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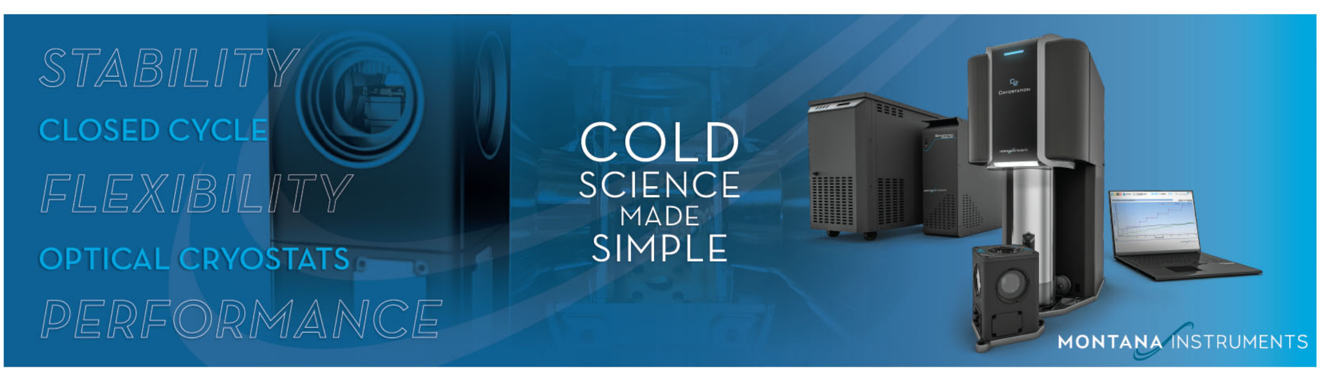

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Weak turbulence of Kelvin waves in superfluid He

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The physics of small-scale quantum turbulence in superfluids is essentially based on knowledge of the energy spectrum of Kelvin waves, E_k. Here we derive a new type of kinetic equation for Kelvin waves on quantized vortex filaments with random large-scale curvature which describes a step-by-step energy cascade over scales resulting from five-wave interactions. This approach replaces the earlier six-wave theory, which has recently been shown to be inconsistent owing to nonlocalization Solving the four-wave kinetic equation, we found a new *local* spectrum with a universal (curvature-independent) exponent, $E_k \propto k^{-5/3}$, which must replace the nonlocal spectrum of the six-wave theory, $E_k \propto k^{-7/5}$ in any future theory, e.g., when determining the quantum turbulence decay rate, found by Kosik and Svistunov under an incorrect assumption of locality of energy transfer in six-wave interactions. *© 2010 American Institute of Physics.*
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I. PHYSICAL BACKGROUND

Turbulence in superfluids[1,2] is a most fascinating natural phenomena, in which the transition from the laws of classical physics to quantum mechanics proceeds gradually as energy passes from large to small scales along a turbulent cascade. This coexistence of classical and quantum physics in the same system and their interplay is a fundamental consequence of the absence of viscosity, which serves in classical turbulence to quench the energy cascades over scales which are still large enough to be classical. In superfluids, on the other hand, when the temperature is close to absolute zero, this quenching mechanism is absent, and the energy flux unavoidably approaches scales where quantization of the vortex circulation (discovered by Feynman[3]) is essential. Recently, there have been significant advances in experimental techniques allowing studies of turbulence in various systems such as ^{3}He,[4,5] ^{4}He[6,7] and Bose–Einstein condensates of supercold atoms.[8,9] Experimental devices are often not small enough to probe the transitional and quantum scales directly. The impressive advances in numerical simulations[10,11] have been very important because they provide access to some characteristics of turbulence that are not yet available experimentally. In the zero-temperature limit, one of the most interesting questions is the nature of the energy dissipation, specifically the mechanisms for transfer of energy down to tiny (almost atomic) scales where vortices can radiate their energy away by emitting phonons.

A commonly accepted model of superfluid turbulence consists of a randomly moving tangle of quantized vortex lines which can be characterized by a mean intervortex distance ℓ and a vortex core radius $a \leqslant \ell$. The vortex core radius has atomic dimensions and the conventional description for fluid media fails in it. There are two ways to deal with the vortex core. The first is a "microscopic" model in which the core is resolved: it is based on the Gross–Pitaevski equation,

$$\frac{\partial \Psi}{\partial t} + \nabla^2 \Psi - \Psi |\Psi|^2 = 0, \tag{1}$$

where Ψ is so-called condensate wave function. This model has been systematically derived for Bose–Einstein condensates of super-cold atoms, but not for liquid helium. Nevertheless, it is frequently used for describing superfluid flows in helium because it incorporates several essential features of these superfluids, i.e., vortex quantization, acoustic waves (phonons) in presence of a condensate, and a description of the gradual (nonsingular) reconnection of vortex lines.

The Gross–Pitaevski equation can, however, be costly to study, and one often resorts to using the so-called Biot–Savart formulation of the Euler equations for ideal classical fluids, exploiting the fact that far away from the vortex cores the Gross–Pitaevski dynamics is isomorphic to the ideal classical flow via a Madelung transformation. In the Biot–Savart model, the vortices are postulated in terms of a cutoff in the equations for the vortex line elements. The equations used here are

$$\dot{\mathbf{r}} = \frac{\kappa}{4\pi} \int \frac{d\mathbf{s} \times (\mathbf{r} - \mathbf{s})}{|\mathbf{r} - \mathbf{s}|^3}, \tag{2}$$

with a cutoff at the core radius a, i.e., integrating over the range $|\mathbf{r}-\mathbf{s}| > a$. Here κ is the circulation quantum. In the following we rely on the Biot–Savart model.

