the KS [8] and KS [7] results could be that both simulations were for unforced freely-evolving WT, which could lead to different spectra at different stages of the evolution. No evidence was presented in either of these simulations why the presented spectrum should be thought of as saturated and universal. It would be much cleaner to consider a forced-dissipated system which could reach a statistically steady state. Also, Ref. [7] mentions a k-space filtering procedure to eliminate noise, but no evidence was presented that this procedure does not change the spectral slope.

Finally, we would like to say that we are surprised by lighthearted attitude of KS to the results of direct calculation of the interaction vertex $V_{1,2,3}^{4,5,6}$ in the full Biot-Savart case done in Ref. [9]. This paper has fixed serious mistakes in the leading order of $V_{1,2,3}^{4,5,6}$ made in the previous papers of KS, and for the first time completed the analytical calculation of this vertex. Since, as we showed in [2], the symmetry arguments do not prevent the linear k-asymptotic in $V_{1,2,3}^{4,5,6}$, the direct calculation appears to be the only way to find the leading-order asymptotic behavior in order to analyze the locality property.

To conclude, there is neither proof of locality nor any refutation of the possibility of linear asymptotic behavior of interaction vertices in the texts [1, 3] of Kozik and Svistunov.

References

1. E. Kozik, B. Svistunov, J. Low Temp. Phys. **161** (2010). doi:10.1007/s10909-010-0242-z, this issue
2. V.V. Lebedev, V.S. L'vov, J. Low Temp. Phys. **161** (2010). doi:10.1007/s10909-010-0215-2, this issue
3. E. Kozik, B. Svistunov, Geometric symmetries in superfluid vortex dynamics. arXiv:1006.0506v1 [cond-mat.other] 2 Jun 2010
4. E. Kozik, B. Svistunov, Phys. Rev. Lett. **92**, 035301 (2004)
5. E. Kozik, B. Svistunov, J. Low Temp. Phys. **156**, 193 (2009)

6. V.S. L'vov, S. Nazarenko, Pis'ma ZhETF **91**, 464–470 (2010)
7. E. Kozik, B. Svistunov, arXiv:1007.4927v1 [cond-mat.other] July (2010)
8. E. Kozik, B. Svistunov, Phys. Rev. Lett. **94**, 025301 (2005)
9. J. Laurie, V.S. Lvov, S. Nazarenko, O. Rudenko, Phys. Rev. B **81**, 104526 (2010)