

Dynamics of the density of quantized vortex lines in superfluid turbulence

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The quantization of vortex lines in superfluids requires the introduction of their density $\mathcal{L}(\mathbf{r}, t)$ in the description of quantum turbulence. The space homogeneous balance equation for $\mathcal{L}(t)$, proposed by Vinen on the basis of dimensional and physical considerations, allows a number of competing forms for the production term \mathcal{P} . Attempts to choose the correct one on the basis of time-dependent homogeneous experiments ended inconclusively. To overcome this difficulty we announce here an approach that employs an inhomogeneous channel flow which is very suitable to distinguish the implications of the various possible forms of the desired equation. We demonstrate that the originally selected form which was extensively used in the literature is in strong contradiction with our data. We therefore present a new form of an inhomogeneous equation for $\mathcal{L}(\mathbf{r}, t)$ that is in agreement with our data and propose that it should be considered for further studies of superfluid turbulence.

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Background. Below the Bose-Einstein condensation temperature $T_\lambda \approx 2.18$ K, liquid ^4He becomes a quantum inviscid superfluid [1–4]. Aside from the lack of viscosity, the vorticity in ^4He is constrained to vortex-line singularities of fixed circulation $\kappa = h/M$, where h is Planck's constant and M is the mass of the ^4He atom. These vortex lines have a core radius $a_0 \approx 10^{-8}$ cm, compatible with the interatomic distance. In generic turbulent states, these vortex lines appear as a complex tangle with a typical intervortex distance ℓ [5].

Recent progress in laboratory [6–22] and numerical studies [4,6,23–28] of superfluid turbulence in superfluid ^3He and ^4He led to a growing consensus that the statistical properties of superfluid turbulence at large scales $R \gg \ell$ are similar to those of classical turbulence. An acceptable theory of these large scale properties [6,7,29–34] is based on the Landau-Tisza two-fluid model [35,36], using “normal” and “superfluid” components of densities ρ_n and ρ_s with velocity fields $\mathbf{u}_n(\mathbf{r}, t)$ and $\mathbf{u}_s(\mathbf{r}, t)$. This model was extended by Hall-Vinen [37] and Bekarevich-Khalatnikov [38] to include a mutual friction between the components, proportional to the vortex-line density \mathcal{L} . This means that a theory of large-scale motions which is affected by the mutual friction requires the inclusion of the dynamics of \mathcal{L} .

The situation changes drastically upon considering the statistical properties on smaller scales, where the quantization of vortex lines becomes crucial. Several statistical characteristics of the vortex tangle become essential for a consistent description. Besides \mathcal{L} , these characteristics involve the mean-square curvature $(\tilde{S})^2$, and the vortex tangle anisotropy parameters I_\parallel and I_ℓ introduced by Schwarz [39]. The most important of these is the vortex-line density $\mathcal{L}(\mathbf{r}, t)$. It is expected that coupling one or more of these quantities to the variables of the two fluid Hall-Vinen-Bekarevich-Khalatnikov equations [37,38] is a minimal requirement for an acceptable theory of quantum turbulence on the smaller scales.

The problem. A phenomenological equation of motion for $\mathcal{L}(t)$ was suggested by Vinen for homogeneous counterflows [40,41]:

$$d\mathcal{L}(t)/dt = \mathcal{P}(t) - \mathcal{D}(t). \quad (1a)$$

Here, the production term $\mathcal{P}(t)$ describes the growth of \mathcal{L} due to the extension of the vortex rings by mutual friction which is caused by the difference between the velocities of the normal fluid and superfluid components (“the counterflow velocity” V_{ns}). The decay term $\mathcal{D}(t)$ is again caused by the mutual friction due to the moving normal fluid components and is assumed to be independent of V_{ns} . Therefore, both terms should be proportional to the dimensionless dissipative mutual friction parameter α . In principle, it is not guaranteed that the equation for \mathcal{L} can be closed via \mathcal{L} and V_{ns} . Such a closure for \mathcal{P} and \mathcal{D} [40] assumes that (beside α , κ , and V_{ns}) the only relevant variable in the problem is the instantaneous value $\mathcal{L}(t)$, while \tilde{S} , I_\parallel , I_ℓ , etc., are to some extent unimportant. Upon accepting this closure idea the dimensional reasoning dictates [5,40,41]

$$\mathcal{P} \Rightarrow \mathcal{P}_{\text{cl}} = \alpha \kappa \mathcal{L}^2 F(x), \quad x \equiv V_{\text{ns}}^2 / \kappa^2 \mathcal{L}, \quad (1b)$$

$$\mathcal{D} \Rightarrow \mathcal{D}_{\text{cl}} = \alpha \kappa \mathcal{L}^2 G(x). \quad (1c)$$

Here, $F(x)$ and $G(x)$ are dimensionless functions of the dimensionless argument x .