Naturally, at scales $L \geqslant \ell$ the discreteness is unimportant and these can be described classically with an energy flux toward smaller scales in accordance with the celebrated Richardson-Kolmogorov cascade. Then energy is transferred through the crossover scale ℓ by some complicated mechanisms,[17–19] thereby exciting smaller scales $\ell < \lambda < a$ which propagate along the individual vortex filaments as waves. These were predicted by Lord Kelvin more than a century ago[12] and experimentally observed in superfluid ^{4}He about 50 years ago. It is believed that Kelvin waves (KWs) play a crucial role in superfluid dynamics, transferring energy from ℓ to a much smaller scale, where it can be dissipated via emission of bulk phonons. Over a wide range of scales KWs are weakly nonlinear and can be treated within the theory of weak-wave turbulence.[16] Such an approach for KWs was initiated in Ref. 13, where a six-wave kinetic equation (KE) was introduced, and a KW spectrum was derived from this equation using dimensional analysis, $E_{KS}(k) \propto k^{-7/5}$. Dimensional analysis of the KE is based on the *assumption* that all integrals in the collision term converge. Physically, this means that energy transfer can be treated as a step-by-step cascade in which energy entering a given range of wave vectors k comes from smaller k' of the same order of magnitude and is transferred to larger k'', again of the order of k. This assumption, first suggested in 1941 by Kolmogorov for hydrodynamic turbulence is often called "localization of energy transfer." The spectrum $E_{KS}(k) \propto k^{-7/5}$ was subsequently used in theoretical constructions of superfluid turbulence, e.g., to describe the classical-quantum crossover range of scales and to explain the dissipation rate in the superfluid turbulence.[17–20] However, it was recently shown[14] that this spectrum is nonlocal and, therefore, unrealizable. This crucial locality check was only possible after the highly nontrivial calculation of the six-wave interaction coefficient had been completed;[14] this took into account important contributions that had been neglected previously and yielded explicit expressions for this coefficient in the relevant asymptotic limits.

In this paper, we exploit the consequences of the nonlocalization of the 6-wave theory, and replace the latter with a new local 5-wave theory of KW turbulence. Our 5-wave theory arises from the 6-wave theory (completed in Ref. 14) in the strongly nonlocal case, when one of the waves in the sextet is much longer than the other five and corresponds to the outer scale—infra-red (IR)—cutoff. We derive a new, local spectrum of the KW turbulence, which should be used to revise the parts of superfluid turbulence theory where the nonlocal spectrum of the 6-wave theory has been used in the past.

II. STATISTICAL DESCRIPTION OF WEAK-WAVE TURBULENCE

Weak-wave turbulence refers to a class of strongly nonequilibrium statistical systems consisting of a large number of weakly nonlinear waves excited in nondissipative (Hamiltonian) dispersive media.[16] These systems constitute a unique example where strongly nonequilibrium statistics can be addressed systematically, and states analogous to Kolmogorov–Richardson cascades of classical turbulence can be obtained analytically. Let us briefly review the theory of weak-wave turbulence with application to the five- and six-wave systems (three- and four-wave processes are absent in KWs) starting from a classical Hamiltonian equation for the complex canonical amplitude of waves $a_{\mathbf{k}} \equiv a(\mathbf{k},t)$ and $a^*_{\mathbf{k}}$ (classical analogues of the Bose-operators of particle creations and annihilation) with a wave vector $\mathbf{k}$,

$$i\frac{\partial a_{\mathbf{k}}}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*_{\mathbf{k}}}. \tag{3}$$

Here $\mathcal{H}$ is a Hamiltonian which for the wave systems is given by

$$\mathcal{H} = \mathcal{H}_{\rm free} + \mathcal{H}_{\rm int}; \quad \mathcal{H}_{\rm free} = \int \omega_k a_{\mathbf{k}} a^*_{\mathbf{k}} d\mathbf{k}, \tag{4}$$

where ω_k is the wave frequency. For KWs, $\omega_k = \Lambda\kappa k^2/4\pi$ where κ is the quantum of circulation. $\mathcal{H}_{\rm int}$ is an effective interaction Hamiltonian for KWs propagating along straight vortex line, which equals

$$\mathcal{H}_{1\leftrightarrow 3} = \frac{1}{6}\int d\mathbf{k}_1 \ldots d\mathbf{k}_4 \delta_1^{2,3,4}[V_1^{2,3,4} a_1 a_2^* a_3^* a_4^* + \text{c.c.}] \tag{5}$$

for four-wave systems or

$$\mathcal{H}_{3\leftrightarrow 3} = \frac{1}{36}\int d\mathbf{k}_1 \ldots d\mathbf{k}_6 \sigma_{1,2,3}^{4,5,6} W_{1,2,3}^{4,5,6} a_1 a_2 a_3 a_4^* a_5^* a_6^* \tag{6}$$

for six-wave systems. Here we use the shorthand notation $a_j \equiv a_{\mathbf{k}_j}$, $k_j = |\mathbf{k}_j|$, $\delta_{1,2}^{3,4,5} \equiv \delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4-\mathbf{k}_5)$ and $\delta_{1,2,3}^{4,5,6} \equiv \delta(\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_4-\mathbf{k}_5-\mathbf{k}_6)$.

