Comment on "Theoretical analysis of quantum turbulence using the Onsager ideal turbulence theory"

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In a recent paper, Tanogami [Phys. Rev. E **103**, 023106 (2021)] proposes a scenario for quantum turbulence where the energy spectrum at scales smaller than the intervortex distance is dominated by a quantum stress cascade, in opposition to Kelvin-wave cascade predictions. The purpose of the present Comment is to highlight some physical issues in the derivation of the quantum stress cascade, in particular to stress that quantization of circulation has been ignored.

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In a recent paper, Tanogami presents a theoretical investigation of quantum turbulence at very low temperatures by adapting standard techniques used in classical hydrodynamics [1]. Following Onsager's ideas of classical turbulence [2], Tanogami proposes a double energy cascade scenario where the energy spectrum $E^v(k)$ behaves as

$$E^v(k) \sim \begin{cases} C_{\text{large}}\, k^{-5/3} & \text{for } \ell_{\text{large}}^{-1} \ll k \ll \lambda^{-1}, \\ C_{\text{small}}\, k^{-3} & \text{for } \lambda^{-1} \ll k \ll \ell_{\text{small}}^{-1}. \end{cases} \tag{1}$$

Here, k is the wave vector, and C_{large} and C_{small} are positive constants. ℓ_{large} is a scale that can be identified with the inertial scale of turbulence and ℓ_{small} is defined using the quantum stress cospectrum (see Ref. [1]). Then, Tanogami defines the *quantum baropycnal work* flux Λ_ℓ^Σ and identifies the scale λ as the scale at which Λ_ℓ^Σ becomes constant. Finally, he associates λ with the mean intervortex distance ℓ_i.

It has been largely accepted by the community that at scales smaller than the intervortex distance, the energy spectrum should display an energy cascade where the physics is dictated by the Kelvin-wave cascade [3–5]. With his derivation, Tanogami questions this cascade and proposes a completely different mechanism. In this Comment we argue that the k^{-3} *quantum stress cascade* is based on unphysical assumptions, and therefore such a scenario cannot take place. Basically, Tanogami's derivation completely ignores the quantization of velocity circulation, a crucial property of quantum turbulence that cannot be neglected at scales smaller than ℓ_i. We describe in the following our main criticisms on the double-cascade scenario proposed in Ref. [1].

The main issue of Tanogami's work is the starting equations used to apply and adapt to the quantum case, the standard techniques used in compressible classical turbulence. It is explicitly stated, already in the abstract, that three-dimensional quantum turbulence is investigated by using the Gross-Pitaevskii (GP) equation, but this statement is highly misleading. The GP equation is indeed an excellent theoretical framework to study low-temperature quantum turbulence because quantum vortices naturally arise as topological defects of the macroscopic wave function. As a consequence, the velocity circulation around a quantum vortex is quantized. However, the actual starting equations of Tanogami's derivation are the dispersive Euler equations, obtained after using the Madelung transformation in the GP equation. This transformation writes the complex wave function ψ as

$$\psi(\mathbf{x}) = \sqrt{\rho(\mathbf{x})}e^{i\theta(\mathbf{x})}, \tag{2}$$

where ρ can be identified with the fluid density and the phase θ defines the velocity field through the relation $\mathbf{v} = (\hbar/m)\nabla\theta$, with $\hbar$ the reduced Planck constant and m the boson mass. Indeed, after introducing (2) into the GP equation, one obtains the continuity equation for the density [Eq. (2) of Tanogami's work] and a modified Bernouilli equation for the phase θ. Only after taking the gradient of the Bernouilli equation and rearranging terms does one obtain the momentum equation for the velocity [Eq. (3) of Tanogami's work]. At the vortex core, ψ vanishes and therefore its phase is not defined. As a consequence, the dynamics of the dispersive Euler equation is mathematically equivalent to one of the GP equation only in the absence of vortices. Moreover, in the presence of vortices, the dispersive Euler equations become meaningless as they are undefined at the vortex locations and therefore cannot be used for predicting the motion of the vortex lines. Such issues were already pointed out by Wallstrom [6], where the inequivalence between the Schrödinger and the hydrodynamic was also discussed. We emphasize that such vortices are singular only in the velocity-density formulation: They cannot be considered weak solutions because they correspond to a perfectly smooth field ψ whose evolution is perfectly regular, unique, energy, momentum, and particle conserving. As a consequence, the Kolmogorov-Onsager ideas of turbulence cannot be directly applied there.