The most delicate issue in this approach is the determination of the functions $F(x)$ and $G(x)$. Vinen [40] assumed that the decay term \mathcal{D} is independent of V_{ns} , leading to $G(x) = C_{\text{dec}}$. On the other hand, Vinen and later authors (see, i.e., [42]) chose \mathcal{P} to be proportional to the mutual friction force $f \propto \alpha |V_{\text{ns}}|$, leading to the proposition that $F(x) \propto \sqrt{x}$, and then

$$\mathcal{P}_{\text{cl}} \Rightarrow \mathcal{P}_1 = \alpha C_1 \mathcal{L}^{3/2} |V_{\text{ns}}|, \quad (2a)$$

where C_1 is a dimensionless constant. Vinen [41] realized that (2a) is not the only possibility. Another choice can follow the spirit of Landau's theory of critical phenomena, considering \mathcal{L} as an order parameter which determines $d\mathcal{L}/dt$ via an analytical function $F(x)$. Then the leading term in the expansion of $F(x) \propto x$, giving

$$\mathcal{P}_{\text{cl}} \Rightarrow \mathcal{P}_2 = \alpha C_2 \mathcal{L} V_{\text{ns}}^2 / \kappa. \quad (2b)$$

Both options (2) are of course dimensionally correct and they predict the same stationary solution, $\mathcal{L}_{\text{st}} \propto V_{\text{ns}}^2$, which is well supported by both experiments and numerical simulations (see, e.g., Ref. [26] and references therein).

In principle, one could hope to distinguish between the different forms of this important equation by comparing their prediction for the *time evolution* from some initial condition toward \mathcal{L}_{st} in the presence of counterflow V_{ns} . Unfortunately, the difference in prediction is too small. Neither Vinen [41] himself nor later [43] experimental attempts succeeded to distinguish between these two forms [48]. We have made our own attempts to distinguish between the two discussed models (2) by numerical simulation of space homogeneous counterflow turbulence, finding inconclusive results as well.

The *proposed resolution* of this old conundrum can be obtained by studying *inhomogeneous flows* such as channel flows in which the various relevant variables have nontrivial profiles. We will argue that in fact none of the equations (2) are correct. We propose yet a third form of \mathcal{P} [corresponding to $f(x) \propto x^{3/2}$]:

$$\mathcal{P}_{cl} \Rightarrow \mathcal{P}_3 = \alpha C_{\text{prod}} \sqrt{\mathcal{L}} V_{ns}^3 / \kappa^2. \quad (3a)$$

Being dimensionally correct, this closure fits the data that are presented below significantly better than either of the equations (2). We are led to a revision of the homogeneous equation of motion for the field $\mathcal{L}(\mathbf{r}, t)$ in the form

$$\frac{\partial \mathcal{L}(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathcal{J}_{cl}(\mathbf{r}, t) = \mathcal{P}_3(\mathbf{r}, t) - \mathcal{D}_{cl}(\mathbf{r}, t). \quad (3b)$$

Here, we have added a vortex-line density flux $\mathcal{J}(\mathbf{r}, t)$. Based on our numerical simulations (see below) we suggest to model $\mathcal{J}(\mathbf{r}, t)$ as follows:

$$\mathcal{J}_{cl}(\mathbf{r}, t) = -C_{\text{flux}} (\alpha/2\kappa) \nabla V_{ns}^2. \quad (3c)$$

Notice that the suggested Eqs. (3) are based on our analysis of counterflow turbulence with laminar normal fluid components. Nevertheless we believe that Eqs. (3) or their modifications may serve as a basis for future studies of inhomogeneous superfluid turbulence in a wide variety of conditions, including the evolution of a neutron-initiated micro big bang in superfluid ^3He [44], turbulent counterflows and pressure-driven superfluid channel and pipe flows of ^4He , turbulent flow of ^3He in rotating cryostat, etc..