These equations effectively describe weakly nonlinear waves of any type,[16] using *only* the relevant dynamical information, that is present in the system Hamiltonian $\mathcal{H}$. The main technical problem is to find $\mathcal{H}$ for a particular complicated physical system. Fortunately, in case of KWs this cumbersome job has been undertaken in Ref. 14.

A statistical description of weakly interacting waves can be obtained[16] in terms of the KE

$$\frac{\partial n(\mathbf{k},t)}{\partial t} = \text{St}(\mathbf{k},t), \tag{7}$$

for the wave action spectrum $n(\mathbf{k},t)$, defined by

$$\langle a(\mathbf{k},t) a^*(\mathbf{k}',t)\rangle = n(\mathbf{k},t)\,\delta(\mathbf{k}-\mathbf{k}'),$$

where $\langle \ldots \rangle$ represents ensemble averaging. The collision integral $\text{St}(\mathbf{k},t)$ can be found in various ways,[16] including the Golden Rule widely used in quantum mechanics. For five- and six-wave processes we have, respectively,

$$\begin{aligned}\text{St}_{1\leftrightarrow 3} = \frac{\pi}{12}\int d\mathbf{k}_1 \ldots d\mathbf{k}_3 \{ &|V_{\mathbf{k}}^{1,2,3}|^2 \delta_{\mathbf{k}}^{1,2,3} \mathcal{N}_{\mathbf{k}}^{1,2,3} \delta(\omega_{\mathbf{k}} - \omega_1 \\ &- \omega_2 - \omega_3) + 3|V_1^{\mathbf{k},2,3}|^2 \delta_1^{\mathbf{k},2,3} \mathcal{N}_1^{\mathbf{k},2,3} \\ &\times \delta(\omega_1 - \omega_{\mathbf{k}} - \omega_2 - \omega_3)\},\end{aligned}$$

$$\mathcal{N}_1^{2,3,4} = n_1 n_2 n_3 n_4 (n_1^{-1} - n_2^{-1} - n_3^{-1} - n_4^{-1}); \tag{8}$$

and

$$\begin{aligned}\text{St}_{3\leftrightarrow 3} = \frac{\pi}{12}\int d\mathbf{k}_1 \ldots d\mathbf{k}_5 &|W_{\mathbf{k},1,3}^{4,5,6}|^2 \delta_{\mathbf{k},1,3}^{4,5,6} \delta(\omega_{\mathbf{k}} + \omega_1 + \omega_2 \\ &- \omega_3 - \omega_4 - \omega_5) n_{\mathbf{k}} n_1 n_2 n_3 n_4 n_5 (n_1^{-1} + n_2^{-1} + n_3^{-1} \\ &- n_4^{-1} - n_5^{-1} - n_6^{-1}).\end{aligned} \tag{9}$$

Scaling solutions of these KE's can be found subject to two conditions obeyed by various wave systems, e.g., gravity and capillary waves on fluid surfaces, Langmuir and ion acoustic waves in plasmas, etc.[16]

Scale-invariance of the wave system, where the frequency of the waves and the interaction coefficients are homogeneous functions of the wave vectors, i.e.,

$$\omega(\lambda,k)=\lambda^{\alpha_2}\omega(k), \quad V(\lambda\mathbf{k}_1,\lambda\mathbf{k}_2;\lambda\mathbf{k}_3,\lambda\mathbf{k}_4,\lambda\mathbf{k}_5) \equiv \lambda^{\alpha_5}V(\mathbf{k}_1,\mathbf{k}_2;\mathbf{k}_3,\mathbf{k}_4,\mathbf{k}_5),$$

and a similar relationship holds for $W_{1,2,3}^{4,5,6}$ with a superscript α_6.

Interaction localization, in the sense that the main contribution to the energy balance of a given k-wave (with wave vector k) originates in its interaction with k'-waves that have $k' \sim k$. Mathematically, this means that all integrals over k_1, k_2 etc. in the KE's (7)–(9) *converge*, so that, in the scale-invariant case, the leading contribution to the collision integral indeed originates from the regions $k_2 \sim k$, $k_3 \sim k$, etc. Note that nonlocal spectra are not solutions of the KE's (7)–(9) and, therefore, are physically irrelevant.

In *scale-invariant* wave systems one seeks *scale-invariant* solutions of the KE's,

$$n(k)=Ak^{-x}, \tag{10}$$

where A is a dimensional number. To find the scaling index x for turbulent spectra with a constant energy flux over the scales, we note that all of the KE's (7)–(9) conserve the total energy of the wave system,

$$\frac{dE}{dt}=0, \quad E \equiv \int E_{\mathbf{k}} d\mathbf{k}, \quad E_{\mathbf{k}} \equiv \omega_{\mathbf{k}} n_{\mathbf{k}}.$$

Therefore, the **k**-space energy density, $E_{\mathbf{k}}$, satisfies a continuity equation

$$\frac{\partial E_k}{\partial t}+\frac{\partial \varepsilon_k}{\partial k}=0.$$

Here ε_k is the energy flux over the scales, expressed in terms of an integral over a sphere of radius k:

$$\varepsilon_k=\int_{k'<k} d\mathbf{k}'\,\omega_{k'}\mathrm{St}(k',t).$$

Assuming localization of the interaction, one estimates the d-dimensional integral $\int d\mathbf{k}$ as k^d, the interaction coefficients

$$V_1^{2,3,4} \sim V_k^{k,k,k} \sim Vk^{\alpha_4},$$

$$W_{1,2,3}^{4,5,6} \sim W_{k,k,k}^{k,k,k} \sim Wk^{\alpha_6}, \tag{11}$$

and $n_k = A_p k^{-x_p}$ for the p-wave interactions. Therefore

$$\varepsilon_k \sim k^{4d}(Vk^{\alpha_5})^2(A_5k^{-x_5})^4, \quad 2 \Leftrightarrow 3 \text{ scattering};$$

$$\varepsilon_k \sim k^{5d}(Wk^{\alpha_6})^2(A_6k^{-x_6})^5, \quad 3 \Leftrightarrow 3 \text{ scattering}. \tag{12}$$

