V.E. Zakharov V.S. L'vov
G. Falkovich

Kolmogorov Spectra of Turbulence I

Wave Turbulence

1. Equations of Motion and the Hamiltonian Formalism

1.1 The Hamiltonian Formalism for Waves in Continuous Media

Equations describing waves in different media and written in natural variables are diverse. For example, the Bloch equations defining the motion of a magnetic moment are totally different from the Maxwell equations for nonlinear dielectrics. The latter radically differ from the Euler equations for compressible fluids. However all of them as well as many other equations describing nondissipative media, possess an implicit or explicit Hamiltonian structure. This was established empirically and is reflected by the fact that all these models may be derived from initial microscopic Hamiltonian equations of motion.

The Hamiltonian method is applicable to a wide class of weakly dissipative wave systems; it clearly manifests general properties of small-amplitude waves. For example, spin, electromagnetic and sound waves are just waves, i.e., medium oscillations, transferred from one point to another. If we are interested only in small-amplitude wave propagation phenomena, such as diffraction, we do not really need to know what it is that oscillates: magnetic moment, electrical field or density. Their respective dispersion law $\omega(\boldsymbol{k})$ contains all the information about the medium properties that is necessary and sufficient for studying the propagation of noninteracting waves. As we shall see now, the $\omega(\boldsymbol{k})$-function is a coefficient in the term of the Hamiltonian which is quadratic with respect to wave amplitudes, i.e., to complex normal variables. The actual Hamiltonian is a power series in these variables that contains all the information about nonlinear wave interactions. Let us consider the transition to such variables using a simple yet very important example.

1.1.1 The Hamiltonian in Normal Variables

A continuous medium of dimensionality d may be defined in the simplest case by a pair of canonical variables $p(\boldsymbol{r},t)$ and $q(\boldsymbol{r},t)$. The canonical equations of motion are expressed as

$$\frac{\partial q(\boldsymbol{r},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta p(\boldsymbol{r},t)}\,, \tag{1.1.1}$$

$$\frac{\partial p(\boldsymbol{r},t)}{\partial t} = -\frac{\delta \mathcal{H}}{\delta q(\boldsymbol{r},t)} \ . \tag{1.1.2}$$

The Hamiltonian $\mathcal{H}$ depends on $p(\boldsymbol{r},t)$ and $q(\boldsymbol{r},t)$ as a functional. The symbols $\delta/\delta q$ and $\delta/\delta p$ designate variational derivatives which are extensions of partial derivatives for the continuous case (Sect. A.1). The formal advantage of the Hamiltonian method is that its equations are symmetric in coordinate q and momentum p. To illustrate this advantage, let us first go over to new canonical variables $Q = \lambda q$, $P = p/\lambda$, choosing the dimensional factor λ in such a way that P and Q have the same dimension. Then we introduce complex variables

$$a = (Q + iP)/\sqrt{2} \ , \tag{1.1.3}$$

$$a^* = (Q - iP)/\sqrt{2} \tag{1.1.4}$$

to obtain

$$\sqrt{2}\frac{\partial a}{\partial t} = \frac{\delta \mathcal{H}}{\delta P} - i\frac{\delta \mathcal{H}}{\delta Q}, \qquad \sqrt{2}\frac{\partial a^*}{\partial t} = \frac{\delta \mathcal{H}}{\delta P} + i\frac{\delta \mathcal{H}}{\delta Q} \ . \tag{1.1.5}$$

Substituting for $\mathcal{H}, P$ and Q, we obtain

$$i\frac{\partial a(\boldsymbol{r},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{r},t)}, \qquad i\frac{\partial a^*(\boldsymbol{r},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta a(\boldsymbol{r},t)} \ . \tag{1.1.6}$$

The second equation follows from the first by complex conjugation. Hence, we have obtained one complex equation instead of two real ones (1.1.1,2). In quantum mechanics such a substitution of variables corresponds to a transition from the coordinate-momentum representation to a representation using creation and annihilation Bose operators. Their classical analogues are complex canonical variables. Obviously, the canonical variables (1.1.3,4) are by no means the only possible variables to choose. A large choice of transformations from one set of variables to another $a, a^* \to b, b^*$ exists, such that the equations of motion retain their canonical form (1.1.6). For a given set of explicit variables the canonicity condition is expressed through the Poisson brackets of two functions

$$\{f(q), g(q')\} = \int d\boldsymbol{r}'' \left[\frac{\delta f(q)}{\delta a^*(\boldsymbol{r}'')}\frac{\delta g(q')}{\delta a(\boldsymbol{r}'')} - \frac{\delta f(q)}{\delta a(\boldsymbol{r}'')}\frac{\delta g(q')}{\delta a^*(\boldsymbol{r}'')}\right]$$

(see Sect. A.2) and has the simple form

$$\{b(q), b(q')\} = 0, \quad \{b(q), b^*(q')\} = \delta(q - q') \ . \tag{1.1.7}$$

To ensure that this is a one-to-one transformation the index q should cover a "complete set" of values, for example, the space R^d. It should be noted that this wide range of possibilities in selecting canonical variables is an important advantage of the Hamiltonian method. It ensures the choice of the most adequate variables for a specific problem. We shall define the canonical variables $a(p)$ in such a way that they determine wave amplitudes and become zero for vanishing waves.