Starting from first principles. To reach the desired Eq. (3b) we denote the coordinates of the quantized vortex lines by $s(\xi, t)$, which is parametrized by the arclength ξ . Schwarz [39] derived the equation of motion for the length of the vortex-line segment $d\xi$:

$$\frac{1}{\delta\xi} \frac{d\delta\xi}{dt} \approx \alpha V_{ns}(s, t) \cdot (s' \times s''). \quad (4a)$$

Here, $s' = ds/d\xi$, $s'' = d^2s/d\xi^2$, and α is the temperature dependent dissipative mutual friction parameter. The counterflow velocity V_{ns} (the arguments s and t are suppressed for notational simplicity) is the difference between the normal fluid velocity V^n and superfluid velocity V^s :

$$V_{ns} \equiv V^n - V^s, \quad V^s = V_0^s + V_{BS}. \quad (4b)$$

The superfluid velocity V^s includes the macroscopic potential part V_0^s , and the Biot-Savart velocity V_{BS} . The latter term is defined by the entire vortex tangle configuration \mathcal{C} :

$$V_{BS} = \frac{\kappa}{4\pi} \int_{\mathcal{C}} \frac{(s - s_1) \times ds_1}{|s - s_1|^3} = V_{LIA}^s + V_{nl}^s. \quad (4c)$$

The integral (4c) is logarithmically divergent when $s_1 \rightarrow s$. It is customary to regularize it by using the vortex core radius a_0 and the mean vortex-line radius of curvature $R \equiv 1/\bar{S}$. The main contribution to V_{BS} originates from integrating over scales between a_0 and R , i.e., $a_0 \leq |s_1 - s| \leq R$. This contribution is known as the local induction approximation (LIA) [39] and is written as

$$V_{LIA}^s = \beta s' \times s'', \quad \beta \equiv (\kappa/4\pi) \ln(R/a_0). \quad (4d)$$

The V_{nl}^s term is nonlocal, being produced by the rest of the vortex configuration \mathcal{C}' with $|s_1 - s| > R$:

$$V_{nl}^s = \frac{\kappa}{4\pi} \int_{\mathcal{C}'} \frac{(s - s_1) \times ds_1}{|s - s_1|^3}. \quad (4e)$$

The next step is to integrate Eq. (4a) over the vortex tangle in a fixed volume Ω which resides in slices between y and $y + \delta y$, going over all x and z . This provides us with the equation of motion for $\mathcal{L}(y, t) \equiv \int_{\mathcal{C}_\Omega} d\xi / \Omega$. This equation is written in the form similar to Eq. (3b),

$$\frac{\partial \mathcal{L}(y, t)}{\partial t} + \frac{\partial \mathcal{J}(y, t)}{\partial y} = \mathcal{P}(y, t) - \mathcal{D}(y, t), \quad (5)$$

with the following identification for the flux \mathcal{J} (toward the walls), production \mathcal{P} , and decay term \mathcal{D} :

$$\mathcal{P}(y, t) = \frac{\alpha}{\Omega} \int_{\mathcal{C}_\Omega} d\xi (V^n - V_0^s - V_{nl}^s) \cdot (s' \times s''), \quad (6a)$$

$$\mathcal{D}(y, t) = \frac{\alpha\beta}{\Omega} \int_{\mathcal{C}_\Omega} d\xi |s''|^2 = \alpha\beta \mathcal{L} \tilde{S}^2, \quad (6b)$$

$$\mathcal{J}(y, t) = \frac{1}{\Omega} \int_{\mathcal{C}_\Omega} d\xi V_{\text{drift}, y} = \frac{\alpha}{\Omega} \int_{\mathcal{C}_\Omega} d\xi V_{ns, x} s'_z. \quad (6c)$$

Here, the production and decay terms come directly from integrating Eq. (4a); they coincide with the corresponding equations in Ref. [39] with the only difference being that in Eqs. (6) the integrals are taken in the slice Ω (between y and $y + \delta y$), while in Ref. [39] the integrals are taken over the entire volume. In the flux term the drift velocity $V_{\text{drift}}(\xi)$ of the vortex-line segment $s(\xi)$ can be found from the vortex filament equations [39] and written as follows:

$$V_{\text{drift}}(\xi) = V^s + \alpha s' \times V_{ns}. \quad (7)$$

The mean value of V^s is oriented along the x direction and it does not contribute to the y component of the flux \mathcal{J} . The y component of the second term in Eq. (7) gives the final expression in Eq. (6c).