For turbulence spectra with a constant energy flux, $\varepsilon_k = \varepsilon = \mathrm{const}$, i.e., $\varepsilon_k \propto k^0$. For the p-wave process this gives the scaling exponent of $n(k)$, x_p, and an energy scaling exponent y_p, $E(k) \propto k^{-y_p}$, with

$$x_p = d+\frac{2\alpha_p}{p-1}, \quad y_p = x_p - \alpha_2. \tag{13}$$

In fact, these expressions are valid for any $p>2$. For the three- and the four-wave processes (with $p=3$ and $p=4$) this gives the well-known results, see, e.g., Ref. 16. Note however, that the 4-wave $1 \leftrightarrow 3$ is considered here for the first time, and it is different from the previously considered standard $2 \leftrightarrow 2$ processes.

III. KELVIN-WAVE TURBULENCE WITH A SIX-WAVE INTERACTION

To consider the KW system, one has to start with the Biot–Savart equations (2), consider an equilibrium state corresponding to an infinitely long straight vortex line, and perturb it with small angle disturbances. This will correspond to a setup of weakly nonlinear KWs which are dispersive, and which can be described using the weak-wave turbulence theory. For this, one has to parametrize the transverse displacement vector of the perturbation by the distance along the unperturbed line, transform to Fourier space, and expand over small perturbation angles and the small parameter $1/\Lambda$, where $\Lambda = \ln(\ell/a) \gg 1$. This expansion in two small parameters is not easy. This is because in the leading order in $1/\Lambda$ the model is integrable, i.e., noncascading, and describing the leading order of the energy transfers requires going to the next order in $1/\Lambda$. A second difficulty is that the lowest order process, the four-wave resonances, are absent in one dimensional systems of this sort with an upward concave dispersion relation. Thus, it is necessary to proceed to the next order in the small nonlinearity, as well. These two facts make finding of the effective interaction Hamiltonian $\mathcal{H}_{\mathrm{int}}$ for KWs difficult. For the six-wave process, which assumes that the underlying vortex is perfectly straight, this task has been undertaken only recently.[14] The effective $3 \leftrightarrow 3$-interaction coefficient W was shown to be

$$W_{1,2,3}^{4,5,6} = \frac{-3}{4\pi\kappa} k_1k_2k_3k_4k_5k_6 F_{1,2,3}^{4,5,6}, \tag{14}$$

where F is a nonsingular dimensionless function of $k_1 \dots k_6$ that is close to unity in the relevant range of its arguments. In particular, $F \to 1$ when one or several of the k's are much smaller than the maximum wave number in the sextet.

Equations (3), (6), and (14) provide *all* the necessary information about KWs needed for the further developments in this paper. Those interested in further details about the derivation of these equations can find them in Ref. 14.

Note that the form of Eq. (14) is to be expected because it corresponds to a very simple physical fact: long KWs (with small k's) can contribute to the energy of a vortex line only when they produce curvature. The curvature, in turn, is proportional to wave amplitude a_k and, at fixed amplitude, is inversely proportional to their wave-length, i.e., $\propto k$. Therefore in the effective equation of motion, each a_j has to be accompanied by k_j, if $k_j \leqslant k$. This statement is reflected in Eq. (14). The cumbersome calculations of Ref. 14 support these reasoning, and additionally provide an explicit numerical factor $-3/4\pi$, along with an explicit expression for F which may be important in further research, and will be required for careful comparison with future experiments or numerical computations.

Equation (14) yields an estimate of $W_{1,2,3}^{4,5,6}$ as Wk^6. Thus, Eq. (12) reproduces the Kozik–Svistunov (KS) scaling for the $3 \leftrightarrow 3$ processes, which is written below with a dimensionless constant C_{KS}:

$$n_{KS} = \frac{C_{KS}\kappa^{2/5}\varepsilon^{1/5}}{k^{17/5}}, \quad E_{KS} = N\frac{C_{KS}\kappa^{7/5}\varepsilon^{1/5}}{k^{7/5}}. \tag{15}$$

We repeat that the KS spectrum (15) is valid only if it is local, i.e., if all the integrals (9) converge, so that $\mathrm{St}_{3\leftrightarrow 3}(k)$ can be estimated as in Eq. (12). However, a detailed analysis (given in Ref. 14 and reproduced briefly below) show that the KS spectrum is nonlocal and, therefore, physically unrealizable. In order to find a valid turbulent KW, we briefly reproduce that analysis, making use of Eq. (14).