Let us expand the Hamiltonian $\mathcal{H}$ in a power series of variables $a(\boldsymbol{r})$ and $a^*(\boldsymbol{r})$ assuming these to be small. The zeroth order term is of no interest for us, since it does not occur in the equation of motion. There are no first-order terms as the medium is assumed to be in equilibrium if the amplitudes of the waves were zero, and, consequently, the Hamiltonian to be minimal at $a = a^* = 0$. Thus the $\mathcal{H}$ expansion starts from the second-order terms:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\text{int}} \ . \tag{1.1.8}$$

The most general form of $\mathcal{H}_2$ is:

$$\begin{aligned}\mathcal{H}_2 = \int d\boldsymbol{r}\,d\boldsymbol{r}'\{&A(\boldsymbol{r},\boldsymbol{r}')a(\boldsymbol{r})a^*(\boldsymbol{r}')\\ &+ (1/2)[B^*(\boldsymbol{r},\boldsymbol{r}')a(\boldsymbol{r})a(\boldsymbol{r}') + \text{c.c}]\} \ .\end{aligned} \tag{1.1.9}$$

Here "c.c." means the complex conjugate of the preceding term.

The $\mathcal{H}_2$ value is real (the Hamiltonian is Hermitian). Therefore

$$A(\boldsymbol{r},\boldsymbol{r}') = A^*(\boldsymbol{r}',\boldsymbol{r}), \quad B(\boldsymbol{r},\boldsymbol{r}') = B(\boldsymbol{r}',\boldsymbol{r}) \ . \tag{1.1.10}$$

Below we shall consider the medium to be spatially homogeneous. This very important assumption is the basis of all following discussions. Because of space homogeneity, the functions $A(\boldsymbol{r},\boldsymbol{r}')$ and $B(\boldsymbol{r},\boldsymbol{r}')$ depend only on the difference of the arguments $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$. Now

$$A(\boldsymbol{R}) = A^*(-\boldsymbol{R}), \quad B(\boldsymbol{R}) = B(-\boldsymbol{R}) \ . \tag{1.1.11}$$

The Hamiltonian can be significantly simplified using the Fourier transform:

$$\begin{aligned} a(\boldsymbol{k}) = a_k &= (1/V)\int a(\boldsymbol{r})\exp(-i\boldsymbol{k}\boldsymbol{r})\,d\boldsymbol{r},\\ a(\boldsymbol{r}) &= \sum_{\boldsymbol{k}} a_k \exp(i\boldsymbol{k}\boldsymbol{r}) \ .\end{aligned} \tag{1.1.12}$$

Here V is the volume of a sample (the propagation medium). We consider the wave vector $\boldsymbol{k}$ to be a discrete variable. This corresponds to imposing periodic space boundary conditions on the wave field $a(\boldsymbol{r}+L) = a(\boldsymbol{r})$, $V = L^d$. If required, one can pass in any conventional manner from the summation over k to the integration

$$(2\pi)^d \sum_{\boldsymbol{k}} = V \int d\boldsymbol{k} \ . \tag{1.1.13}$$

The Fourier transform (1.1.12) is canonical but not unimodal. This means that the canonical equation of motion (1.1.6) retains its canonical form, but the new Hamiltonian differs from the old one by a factor, which is the inverse sample volume

$$V\mathcal{H}(a_{\boldsymbol{k}}, a_{\boldsymbol{k}}^*) = \mathcal{H}\{a(\boldsymbol{r}), a^*(\boldsymbol{r})\} \ , \tag{1.1.14}$$

$$i\frac{\partial a(\boldsymbol{k},t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k},t)} .$$

It is essential that in the new variables $a(k)$ the quadratic part of the Hamiltonian represents a single integral over $d\boldsymbol{k}$

$$\mathcal{H}_2 = \int \{A(\boldsymbol{k})a(\boldsymbol{k},t)a^*(\boldsymbol{k},t) + \frac{1}{2}[B(\boldsymbol{k})a(\boldsymbol{k},t)a(-\boldsymbol{k},t) + B^*(\boldsymbol{k})a^*(\boldsymbol{k},t)a^*(-\boldsymbol{k},t)]\}\, d\boldsymbol{k} , \tag{1.1.15}$$

$$A(\boldsymbol{k}) = \int A(\boldsymbol{R})\exp(i\boldsymbol{k}\boldsymbol{R})\, d\boldsymbol{R} , \quad B(\boldsymbol{k}) = \int B(\boldsymbol{R})\exp(i\boldsymbol{k}\boldsymbol{R})\, d\boldsymbol{R} .$$

In view of (1.1.10), $A(\boldsymbol{k}) = A^*(\boldsymbol{k})$ is a real function and $B(\boldsymbol{k}) = B(-\boldsymbol{k})$ is an even function. The latter means that we can consider $B(\boldsymbol{k})$ to be real as well. Indeed, if $B(\boldsymbol{k}) = |B(\boldsymbol{k})|\exp[i\psi(\boldsymbol{k})]$, then $\psi(\boldsymbol{k}) = \psi(-\boldsymbol{k})$ and one can dispose of $\psi(\boldsymbol{k})$ by substitution $a(\boldsymbol{k}) \to a(\boldsymbol{k})\exp[-i\psi(\boldsymbol{k})/2]$. Let us pose a question: in which case may the Hamiltonian (1.1.15) be diagonalized with respect to the wave vector using a linear transformation

$$a(\boldsymbol{k},t) = u(\boldsymbol{k})b(\boldsymbol{k},t) + v(\boldsymbol{k})b^*(-\boldsymbol{k},t) ; \tag{1.1.16}$$

in other words, is there a way to represent it as

$$\mathcal{H}_2 = \int \omega(\boldsymbol{k})b(\boldsymbol{k},t)b^*(\boldsymbol{k},t)\, d\boldsymbol{k} \ ? \tag{1.1.17}$$