The dispersive Euler equations are however useful to provide a phenomenological hydrodynamic interpretation of the GP equation at scales where the quantization of circulation can be ignored, that is, at scales much larger than the intervortex distance ℓ_i. In order to give some meaning to the dispersive Euler equation at scales smaller than ℓ_i, such equations have to be supplied with information about the location of quantum vortices, that reduces in these variables to extremely complex moving boundary conditions on three-dimensional curves where density vanishes and circulation is quantized. Such constraints have been ignored in Tanogami's work, and could hardly be incorporated. Indeed, even if ρ and $\mathbf{v}$ are such that at $t = 0$ they represent a quantum turbulent state, at $t > 0$ the solutions of the dispersive Euler equations, if some meaning could be given to them, would most likely become a more general type of flow, not necessarily representing a quantum turbulent state with quantized vortices. For the reasons we have presented, Tanogami's work could provide, in principle, a rigorous derivation of the energy spectrum based on the Onsager conjecture and a phenomenological model of quantum turbulence, only at scales $k \ll \ell_i^{-1}$, i.e., at scales where quantum turbulence displays a classical behavior. Therefore the quantum stress cascade has no physical relevance and should not be considered as a possible scenario.

In more specific terms, in the main text Tanogami's derivation assumes some regularity of the velocity field that does not hold for quantum vortices. It is assumed that the velocity field is Hölder continuous with exponent $h \in (0, 1]$, i.e.,

$$\delta\mathbf{v}(\mathbf{r};\mathbf{x}) = \mathbf{v}(\mathbf{r}+\mathbf{x}) - \mathbf{v}(\mathbf{x}) = O(|\mathbf{r}|^h), \tag{3}$$

for $\ell = |\mathbf{r}| \to 0$. Note that an infinitely thin isolated quantum vortex leads to a regularity of $h = -1$ [which in terms of the energy spectrum corresponds to $E^v(k) \sim k^{-1}$ scaling [7,8]]. Note that a quantum vortex corresponds to a regularity of $h = -1$. In order to overcome this issue, the author considers a domain Ω that excludes any possible point $\mathbf{x}$ having a local Hölder exponent with $h < 0$. In doing so, all vortices are excluded from the domain. Therefore, only far-field contributions of the velocity field are retained. Even if the notion of quantized vortices was somehow introduced in the dynamics of the dispersive Euler equations, excluding the energy of this domain would certainly miss most contributions arising from Kelvin waves. To further avoid this possible issue, the author devotes one Appendix to generalizing the calculations using Besov spaces. In this approach, only the L^p norm of the velocity increments is assumed to have such restricted regularity, allowing in principle the velocity field to have a local negative Hölder exponent. Such an assumption is far from being justified physically as the vortex line density should be vanishingly small in order to fulfill such a restriction. Moreover, note that the author restricts in Eq. (87) the regularity of density gradients to have a Hölder exponent $\beta \in (0, 1)$. Contrary to the velocity field, the density and its gradients are completely regular fields in the vortex, so β is larger than or equal to 1. Note that for a quantum vortex, the velocity diverges but the density simultaneously vanishes at its position in such a way that the wave function, and the momentum and energy densities, are completely regular fields. This singularity is only a trivial consequence of the Madelung transformation not being defined at the vortex core.

In addition, we would like to remark that the quantum stress cascade predicted by Tanogami results from the quantum pressure term of the dispersive Euler equation. Such a term is independent of the GP nonlinearity and therefore the same analysis could be applied to the (linear) Schrödinger equation. With this equation being linear, we cannot expect an energy transfer along scales. Perhaps the k^{-3} law is a consequence of the regularity of the wave function at small scales, in agreement with Tanogami's choice of a local Hölder exponent $h = 1$ in this range. Such a scaling could emerge at scales $\ell \ll \xi$, however, as mentioned in his work, it could be much shallower due to nonlinearity depletion.

Finally, from a practical point of view, if such a quantum stress cascade displaying a k^{-3} scaling exits at scales between ℓ_i and the healing length ξ, it should be overwhelmed by Kelvin waves (KWs), which display much less steep spectra [3–5]. Also, we would like to remark that there exist several works using the GP equation where the KW cascade has been observed, which we summarize in the following. In Ref. [9], the KW cascade was observed by directly tracking perturbed straight vortex lines. In that setting, there was no Kolmogorov cascade as by construction $\ell_i \sim \ell_{\mathrm{large}}$, but the KW cascade was observed to be compatible with the weak wave turbulence predictions. Later, in Ref. [10], KWs were studied in turbulent tangles (exhibiting a Kolmogorov spectrum) by using spatiotemporal filtering. Then, by tracking large vortex rings of turbulent quantum tangles, Villois *et al.* [11] observed the development of a KW cascade with a spectrum supporting the L'vov and Nazarenko prediction [4] for more than one decade. Finally, several simulations, using resolutions up to 4096^3 collocations points, have observed a Kolmogorov scaling range, followed by a KW cascade at scales smaller than ℓ_i [12–14]. In particular, in Müller and Krstulovic [14], the L'vov and Nazarenko prediction was observed, including the scaling with the energy flux. In summary, we believe that there is enough evidence supporting the scenario where in quantum turbulence the Kolmogorov energy cascade is followed by the KW energy cascade.

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