Numerical simulations were set up in a three-dimensional planar channel geometry [see Fig. 1(a)] of half-width h with a prescribed time-independent profile of the streamwise projection of the normal velocity $V_x^n(y)$. To find the vortex tangle configurations we used the vortex filament method, taking into account the potential flow V_0^s to maintain the counterflow condition. Details of the simulation method can be found in Refs. [26,39]. Here we used the reconnection method [45] and the line resolution $\Delta\xi = 1.6 \times 10^{-3}$ cm. Periodic conditions were used in the streamwise and spanwise directions. In the wall-normal y direction the boundary conditions on V_y^s are $V_y^s(\pm h) = 0$. Finally, $s'(\pm h) = (0, \pm 1, 0)$ at the solid walls. Periodically wrapped replicas of the tangle configuration were

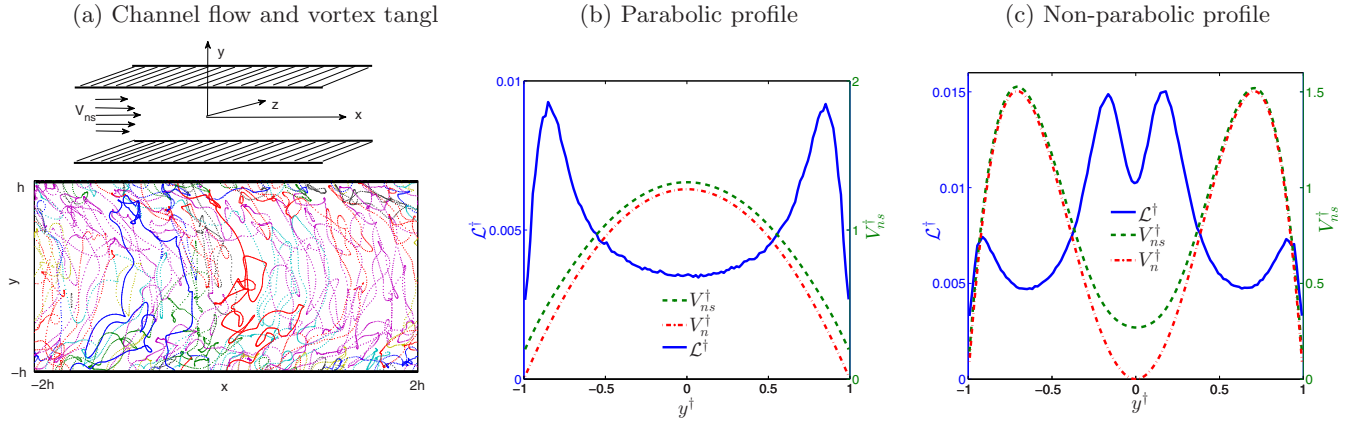


FIG. 1. (Color online) Numerical setup and obtained profiles of counterflow velocity and vortex-line density. (a) Upper figure: Plane channel flow geometry $L_x \times L_y \times L_z$, $L_x = 4h$, $L_y = 2h$, $L_z = 2h$, $h = 0.05$ cm. Lower figure: Projection of a vortex tangle on the (x, y) plane, orthogonal to the walls. Different colors are used to distinguish vortex lines. (b), (c) Prescribed parabolic (b) and nonparabolic (c) normal velocity profiles $V_n^\dagger(y^\dagger)$ ($-\cdot-$), the resulting counterflow profiles $V_{ns}^\dagger(y^\dagger)$ ($-$), and the resulting profiles of $\mathcal{L}^\dagger(y^\dagger)$ ($-$) in dimensionless units defined by Eqs. (8). $T = 1.6$ K.

used in the x and z directions, with reflected configurations in the y direction.