IV. NONLOCALIZATION OF ENERGY TRANSFER IN SIX-WAVE INTERACTIONS

Let us now check if the KS spectrum (15) is local or not. For this, we consider the $3\leftrightarrow 3$ collision term (9) for KW with the interaction amplitude $\mathcal{W}_{1,2,3}^{4,5,6}$ as in Eq. (14) and $n(k)$ as in Eq. (10). In this case $\int d\mathbf{k}_j$ are one dimensional integrals $\int_{-\infty}^{\infty} dk_j$. In the IR region $k_1 \ll k, k_j$, $j=2,3,4,5$, we have $F\approx 1$ and the integral over k_1 scales as

$$\Psi \equiv \frac{2}{\kappa}\int_{1/\ell} k_1^2 n(k_1) dk_1 = \frac{2A}{\kappa}\int_{1/\ell} k_1^{2-x} dk_1. \tag{16}$$

The lower limit 0 in Eq. (16) is replaced by $1/\ell$, where ℓ is the mean inter-vortex separation ℓ, at which the approximation of noninteracting vortex lines fails and one expects a cutoff of the power-law behavior of Eq. (10). The factor 2 in Eq. (16) reflects the fact that the ranges of positive and negative $\mathbf{k}_1$ give equal contributions, and the factor $1/\kappa$ is introduced to make the parameter Ψ dimensionless. Ψ has the significance of the mean-square angular deviation of the vortex lines from straight. Therefore $\Psi \lesssim 1$; for highly polarized vortex lines $\Psi \ll 1$.

The integral (16) clearly has an IR-divergence if $x>3$, which is the case for the KS spectrum (15) with $x_6=17/5$. Note that all the similar integrals over $\mathbf{k}_2$, $\mathbf{k}_3$, $\mathbf{k}_4$, and $\mathbf{k}_5$ in Eq. (9) also diverge in exactly the same way as Eq. (16). Moreover, when two of the wave numbers belonging to the same side in the sextet approach zero simultaneously, each of them will yield an integral similar to Eq. (16), and the net result will be the product of these integrals, i.e., a stronger singularity than in the case of just one small wave number. On the other hand, small wave numbers which are on the opposite sides of the resonant sextet do not lead to a stronger divergence because of an extra reduction in Eq. (9) from the condition $n_1^{-1}+n_2^{-1}+n_3^{-1}-n_4^{-1}-n_5^{-1}-n_6^{-1}$.

The divergence of the integrals in Eq. (9) means that the KS spectrum (15) is not a solution of the KE (9) and is, therefore, unrealizable. Another, self-consistent solution of this KE must be found.

V. EFFECTIVE FOUR-WAVE THEORY OF KW TURBULENCE

The nonlocalization of the six-wave theory is a serious problem. It means that the dominant sextets contributing to $3\leftrightarrow 3$-scattering are those for which two of the wave numbers from the same side of the six-wave resonance conditions

$$\omega_k + \omega_1 + \omega_2 = \omega_3 + \omega_4 + \omega_5,$$

$$\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4 + \mathbf{k}_5. \tag{17}$$

are very small, with $k_j \lesssim 1/\ell$. Thus these equations effectively become

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3, \quad \mathbf{k}_1 = \mathbf{k} + \mathbf{k}_2 + \mathbf{k}_3,$$

$$\mathbf{k}_2 = \mathbf{k} + \mathbf{k}_1 + \mathbf{k}_3, \quad \text{or} \quad \mathbf{k}_3 = \mathbf{k} + \mathbf{k}_2 + \mathbf{k}_1, \tag{18}$$

and, with the respective conditions for the frequencies, implies a 4-wave process of the $1\leftrightarrow 3$ type. In other words, one can interpret such nonlocal sextets on straight vortex lines as quartets on curved vortices, with the slowest modes in the sextet responsible for the large-scale curvature R of the underlying vortex line in the 4-wave approach.

To derive an effective 4-wave KE, we begin with the 6-wave collision integral (9) and find the leading contributions to it when the spectrum n_k is steeper than k^{-3} in the IR region. There are four of these. The first originates in the region where k_1 and k_2 are much smaller than the rest of the k_j's. The other three other contributions originate on the other side of the sextet: regions where either k_3 and k_4, or k_3 and k_5, or k_4 and k_5 are small. These contributions are equal, so it is sufficient to find just one of them and multiply the result by three. In particular, the sum of the four contributions can be written exactly in the form of a $1\leftrightarrow 3$-collision term (8) with an effective $1\leftrightarrow 3$-interaction amplitude

$$V_1^{2,3,4} = -3\Psi \mathbf{k}_1\mathbf{k}_2\mathbf{k}_3\mathbf{k}_4/(4\pi\sqrt{2}), \tag{19}$$

because, as shown in Ref. 14,

$$\lim_{k_1\to\infty} F(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3|\mathbf{k}_4\mathbf{k}_5,\mathbf{k}_6) = 1.$$

In deriving Eq. (8) with $V_1^{2,3,4}$ and Eq. (19), we took only the leading contributions in the respective IR regions, factorized the integrals over these wave vectors as in Eq. (16) and took only the zeroth order terms with respect to the small wave vectors (by setting these wave numbers equal to zero) in the rest of Eq. (9).

Equation (8) with $V_1^{2,3,4}$ as in Eq. (19) is an effective 4-wave KE, which we were aiming to obtain. This KE corresponds to interacting quartets of KWs propagating along a vortex line having a random large-scale curvature $R\lesssim \ell$. Equation (19) estimates $V_1^{2,3,4}$ as $\mathcal{V}k^4$ with $\mathcal{V}\sim\Psi$. Using this scaling in Eq. (12), we arrive at a spectrum for the $1\leftrightarrow 3$ processes with scaling exponents $x_4=11/3$ and $y_5=5/3$,

$$n_{LN} = \frac{C_{LN}\varepsilon^{1/3}}{\Psi^{2/3}k^{11/3}} \Rightarrow E_{LN} = \frac{C_{LN}\Lambda\kappa\varepsilon^{1/3}}{\Psi^{2/3}k^{5/3}}, \tag{20}$$

the local $(1\leftrightarrow 3)$ L'vov-Nazarenko (LN) spectrum.