First, let us derive the canonicity conditions for this transformation. On the one hand,

$$\frac{\partial a(\boldsymbol{k},t)}{\partial t} = u(\boldsymbol{k})\frac{\partial b(\boldsymbol{k},t)}{\partial t} + v(\boldsymbol{k})\frac{\partial b^*(-\boldsymbol{k},t)}{\partial t} = i\frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k},t)} .$$

On the other hand, one should require $\partial b/\partial t$ to equal $i\delta\mathcal{H}/\delta b^*$ and

$$\begin{aligned}\frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k})} &= u(\boldsymbol{k})\frac{\delta \mathcal{H}}{\delta b^*(\boldsymbol{k})} - v(\boldsymbol{k})\frac{\delta \mathcal{H}}{\delta b(-\boldsymbol{k})} \\ &= u(\boldsymbol{k})\left[\frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k})}u^*(\boldsymbol{k}) + v(-\boldsymbol{k})\frac{\delta \mathcal{H}}{\delta a(-\boldsymbol{k})}\right] \\ &\quad - v(\boldsymbol{k})\left[\frac{\delta \mathcal{H}}{\delta a^*(\boldsymbol{k})}v^*(\boldsymbol{k}) + u(-\boldsymbol{k})\frac{\delta \mathcal{H}}{\delta a(-\boldsymbol{k})}\right] .\end{aligned}$$

Thus, the canonicity conditions take the following form

$$|u(\boldsymbol{k})|^2 - |v(\boldsymbol{k})|^2 = 1, \quad u(\boldsymbol{k})v(-\boldsymbol{k}) = u(-\boldsymbol{k})v(\boldsymbol{k}) . \tag{1.1.18}$$

The parameter $u(\boldsymbol{k})$ may be chosen to be real without loss in generality, which simply implies a choice of phase for complex variable $b(\boldsymbol{k})$. Since the value of $v(\boldsymbol{k})$ may also be chosen to be real [see (1.1.19)] it is convenient to set

$$u(\boldsymbol{k}) = \cosh[\zeta(\boldsymbol{k})], \quad v(\boldsymbol{k}) = \sinh[\zeta(\boldsymbol{k})].$$

According to (1.1.18), $\zeta(\boldsymbol{k})$ is a real even function. Substituting (1.1.16) into (1.1.15) and comparing to (1.1.17) we obtain after symmetrization with respect to $\boldsymbol{k}$ and $-\boldsymbol{k}$:

$$\begin{aligned}\omega(\boldsymbol{k}) = A(\boldsymbol{k})\cosh^2[\zeta(\boldsymbol{k})] + A(-\boldsymbol{k})\sinh^2[\zeta(\boldsymbol{k})] \\ + 2B(\boldsymbol{k})\sinh[\zeta(\boldsymbol{k})]\cosh[\zeta(\boldsymbol{k})] ,\end{aligned} \tag{1.1.19a}$$

$$\begin{aligned}0 = [A(\boldsymbol{k}) + A(-\boldsymbol{k})]\sinh[\zeta(\boldsymbol{k})]\cosh[\zeta(\boldsymbol{k})] \\ + B(\boldsymbol{k})\left[\cosh^2[\zeta(\boldsymbol{k})] + \sinh^2[\zeta(\boldsymbol{k})]\right] .\end{aligned} \tag{1.1.19b}$$

Dividing $A(\boldsymbol{k})$ into even and odd parts

$$\begin{aligned}A(\boldsymbol{k}) &= A_1(\boldsymbol{k}) + A_2(\boldsymbol{k}), \qquad A(-\boldsymbol{k}) = A_1(\boldsymbol{k}) - A_2((\boldsymbol{k}) \\ A_1(-\boldsymbol{k}) &= A_1(\boldsymbol{k}), \qquad A_2(-\boldsymbol{k}) = -A(\boldsymbol{k}) ,\end{aligned}$$

and substituting the respective expressions for $B(\boldsymbol{k})$, we obtain

$$\omega(\boldsymbol{k}) = A_2(\boldsymbol{k}) + \frac{A_1(\boldsymbol{k})}{\cosh[2\zeta(\boldsymbol{k})]}$$

for the frequency. Thus, the sign of an even part of $\omega(\boldsymbol{k})$ coincides with that of even part of $A(\boldsymbol{k})$. Expressing $\cosh[2\zeta(\boldsymbol{k})]$ from (1.1.19b) we obtain

$$\omega(\boldsymbol{k}) = A_2(\boldsymbol{k}) + \operatorname{sign} A_1(\boldsymbol{k})\sqrt{A_1^2(\boldsymbol{k}) - B^2(\boldsymbol{k})} . \tag{1.1.20}$$

One can see that it is only for real $\omega(\boldsymbol{k})$ possible to find a diagonalizing transformation. In the variables $b(\boldsymbol{k},t)$ the equations of motion become trivial

$$\frac{\partial b(\boldsymbol{k},t)}{\partial t} + i\omega(\boldsymbol{k})b(\boldsymbol{k},t) = 0$$

and have the solution $b(\boldsymbol{k},t) = b(\boldsymbol{k},0)\exp[i\omega(\boldsymbol{k})t]$; thus it is evident that real $\omega(\boldsymbol{k})$ implies the stability of the medium against an exponential growth in the wave amplitudes.