Having selected a stationary profile of $V_x^n(y) \equiv V_n(y)$, we started with a set of arbitrary oriented circular vortex rings and solved the equation for the vortex-line evolution. For the obtained dense vortex tangle we found the time-averaged profile $\langle V_x^s(y) \rangle_t \equiv V_s(y)$. In all cases we ran the simulations until we obtained steady mean profiles. We take temperature $T = 1.6$ K with mutual friction coefficients $\alpha = 0.098$, $\alpha' = 0.016$ [46].

We begin with a parabolic profile for $V_n(y)$. The profiles of $V_{ns}(y)$ and $\mathcal{L}(y)$ in the dimensionless form,

$$y^\dagger \equiv y/L, \quad V^\dagger \equiv V/\sqrt{\langle V_{ns}^2 \rangle}, \quad \mathcal{L}^\dagger \equiv \kappa^2 \mathcal{L} / \langle V_{ns}^2 \rangle, \quad (8)$$

are shown in Fig. 1(b). Taking integrals in Eqs. (6) over the numerically found vortex tangle configuration, we can compute the production, decay, and flux terms, denoted as

$\mathcal{P}_{\text{num}}(y, t)$, $\mathcal{D}_{\text{num}}(y, t)$, and $\mathcal{J}_{\text{num}}(y, t)$, respectively. Then we compare them with their various closure versions, $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_{\text{cl}}, \mathcal{D}_{\text{cl}}$, and \mathcal{J}_{cl} .

From the theoretical point of view the questions are as follows: Can we approximate the integrals in Eqs. (6) only in terms of the counterflow velocity V_{ns} and \mathcal{L} itself, or would the integrals produce other dynamical variables that should require further coupled equations to close the system? Is closure possible in general, or only in some conditions?

Assuming that closure is allowed, dimensional considerations presented us with different versions for the production term $\mathcal{P}_n(y)$ given by Eqs. (2a), (2b), and (3a) for $n = 1, 2$, and 3. With the widely accepted approximation [39] that $R = 1/\tilde{S}$ is proportional to the intervortex distance $\ell = 1/\sqrt{\mathcal{L}}$, one gets from Eqs. (6b) and (4d) the closure (1c) for \mathcal{D} with

$$\mathcal{C}_{\text{dec}} = c_2^2 \ln(R/a_0)/4\pi, \quad \text{where } c_2 \equiv \ell \tilde{S}. \quad (9)$$