VI. LOCAL STEP-BY-STEP ENERGY TRANSFER WITH FOUR-WAVE $1\leftrightarrow 3$ INTERACTIONS

Mathematically, the localization of energy transfer in $1\leftrightarrow 3$-wave processes means convergence of the multidimensional integral in the corresponding collision term of Eq. (8). Here we show that convergence in Eq. (8) is a delicate issue that cannot be settled only on the basis of power-law arguments, because this would yield a divergence.

A. Proof of the infrared convergence

We now show that in the IR region, when at least one of the wave vectors, say k_2, is much smaller then k, only a quadrupole cancellation of the largest, next largest, and next two sub-leading contributions seem to yield a final, convergent result for the collision term (8). The three integrals in Eq. (8) are restricted by two conservation laws, specifically

$$1 \to 3: \quad \mathbf{k}=\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3, \quad k^2=k_1^2+k_2^2+k_3^2 \tag{21}$$

in the first term, and by

$$3 \to 1: \quad \mathbf{k}+\mathbf{k}_2+\mathbf{k}_3=\mathbf{k}_1, \quad k^2+k_2^2+k_3^2=k_1^2 \tag{22}$$

in the second term. Therefore, only one integration, say with respect to $\mathbf{k}_2$, remains in each term.

In the IR region $k_2 \ll k_1 \ll k$, from Eqs. (21) and (22), we find for the $1\to 3$ and $3 \to 1$ terms that

$$1 \to 3: \quad \mathbf{k}_1=\mathbf{k}-\frac{k_2^2}{\mathbf{k}_1+\mathbf{k}_2} \approx \mathbf{k}-\frac{k_2^2}{\mathbf{k}},$$

$$1 \to 3: \quad \mathbf{k}_3=-\frac{\mathbf{k}_1\mathbf{k}_2}{\mathbf{k}_1+\mathbf{k}_2} \approx -\mathbf{k}_2, \tag{23}$$

and

$$3 \to 1: \quad \mathbf{k}_1=\mathbf{k}+\frac{k_2^2}{\mathbf{k}+\mathbf{k}_2} \approx \mathbf{k}+\frac{k_2^2}{\mathbf{k}},$$

$$3 \to 1: \quad \mathbf{k}_3=-\frac{\mathbf{k}\mathbf{k}_2}{\mathbf{k}+\mathbf{k}_2} \approx -\mathbf{k}_2. \tag{24}$$

These equations imply three important points:

1) in both cases in the leading order $k_3 \simeq k_2$, i.e., when $k_2 \ll k$, k_3 is also small;
2) the difference between $\mathbf{k}_1$ and $\mathbf{k}$ is of the second order in the small k_2: $|\mathbf{k}_1-\mathbf{k}| \simeq k_2^2/k$;
3) these leading contributions to $\mathbf{k}_1-\mathbf{k}$ have the same modulus but different signs in the $1 \leftrightarrow 3$ term and in the $3 \leftrightarrow 1$ term.

Therefore, in the leading order the expressions for N in Eq. (8) can be written as

$$\mathcal{N}_k^{1,2,3} \simeq -x(k_2/k)^2 n_k n_2 n_3 \simeq -\frac{xA^3}{k^{(x+2)}} k_2^{2(1-x)}, \tag{25}$$

$$\mathcal{N}_1^{k,2,3} \simeq +x(k_2/k)^2 n_k n_2 n_3 \simeq +\frac{xA^3}{k^{(x+2)}} k_2^{2(1-x)}, \tag{26}$$

where we have substituted n_j from Eq. (10). It is important that these estimates (in the leading order) have *the same magnitude and different signs*.

The next step is to compute the integrals

$$I_{1\to 3} \equiv \int d\mathbf{k}_1 d\mathbf{k}_3 \delta(\mathbf{k}-\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\,\delta(k^2-k_1^2-k_2^2-k_3^2)$$
$$=\frac{|\mathbf{k}+\mathbf{k}_2|}{2|k^2+2\mathbf{k}\mathbf{k}_2-k_2^2|} \to \frac{1}{2k}; \tag{27}$$

$$I_{3\to 1} \equiv \int d\mathbf{k}_1 d\mathbf{k}_3 \delta(\mathbf{k}+\mathbf{k}_2+\mathbf{k}_3-\mathbf{k}_1)\,\delta(k^2+k_2^2+k_3^2-k_1^2)$$
$$=\frac{1}{2|\mathbf{k}+\mathbf{k}_1|} \to \frac{1}{2k}, \tag{28}$$

that is, in the leading order these results coincide and do not contain the small parameter.

Now we can find the contributions to $\mathrm{St}_{1\leftrightarrow 3}$, given by Eq. (8), from the region $k_2 \ll k$. According to Eq. (19) we can write $V_k^{1,2,3}=V_1^{k,2,3}=V\mathbf{k}\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3$. Using our estimates (25) and (26) for N and Eqs. (27) and (28) we obtain

$$1 \to 3: \quad \mathrm{St}_{1\to 3}^{k_2 \ll k} \approx -\frac{x\pi V^2 A^3}{24k^{x-1}} \int k_2^{2(3-x)} d\mathbf{k}_2; \tag{29}$$

$$3 \to 1: \quad \mathrm{St}_{3\to 1}^{k_2 \ll k} \approx +\frac{3x\pi V^2 A^3}{24k^{x-1}} \int k_2^{2(3-x)} d\mathbf{k}_2. \tag{30}$$

One can see that, despite the substantial extent of cancellations in the estimates for N, the integrals in Eqs. (29) and (30) diverge if $x \geqslant 3.5$, which holds in the case of the LN-scaling exponent $x=11/3$.