In most physical situations, the Hamiltonian is the wave energy density whose sign coincides, by virtue of (1.1.17), with that of the frequency $\omega(\boldsymbol{k})$.

In general, wave excitation increases the energy of the medium, which implies that the function $\omega(\boldsymbol{k})$ is usually positive. A negative value of $\omega(\boldsymbol{k})$ indicates that the energy of the medium decreases with increasing wave excitation. This is possible in systems that are far from thermodynamic equilibrium, for example, in plasmas containing a flux of particles. In that case a_k describes the negative-energy waves. It should be borne in mind that the Hamiltonian has been formally defined to an accuracy of a sign, since the transformation $\mathcal{H} \leftrightarrow -\mathcal{H}$, $a \leftrightarrow a^*$ is possible. Therefore the negative-energy waves may be considered only in the case when the $\omega(\boldsymbol{k})$ function changes sign in the $\boldsymbol{k}$-space. What is the connection between a change of sign in $\omega(\boldsymbol{k})$ and wave instability? As one can see from

(1.1.20), a change of sign in the even part of $\omega(\boldsymbol{k})$ means that there is a surface on which $A_1(\boldsymbol{k}) = 0$. In the general position on this surface $B(\boldsymbol{k}) \neq 0$. This implies that at least in the vicinity of the zero surface of $A_1(\boldsymbol{k})$ the square of the frequency is negative and the medium is unstable. Thus, if the even part of the $\omega(\boldsymbol{k})$-function is sign-alternating and the $B(\boldsymbol{k})$-function does not identically vanish wherever $A_1(\boldsymbol{k}) = 0$, the k-space contains a field of linear instability. Given this instability, it is impossible to transform the Hamiltonian into the form (1.1.17). In this case, however, the Hamiltonian (1.1.15) reduces to the simple form

$$\mathcal{H}_2 = \int C(\boldsymbol{k})[b(\boldsymbol{k},t)b(-\boldsymbol{k},t) + b^*(\boldsymbol{k},t)b^*(-\boldsymbol{k},t)]\,d\boldsymbol{k}\ ,$$
$$C(\boldsymbol{k}) = C(-\boldsymbol{k}) = C^*(\boldsymbol{k})\ .$$

From the analogy to quantum-mechanics it is evident that such a Hamiltonian describes the creation of a pair of quasi-particles from the vacuum (and the reverse process), thus representing just such an unstable medium. Summarizing, the canonical transformations (1.1.16) allow us to eliminate the term with the least factor (in magnitude) in the Hamiltonian (1.1.15).

Up to now, we have considered the case of a medium containing only one type of waves described by a single dispersion law $\omega(\boldsymbol{k})$. We can examine, without any essential complications, a more general case with a medium having several types of waves. In this case, the medium is described by a set of equations

$$\frac{\partial q_j}{\partial t} = \frac{\delta\mathcal{H}}{\delta p_j}, \quad \frac{\partial p_j}{\partial t} = -\frac{\delta\mathcal{H}}{\delta q_j}, \quad j = 1,\ldots n\ .$$

Going over to complex variables $a_j = (1/\sqrt{2})(q_j + ip_j)$, we have for the quadratic part of the Hamiltonian

$$\begin{aligned}\mathcal{H}_2 = \sum_{i,j}^{n} \int d\boldsymbol{r}\,\big[A_{ij}(\boldsymbol{r}-\boldsymbol{r}_1)a_i(\boldsymbol{r},t)a_j^*(\boldsymbol{r}_1,t)\\ + (1/2)\big(B_{ij}(\boldsymbol{r}-\boldsymbol{r}_1)a_i(\boldsymbol{r},t)a_j(\boldsymbol{r}_1,t) + \text{c.c.}\big)\big]\ .\end{aligned} \tag{1.1.21}$$

Now

$$A_{ij}(\boldsymbol{R}) = A_{ji}^*(-\boldsymbol{R})\ , \qquad B_{ij}(\boldsymbol{R}) = B_{ji}(-\boldsymbol{R})\ .$$

Normal variables are introduced by diagonalizing the Hamiltonian (1.1.21), which results in the reduction of the Hamiltonian for a stable medium to the form

$$\mathcal{H} = \sum_j \int \omega_j(\boldsymbol{k})b_j(\boldsymbol{k},t)b_j^*(\boldsymbol{k},t)\,d\boldsymbol{k}\ . \tag{1.1.22}$$

Diagonalization may be accomplished if all $\omega_j(\boldsymbol{k})$ have the same sign and do not identically coincide.