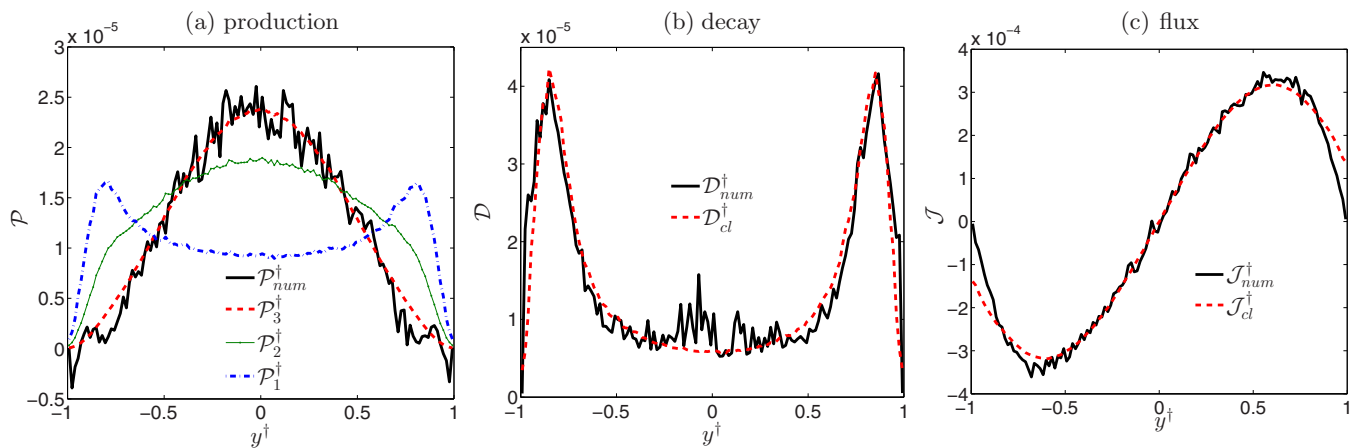


FIG. 2. (Color online) For the parabolic normal velocity profile: A comparison between production (\mathcal{P}), decay (\mathcal{D}), and flux (\mathcal{J}) as obtained from numerical simulations and those obtained using their closure versions. Comparison of the numerical data ($-$) to the competing production forms of the Vinen equation (a): ($-\cdot-$) \mathcal{P}_1 , Eq. (2a), ($-$) \mathcal{P}_2 , Eq. (2b), and ($- -$) \mathcal{P}_3 , Eq. (3b). Comparison of the numerical data for decay (b) and flux term (c) with Eq. (1c) for \mathcal{D} and Eq. (3c) for \mathcal{J} ($- -$).

For the flux term (3c) we suggest (in the channel geometry)

$$\mathcal{J}_{cl}(y,t) = -(\alpha/2\kappa)C_{flux}\partial V_{ns}^2/\partial y. \quad (10)$$

In Fig. 2 we compare the numerical integrals (6) (shown as thick solid black lines) with corresponding closures. The dotted-dashed blue line in Fig. 2(a) shows the Vinen prediction $\mathcal{P}_1(y)$ [Eq. (2a)], while the thin solid green line shows the alternative form $\mathcal{P}_2(y)$ [Eq. (2b)]. By the dashed red line we show the prediction $\mathcal{P}_3(y)$, which is evidently superior to the other two. From this data we can conclude that Eq. (3b) is the one that should be used in the present inhomogeneous case.

Figure 2(b) shows that the numerical integral (6a) and the commonly used form (1c) for the decay term \mathcal{D} practically coincide, meaning that c_2 , defined by Eq. (9), is indeed y independent. Figure 2(c) also demonstrates very good agreement between $\mathcal{J}_{num}(y)$ and $\mathcal{J}_{cl}(y)$ given by Eqs. (6c) and (10).

Realizing that the good match between the numerical data and Eq. (3b) may be accidental due to particular chosen numerical parameters, we repeated the simulations with other magnitudes of the counterflow velocities. We found again a good agreement between the numerical profiles $\mathcal{P}_{num}(y)$, $\mathcal{P}_{num}(y)$, and $\mathcal{J}_{num}(y)$ with the corresponding closures $\mathcal{P}_3(y)$, $\mathcal{D}_{cl}(y)$, and $\mathcal{J}_{cl}(y)$. Consistency requires that the numerical constants C_{prod} , C_{dec} , and C_{flux} in these closures should be V_{ns} independent. Table I shows that this is the case only for C_{prod} , while C_{dec} and C_{flux} approximately depend on V_{ns} as $C_{dec} \propto 1/\langle V_{ns}^2 \rangle$ and $C_{flux} \propto \langle V_{ns}^2 \rangle$, where $\langle V_{ns}^2 \rangle$ is the mean square of V_{ns} across the channel (which, in its turn, $\propto V_n^2$). From these facts we can conclude that our closure (3a) for the production term is confirmed, while the traditional closure (1c) for the decay term and simple closure (10) seems to be questionable, although they reproduce well the profiles $\mathcal{D}_{num}(y)$ and $\mathcal{J}_{num}(y)$ in the parabolic case.

To clarify further the quality of the discussed closures, we imposed a nonparabolic normal velocity profile with two maxima and zero on the centerline, shown in Fig. 1(c).

TABLE I. Closure constants for different values of the normal velocity maximum V_n for parabolic (lines 1–3) and nonparabolic (line 4) profiles.

	V_n (cm/s)	$\sqrt{\langle V_{ns}^2 \rangle}/V_n$	C_{prod} ($\times 10^3$)	C_{dec}	C_{flux}	$C_{dec} \langle V_{ns}^2 \rangle$ [(cm/s) 2]	$C_{flux}/\langle V_{ns}^2 \rangle$ [(s/cm) 2]
1	1.0	0.788	1.8	6.9	0.088	4.31	0.145
2	1.2	0.785	1.8	4.9	0.124	4.35	0.147
3	1.5	0.780	1.9	3.2	0.210	4.32	0.146
4	1	1.247	2.2	2.3	0.048	3.62	0.039