Nevertheless, the following must be taken into account: the $1 \leftrightarrow 3$ contribution to the collision integral has three identical divergent regions: $k_2 \sim k_3 \ll k_1 \approx k$, $k_1 \sim k_3 \ll k_2 \approx k$ and $k_2 \sim k_1 \ll k_3 \approx k$, while Eq. (29) gives an estimate of only the first. Therefore, the total contribution is

$$\mathrm{St}_{1\to 3}^{IR}=3\mathrm{St}_{1\to 3}^{k_2 \ll k} \approx -\frac{3x\pi V^2 A^3}{24k^{x-1}} \int k_2^{2(3-x)} d\mathbf{k}_2, \tag{31}$$

while the $3 \leftrightarrow 1$ contribution has only one divergent region $\mathbf{k}_1 \approx \mathbf{k}$. Therefore,

$$\mathrm{St}_{1\to 3}^{IR}=\mathrm{St}_{1\to 3}^{k_2 \ll k} \approx +\frac{3x\pi V^2 A^3}{24k^{x-1}} \int k_2^{2(3-x)} d\mathbf{k}_2, \tag{32}$$

i.e., exactly the same result as in Eq. (29), but with a different sign. The divergent contributions of Eqs. (29) and (30), therefore, cancel each other, so the next order has to be taken into account.

Note that next order terms in the expansion in $k_2 \ll k$ yield the already convergent integral

$$\mathrm{St}_{1\leftrightarrow 3}^{IR} \propto \int_0^{k_{IR} \ll k} \mathbf{k}_2 k_2^{(6-2x)} d\mathbf{k}_2, \tag{33}$$

with the LN exponent $x=11/3$. Moreover, the excitation of KWs is typically symmetrical in $\mathbf{k} \leftrightarrow -\mathbf{k}$. In this case, this integral has an odd integrand so it equals zero. Then the leading contribution to the $1 \leftrightarrow 3$-collision term in the IR region can be summarized as follows:

$$\mathrm{St}_{1\leftrightarrow 3}^{IR} \sim \frac{V^2 A^3}{k^{x+1}} \int_0^{k_{IR} \ll k} k_2^{2(4-x)} d\mathbf{k}_2 \propto k_{IR}^{9-2x}. \tag{34}$$

IR convergence requires $x < 9/2$. With an exponent $x=9/3$ in the LN spectrum, this gives

$$\mathrm{St}_{1\to 3}^{IR} \propto k_{IR}^{5/3} \equiv k_{IR}^{\delta_{IR}}.$$

Here we introduce an "IR convergence reserve:" $\delta_{IR}=5/3$.

B. Proof of the ultraviolet convergence

Convergence of the integral (8) in the UV region can be established in a similar manner when one of the wave vectors, say $k_2 \gg k$.

Note, first of all, that in the $1\to 3$ term of Eq. (8), there is no UV region, because, according to the second of Eqs. (21) we have $k_j \leqslant k$. In the $3\to 1$ term, Eq. (22) can be satisfied in the leading order if we take $\mathbf{k}_2 \simeq \mathbf{k}_1$; $k_2 \geqslant k_{UV} \gg k$ (the case $\mathbf{k}_3 \simeq \mathbf{k}_1$; $k_3 \geqslant k_{UV}$ gives an identical result). Using the parametrization $\mathbf{k}_1 = \mathbf{k} + k_2^2/(\mathbf{k}+\mathbf{k}_2)$, $\mathbf{k}_3 = -\mathbf{k}\mathbf{k}_2/(\mathbf{k}+\mathbf{k}_2)$ (cf. (26)), we get some cancelations in $\mathcal{N}_1^{k,2,3}$ and the result in the leading order is

$$N_1^{k,2,3} \propto x(x+1)\left(\frac{k_2}{k}\right)^{-2-x} - \left(\frac{k_2}{k}\right)^{-2x}. \qquad (35)$$

Further, similarly to Eqs. (27) and (28), we obtain $I_{3\to 1} \simeq 1/k_2$. As before, the interaction coefficient $V \propto k_2^2$ or $V^2 \propto k_2^4$. Combining the powers of k_2 yields

$$\mathrm{St}_{1\leftrightarrow 3}^{UV} \propto k_{UV}^{V}, \quad y = \max(-2x+4, -x+2). \qquad (36)$$

UV convergence requires that $y<0 \Rightarrow x>2$. One concludes that in the case $x=11/3$, $\mathrm{St}_{1\to 3}^{UV} \propto k_{UV}^{-5/3} \equiv k_{UV}^{-\delta_{UV}}$, where we introduce a "UV convergence reserve," $\delta_{UV}=5/3$.