1.1.2 Interaction Hamiltonian for Weak Nonlinearity

In various problems of nonlinear wave dynamics, the wave amplitude may be defined by a natural dimensionless parameter ξ. For sound waves, this parameter is represented by the ratio of the density variation in the sound wave to the average density of the medium; for fluid surface waves it is the ratio of the wave height to wavelength. For spin waves, ξ is the precession angle of the magnetic moment. For ξ of the order of unity, phenomena specific for each of the above media arise: sound turns into shock waves, fluid surface waves form whitecaps and in ferromagnets an inversion of magnetization occurs, that is, a movable domain wall. Obviously, consideration of all these phenomena from a general viewpoint is not always constructive. If, however, the wave's nonlinearity parameter ξ is small, the characteristic features of the medium are negligible, and the wave dynamics may be described in general terms by expanding the Hamiltonian in terms of canonical variables. Let us now look in greater detail at the expansion we started to analyze in the preceding subsection. Suppose we have only a single wave type in a stable medium. Then the first term of the Hamiltonian expansion has the form (1.1.17), and the corresponding equation of motion may be written as

$$\frac{\partial b(\boldsymbol{k},t)}{\partial t} + i\omega(\boldsymbol{k})b(\boldsymbol{k},t) = 0, \quad b(\boldsymbol{k},t) = b(\boldsymbol{k})\exp[-i\omega(\boldsymbol{k})t]\ .$$

At this level of sophistication, waves in different media are only distinguished by their dispersion laws $\omega(\boldsymbol{k})$. All information about the wave interaction is contained in the higher coefficients of the expansion of $\mathcal{H}$ in a power series of b:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\text{int}}, \qquad \mathcal{H}_{\text{int}} = \mathcal{H}_3 + \mathcal{H}_4 + \ldots\ . \tag{1.1.23}$$

The physical meaning of $\mathcal{H}_3$ and $\mathcal{H}_4$ is easy to understand by analogy with quantum mechanics. The Hamiltonian $\mathcal{H}_3$ describes three-wave processes:

$$\begin{aligned}\mathcal{H}_3 = \frac{1}{2}\int (V_q b_1 b_2 b_3 + \text{c.c.})\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3)\,d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3\\ + \frac{1}{6}\int (U_q b_1^* b_2^* b_3^* + \text{c.c.})\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\,d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3\ .\end{aligned} \tag{1.1.24a}$$

Here and below a shorthand notation is be used: b_1, b_2 are $b(\boldsymbol{k}_1,t), b(\boldsymbol{k}_2,t)$; $q = (\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$ and thus $V_q = V_{123} = V(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)$. The first term in (1.1.24) defines the decay processes $1 \to 2$ and the reverse confluence processes $2 \to 1$. The second term describes mutual annihilation of three waves or their creation from vacuum.

The Hamiltonian $\mathcal{H}_4$ describes processes involving four waves:

$$\mathcal{H}_4 = (1/4)\int W_p b_1^* b_2^* b_3 b_4 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4$$
$$+ \int (G_p b_1 b_2^* b_3^* b_4^* + \text{c.c.})\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4$$
$$+ \int (R_p^* b_1 b_2 b_3 b_4 + \text{c.c})\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 ,$$
$$\text{with} \quad p = (\boldsymbol{k}_1 \boldsymbol{k}_2 \boldsymbol{k}_3 \boldsymbol{k}_4) . \qquad (1.1.24b)$$

The coefficients of the interaction Hamiltonian have the following obvious properties

$$\begin{aligned} V(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) &= V(\boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_2), \\ U(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) &= U(\boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_2) = U(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}_3), \\ G(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) &= G(\boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_2, \boldsymbol{k}_4) = G(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_4, \boldsymbol{k}_3), \\ W(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4) &= W(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_4) \\ &= W(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_4, \boldsymbol{k}_3) = W^*(\boldsymbol{k}_3, \boldsymbol{k}_4, \boldsymbol{k}_1, \boldsymbol{k}_2), \\ R(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_3) &= R(\boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}_3, \boldsymbol{k}_4) = R(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_4, \boldsymbol{k}_3) \\ &= R(\boldsymbol{k}_3, \boldsymbol{k}_2, \boldsymbol{k}_1, \boldsymbol{k}_4) = R(\boldsymbol{k}_4, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_1) . \end{aligned} \qquad (1.1.25)$$

The last equation for $W(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4)$ follows since the Hamiltonian is real.

But to which order in b, b^* should we expand the Hamiltonian $\mathcal{H}$? The answer turns out to be as general as the question: "$\mathcal{H}_5$ and higher-order terms should normally not be taken into account." This can be explained as follows: Since the expansion uses a small parameter, every subsequent term is smaller than the preceding one, and the dynamics of the wave system is determined by the very first term of $\mathcal{H}_{\text{int}}$, i.e., normally $\mathcal{H}_3$. However, three-wave processes may turn out to be "nonresonant" which means that the spatio-temporal synchronization condition (or, in terms of quasi-particles, the momentum-energy conservation law)

$$\omega(\boldsymbol{k} + \boldsymbol{k}_1) = \omega(\boldsymbol{k}) + \omega(\boldsymbol{k}_1) \qquad (1.1.26)$$

cannot be satisfied. Let d be the dimensionality of the medium and $\boldsymbol{k}$ the vector in d-meric space ($d > 1$). Equation (1.1.26) specifies the hypersurface of dimension $2d - 1$ in the $2d$-meric space of vectors $\boldsymbol{k}, \boldsymbol{k}_1$. If this surface does in fact exist [i.e., $\omega(\boldsymbol{k})$ is real], the dispersion law $\omega(\boldsymbol{k})$ is of the decay type. If (1.1.26) has no real solutions, the dispersion law is of the nondecay type.