Although this profile looks strange, it may be realized in a counterflow experiment with nonhomogeneous heating in a short enough channel [47]. For this profile we again computed the counterflow and vortex-line density profiles shown in Fig. 1(c). Next, we repeated all the steps described before, and found again that our proposed form Eq. (3b) fits the data much better than the other two forms, as seen in Fig. 3(a). Figures 3(b) and 3(c) show that the closures for the decay and the flux terms, \mathcal{D}_{cl} and \mathcal{J}_{cl} , are running into trouble, reflecting the numerical profiles only very roughly. In particular, this means that $c_2^2 \equiv (\ell\tilde{S})^2$, which is involved in the closure (1c) via Eq. (9), varies across the channel considerably, as follows from Fig. 3(b). Recall that in the parabolic case c_2^2 is y independent, although it depends on $\langle V_{ns}^2 \rangle$ approximately as $1/\langle V_{ns}^2 \rangle$. Consequently, to improve the closure for the decay term, one needs to involve an additional balance equation for the tangle curvature \tilde{S} . Similarly, our analysis shows that to improve the closure for the flux term, one needs to involve information about the tangle anisotropy.

Conclusion. The suggested closure (3a) for the production $\mathcal{P}_{prod}(y)$ using the counterflow velocity and the vortex-line density profiles can be considered as highly promising. However, the closures for the decay and flux are sensitive, and they may require accounting for additional tangle characteristics. The

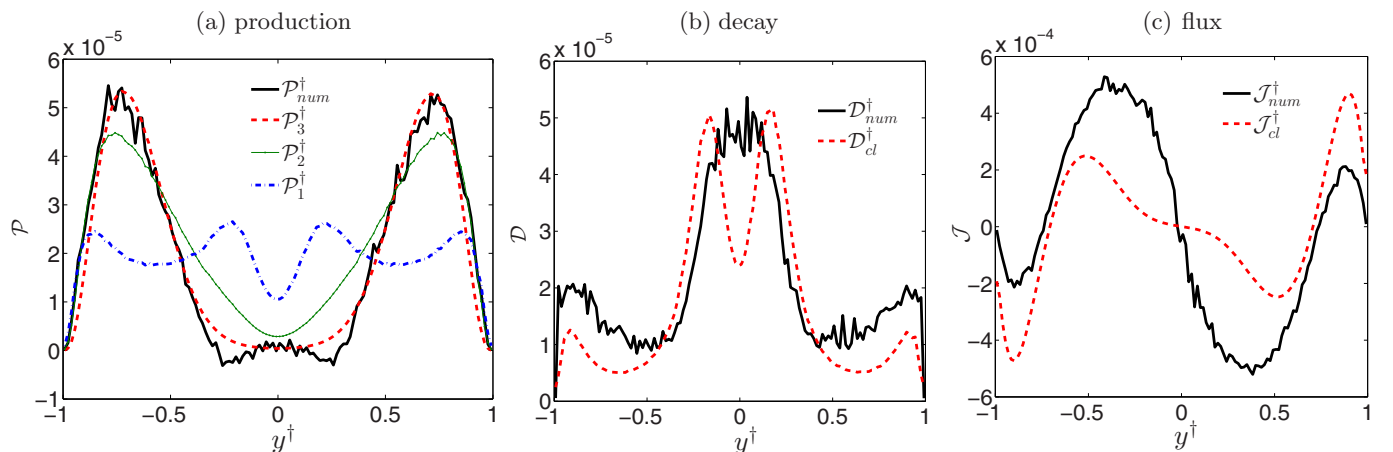


FIG. 3. (Color online) For nonparabolic normal velocity profile: A comparison between production (\mathcal{P}), decay (\mathcal{D}), and flux (\mathcal{J}) as obtained from numerical simulations and those obtained using their closure versions. Comparison of the numerical data (—) to the various allowed production forms of the Vinen equation (a): (· · ·) \mathcal{P}_1 , Eq. (2a), (—) \mathcal{P}_2 , Eq. (2b), and (—) \mathcal{P}_3 , Eq. (3b). Comparison of the numerical data for decay (b) and flux term (c) with Eq. (1c) for \mathcal{D} and Eq. (3c) for \mathcal{J} (—).

first candidate is the tangle curvature; the tangle anisotropy also can be important. Much more work in this direction is required to develop a consistent theory of wall-bounded superfluid turbulence.

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