C. Counterbalanced interaction localization

In particular, $\delta_{IR}=\delta_{UV}$. This equality is not accidental. The observed "counterbalanced" IR-UV localization is a consequence of the scale-invariance of the problem. Indeed, for given values of $k_{IR} \ll \tilde{k} \ll k_{UV}$ the IR-energy flux $k_{IR} \Rightarrow \tilde{k}$ (from the IR region $k \leqslant k_{IR}$ toward the region $\sim k$) should scale as $(k_{IR}/\tilde{k})$ in exactly the same way as the UV-energy flux $\tilde{k} \Rightarrow k_{UV}$ (from the $\tilde{k}$-region toward the UV-region $k \leqslant k_{UV}$) scales as $\tilde{k}/k_{UV}$. This is because the UV-flux $\tilde{k} \Rightarrow k_{UV}$ from the $\tilde{k}$-region can be regarded as equal to the IR flux toward the k_{UV}-region. Remembering that the IR-energy flux $k_{IR} \Rightarrow \tilde{k}$ scales as $(k_{IR}/\tilde{k})^{\delta_{IR}}$, while the UV-flux $k \Rightarrow k_{UV}$ is proportional to $(\tilde{k}/k_{UV})^{\delta_{UV}}$, one immediately concludes that δ_{IR} should be equal to δ_{UV}.

The overall conclusion is that the collision term $\mathrm{St}_{1\leftrightarrow 3}$ is convergent in both the IR and the UV regions for $x=11/3$ and that *energy transfer in the* $1\leftrightarrow 3$ *kinetic equation is local*.

VII. DISCUSSION

* In this paper we have revised the theory of superfluid turbulence in the quantum range of scales where the turbulent cascade is caused by nonlinear interactions of weak Kelvin waves on quantized vortex lines. In particular, we have addressed the problem that the previously used KS spectrum is nonlocal, i.e., is a mathematically invalid and physically irrelevant solution.

* We have presented a new effective theory of Kelvin wave turbulence consisting of wave quintets interacting on vortex lines with random large-scale curvature. This four-wave theory replaces the nonlocal six-wave theory. We have derived an effective four-wave kinetic system of Eqs. (8) and (19), and solved it to obtain a new wave spectrum (20). We have proved that this new spectrum is local, and therefore is a valid solution of the kinetic equation, which should replace the nonlocal (and therefore invalid) Kosik-Svistunov spectrum (15) in the theory of quantum turbulence. In particular, it is now necessary to revise the theory of the classical-quantum crossover scales and its predictions for the turbulent dissipation rate (Refs. 17–20). This kind of revision is also needed for analyzing laboratory experiments and numerical simulations of superfluid turbulence, which have relied on the un-physical KS spectrum (15) over the past five years.

* The difference between the LN-exponent $-5/3$ (see Eq. (20)) and the KS-exponent $-7/5$ (see Eq. (15)) is $4/15$, which is rather small. This may explain why the previous numerical experiments seem to agree with the KS spectrum obtained numerically in Ref. 15. However, one can also see by inspection that these results also agree with the LN slope.

The different physics results in different expressions for the dimensional pre-factors in the KS and LN spectra and, in particular, a different dependence on the energy flux ε, as well as an extra dependence on the large-scale behavior (through Ψ) in Eq. (20). Careful examination of these prefactors is necessary in future numerical simulations in order to test the predicted dependences. This sort of numerical simulations can be done efficiently using the local nonlinear equation (LNE) suggested in Ref. 14 and based on a detailed analysis of nonlinear KW interactions,

$$i\frac{\partial \tilde{w}}{\partial t} + \frac{\kappa}{4\pi}\frac{\partial}{\partial z}\left\{\left[\Lambda - \frac{1}{4}\left|\frac{\partial \tilde{w}}{\partial z}\right|^4\right]\frac{\partial \tilde{w}}{\partial z}\right\} = 0. \qquad (37)$$

The LNE model is similar, but not identical, to the truncated LIA model of[15] (these models become asymptotically identical for weak KWs).

* The pre-factors in the KS spectrum (15) and in the LN spectrum (20) contain very different numerical constants C: a constant of order unity in LN ($C_{\mathrm{LN}} \sim 1$, yet to be found) and a zero constant in KS ($C_{\mathrm{KS}} \equiv 0$ as a formal consequence of its nonlocalization). Also we should take note of a mysteriously very small numerical factor 10^{-5} in Eq. (16) for the energy flux in Ref. 13, that has no physical justification. Actually, nonlocalized energy transfer over scales means that this number should be very large, rather than very small. This emphasizes the confusion about, and highlights the need for numerical re-evaluation of, the prefactor in the spectrum.

To conclude comparison between the KS and LN approach we note that the drastic difference in the numerical prefactors constitutes an important difference between the KS and the LN spectra for practical analysis of experimental data, while the difference between the underlying physics of the local and nonlocal energy cascades, which leads to the difference between the spectral indices, is important from a fundamental, theoretical standpoint.

* In this paper, an effective local five-wave kinetic equation has been derived from the six-wave kinetic equation by exploiting the nonlocalization of the latter. Strictly speaking, this derivation is valid only when the six-wave kinetic equation is valid, i.e., when all the scales are weakly nonlinear, including the ones at the infra-red cutoff. However, the resulting five-wave kinetic equation is likely to be applicable more widely, when only the small scales, and not the large scales, are weak. A similar picture has been seen previously in the case of nonlocal turbulence of Rossby/drift

waves in Ref. 22 and for nonlocal MHD turbulence in Ref. 23. In the future we plan to attempt to derive the five-wave kinetic equation directly from the dynamical equations for Kelvin waves, which should allow us to extend its applicability to the case of strong large scales.

* Finally, we note that the theory proposed here may potentially be useful for other one-dimensional physical systems, such as optical fibers, where nonlinear interactions of one-dimensional wave packages become important with increasing network capacity.

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