In isotropic media, $\omega(\boldsymbol{k})$ is a function of k only. Let $\omega(0) = 0$, $\omega' = \partial\omega(\boldsymbol{k})/\partial k > 0$. In this important case a simple criterion for the decay may be formulated: the dispersion law is of the decay type if $\omega'' = \partial^2\omega(\boldsymbol{k})/\partial k^2 > 0$ and of the nondecay type if $\omega'' < 0$. This criterion has a clear geometric meaning. Consider, for example, the case with $d = 2$. The dispersion law $\omega(\boldsymbol{k})$ then specifies the surface of rotation in a three-dimensional space $\omega, \boldsymbol{k}_x, \boldsymbol{k}_y$. In Fig. 1.1 this is the surface S for $\omega(\boldsymbol{k})$ and S_1 for $\omega(\boldsymbol{k}_1)$. It is seen that (1.1.26) is satisfied if these surfaces intersect, then all the three points $(\omega, \boldsymbol{k})$, (ω_1, k_1) and

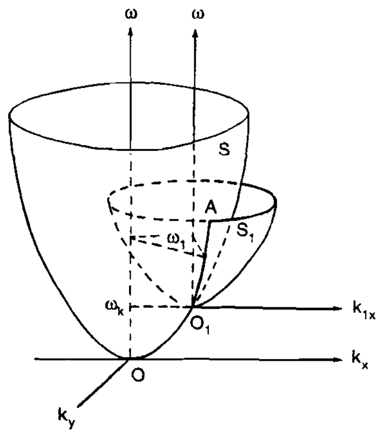

Fig. 1.1. Intersection of the frequency surfaces of the convex function $\omega(k)$

$\big(\omega(\boldsymbol{k} + \boldsymbol{k}_1), \boldsymbol{k} + \boldsymbol{k}_1\big)$ lie on the intersection line $O_1 A$. Surface section requires a convex function, i.e., $\omega'' > 0$. For power functions $\omega(\boldsymbol{k}) \propto k^\alpha$, the dispersion law is of the decay type if $\alpha > 1$. In the limiting case at $\alpha = 1$, decays are only allowed for collinear vectors.

Four-wave processes are always permitted. This is evident from the conservation laws for scattering processes

$$\omega(\boldsymbol{k}) + \omega(\boldsymbol{k}') = \omega(\boldsymbol{k} + \boldsymbol{\kappa}) + \omega(\boldsymbol{k}' - \boldsymbol{\kappa}), \qquad (1.1.27)$$

which are allowed at $\kappa \to 0$ for any dispersion laws. Consequently, it is $\mathcal{H}_4$ that will govern the dynamics of wave systems with a nondecay dispersion law. Subsequent expansion terms $\mathcal{H}_5, \ldots$ will define small and, generally, insignificant corrections. However, nature is certainly more interesting than this formal scheme. Some problems may involve an additional small parameter (apart from the nonlinearity level). For example, for spin waves in magnets (Sect. 1.4), not only $\mathcal{H}_3$ but also $\mathcal{H}_4$ should be taken into account, even in the decay region of the spectrum. The Hamiltonian $\mathcal{H}_3$ arises only because of the magnetic dipole-dipole interaction and has relativistic smallness as compared with the Hamiltonian $\mathcal{H}_4$ which results from the exchange interaction. The authors are aware of problems where one has to take into consideration the fifth- and even sixth-order terms (for example, while considering an approximate conservation of the wave action integral, see below). These problems are, however, rather specialized and go beyond the frame of this book. Hence, it is usually sufficient to expand the Hamiltonian in the decay region of the spectrum up to third-order, and in the nondecay region up to fourth-order terms.

1.1.3 Dynamic Perturbation Theory. Elimination of Nonresonant Terms

It is intuitively clear that in the case of a nondecay dispersion law, the Hamiltonian $\mathcal{H}_3$ describing three-wave processes may turn out to be irrelevant in some respect. We shall show now that in this case one can go over to new canonical variables c_k, c_k^*, such that $\mathcal{H}_3\{c_k, c_k^*\} = 0$. This is possible because the dynamic system under consideration, the weakly nonlinear wave field, is close to a completely integrable dynamic system (a set of noninteracting oscillators). Traditionally classical perturbation theory is employed to handle systems close to completely integrable ones. In this procedure a canonical transformation is derived by sequentially excluding the nonintegrable terms from the Hamiltonian. It is known that the procedure may encounter the problem of "small resonance denominators"; then the only terms to be excluded from the Hamiltonian are those for which the resonance condition is not satisfied. As shown by *Zakharov* [1.1], we can in this case to some extent apply classical perturbation theory.

Let us demonstrate such a procedure using a simple example. Consider the expansion of the one-wave Hamiltonian:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 = \omega bb^* + \frac{V}{2}(b^2 b^* + b^{*2} b) + \frac{U}{6}(b^3 + b^{*3}) + \frac{W}{4} b^2 b^{*2} + G(b^3 b^* + bb^{*3}) + R(b^4 + b^{*4}).$$

We assume every subsequent term to be smaller than the preceding one, i.e. $\mathcal{H}_2 \gg \mathcal{H}_3 \gg \mathcal{H}_4$. Since one needs to eliminate the $\mathcal{H}_3$ term without changing $\mathcal{H}_2$, the transformation must be close to the unity transformation. Thus it is reasonable to search for the transformation in the form of an expansion, which starts from a linear term:

$$b = c + A_1 c^2 + A_2 cc^* + A_3 c^{*2} + B_1 c^3 + B_2 c^* c^2 + B_3 cc^{*2} + B_4 c^{*3} + \dots \quad (1.1.28a)$$

Why do we take the following (c^3-order) terms into account? Taking only the linear and quadratic terms in (1.1.28a) into account is indeed sufficient to eliminate the $\mathcal{H}_3$ term. In that case the fourth-order terms govern the nonlinear interaction. Due to the transformation they will acquire additional terms. To derive these new terms c^3-order terms in (1.1.28a) have to be taken into account. Moreover, we will use them to exclude the last two terms describing the $1 \to 3$ and $0 \to 4$ processes in the Hamiltonian (1.1.24b).

Thus we look for seven coefficients: $A_1, A_2, A_3, B_1, \dots, B_4$. The canonicity condition (1.1.7) is expressed through the Poisson brackets and has the form

$$\{bb^*\} = \frac{\partial b}{\partial c}\frac{\partial b^*}{\partial c^*} - \frac{\partial b^*}{\partial c}\frac{\partial b}{\partial c^*} = 1.$$

Computing the Poisson bracket to an accuracy of c^3-order terms, we obtain three equations

$$A_2 = -2A_1, \quad B_2 = A_3^2 - A_1^2, \quad B_3 + 3B_1 = 2A_2(A_3 - A_1).$$

Substituting (1.1.28a) into the Hamiltonian and demanding that all nonlinear terms (except $c^2 c^{*2}$) vanish, we have four equations

$$2\omega(A_1 + A_2) + V = 0, \quad 6\omega A_3 + U = 0,$$
$$\omega B_4 + \omega A_1 A_3 + \frac{1}{2} V A_3 + \frac{1}{2} U A_1 + R = 0,$$
$$\omega(A_1 A_2 + A_2 A_3 + B_1 + B_3) + V A_3 + \frac{1}{2} U A_2 + G = 0.$$

From these it is easy to find the transformation coefficients

$$\begin{aligned} A_1 &= \frac{V}{2\omega}, & A_2 &= -\frac{V}{\omega}, & A_3 &= -\frac{U}{6\omega}, \\ B_1 &= \frac{V^2}{4\omega^2} + \frac{VU}{6\omega^2} + \frac{G}{2\omega}, & & & B_2 &= \frac{U^2}{36\omega^2} - \frac{V^2}{4\omega^2}, \\ B_3 &= \frac{V^2}{4\omega^2} + \frac{7UV}{12\omega^2} - \frac{3G}{2\omega}, & & & B_4 &= -\frac{UV}{12\omega^2} - \frac{R}{\omega}. \end{aligned} \quad (1.1.28b)$$

In the new variables the Hamiltonian has the simple form

$$\mathcal{H} = \omega cc^* + \frac{1}{4} T c^2 c^{*2}, \qquad T = W - \frac{3V^2}{\omega} - \frac{U^2}{3\omega}.$$

It is easily seen that neglect of the cubic terms in (1.1.28a), would have given wrong values of the additional interaction coefficients supplementing W.

Following the same pattern, let us return to the general case and use a transformation in the form of a power series to eliminate cubic and nonresonant fourth-order terms. In the new variables the Hamiltonian of interaction describes $2 \to 2$ processes (for details see Sect. A.3):

$$\mathcal{H}_4 = \frac{1}{4} \int T_p c_1^* c_2^* c_3 c_4 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4, \quad p = (\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3, \boldsymbol{k}_4), \quad T_p = W_p + T_p'. \quad (1.1.29a)$$

$$\begin{aligned} T_p' = &- \frac{U_{-1-212} U_{-3-434}}{\omega_3 + \omega_4 + \omega_{3+4}} + \frac{V_{1+212}^* V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}} \\ &- \frac{V_{131-3}^* V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{V_{242-4}^* V_{313-1}}{\omega_{3-1} + \omega_1 - \omega_3} \\ &- \frac{V_{232-3}^* V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} - \frac{V_{141-4}^* V_{323-2}}{\omega_{3-2} + \omega_2 - \omega_3}. \end{aligned} \quad (1.1.29b)$$

Here $(j \pm i) = \boldsymbol{k}_j \pm \boldsymbol{k}_i$.

Note that (1.1.29b) is true on the resonant surface

$$\omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2) = \omega(\boldsymbol{k}_3) + \omega(\boldsymbol{k}_4), \qquad \boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}_3 + \boldsymbol{k}_4$$

only, where the coefficient T'_p has the same properties (1.1.25) as W_p. The necessity of taking cubic terms into account for the transformation to yield the correct value of the four-wave interaction coefficient was first pointed out by *Krasitskii* [1.2].

Let us discuss the singularities of (1.1.29). The denominators become zero on the resonance surfaces of three-wave processes:

$$\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2 = 0, \qquad \omega_k + \omega_1 + \omega_2 = 0 \tag{1.1.30a}$$

and

$$\boldsymbol{k} = \boldsymbol{k}_1 + \boldsymbol{k}_2, \qquad \omega_k = \omega_1 + \omega_2 \,. \tag{1.1.30b}$$

The conditions (1.1.30a) can only be satisfied if the medium allows for negative-energy waves. Without such waves, denominators of the type (1.1.30a) do not vanish and the corresponding terms in the three-wave Hamiltonian may be eliminated. The condition of nonzero denominators of the type (1.1.30b) coincides with the nondecay condition for the dispersion law $\omega(k)$. In the nondecay case, the cubic terms in the Hamiltonian may thus be completely excluded. The same holds for the terms in the fourth-order Hamiltonian (1.1.24b) differing in their form from (1.1.29). Prohibition of the $1 \to 2$ and $2 \to 1$ processes implies in general that the $1 \to 3$ and $3 \to 1$ processes are not feasible. One can definitely state that the interaction Hamiltonian of type (1.1.29) is a fundamental model for considering nonlinear processes in media that obey a nondecay dispersion law. Additional terms in it may be interpreted as scattering processes that arise in the second order perturbation theory for three-wave processes. In that case, a virtual forced wave appears at an intermediate stage for which the resonance condition is not satisfied. In this interpretation, every term in (1.1.29b) may be juxtaposed with a picture (see Fig. 1.2) to illustrate which particular process is meant.

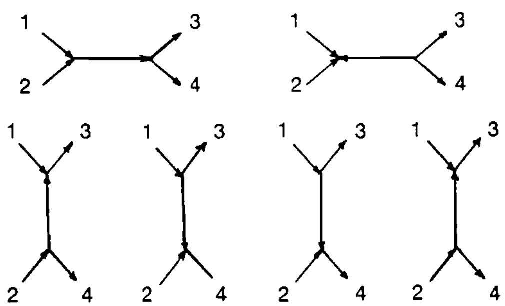

Fig. 1.2. The processes corresponding to different terms in (1.1.29b)

On the resonance manifold (1.1.27) an attempt to exclude the Hamiltonian term (1.1.29) by using a canonical transformation will lead to small denominators describing wave-wave scattering. These processes are allowed irrespective of the type of the dispersion law; hence, it is impossible to exclude this term from the Hamiltonian. The Hamiltonian (1.1.29) may be said to describe wave scattering.

Scattering processes possess an important feature: they do not change the total number of waves. Therefore the equations of motion corresponding to Hamiltonian (1.1.29) preserve one more integral besides energy, namely

$$N = \int c_k^* c_k \, d\boldsymbol{k} \,,$$

which gives the total number of quasi-particles. This is known as the wave action integral. A complete system that has to include minor effects associated with higher-order processes generally preserves the N value only approximately.

One can similarly exclude the cubic terms corresponding to nondecay three-wave processes in the Hamiltonian specifying wave interactions of several types.

1.1.4 Dimensional Analysis of the Hamiltonian Coefficients

As stated above, at weak nonlinearity the Hamiltonian of a system of interacting waves has the standard form (1.1.17, 24, 25). Is it possible to evaluate the coefficients ω_k, V_{123}, T_{1234} without considering the specific nature of every particular problem and to understand, e.g., how these values depend on the wave vectors? The answer is positive if the parameters specifying waves of this type do not yield a quantity with the dimension of a length. In this case the problem is said to possess complete self-similarity (first-order self-similarity). The Hamiltonian coefficients are estimated by dimensional considerations.

Let us first obtain the dimensions of the canonical variables b_k and the coefficients of the interaction Hamiltonian V_{123} and T_{1234}. Bearing in mind that $\mathcal{H}$ has the dimension of an energy density and ω_k the one of a frequency, the dimension of b_k is found by (1.1.18).

$$[\mathcal{H}] = \mathrm{g} \cdot \mathrm{cm}^{2-d} \cdot \mathrm{s}^{-2}, \quad [\omega] = \mathrm{s}^{-1}, \quad [b_k] = \mathrm{g}^{1/2} \cdot \mathrm{cm} \cdot \mathrm{s}^{-1/2} \,. \tag{1.1.31}$$

Here d is the dimensionality of the medium, cm = centimeter, g = gramm, s = second. In view of the fact that $[\omega] = [V_{123} b] = [T_{1234} b^2]$, it is easy to establish

$$[V_{123}] = \mathrm{g}^{-1/2} \cdot \mathrm{cm}^{d/2-1} \cdot \mathrm{s}^{-1/2}, \quad [T_{1234}] = \mathrm{g}^{-1} \cdot \mathrm{cm}^{d-2} \,. \tag{1.1.32}$$

As to be expected, the dimension of Vb^2 (here V is the volume of the system!) coincides with the one of Planck's constant $\hbar$. Naturally, our classical approach is true when the quantum-mechanical occupation numbers $N(k) = Vb^2/\hbar$ are large compared with unity. On the other hand, wave amplitudes b_k should not be too large for the interaction Hamiltonian $\mathcal{H}_{\mathrm{int}}$ to remain small compared with $\mathcal{H}_2$. This gives an upper estimate for b_k. It may be schematically written as

$$\omega_k \gg V_{kkk} \sum_{k'} b_{k'} \,. \tag{1.1.33}$$

If we introduce the dimensionless wave amplitude

$$\xi_k = b_k / B_k, \quad B_k = |\omega_k / V_{kkk}|, \tag{1.1.34}$$

the weak nonlinearity condition may be written as

$$\xi_k \ll 1 . \tag{1.1.35}$$

Now we can discuss some particular examples.

Sound in Continuous Media. As parameters the equations of motion for this problem may include only the medium's density ϱ and the elasticity coefficient κ with the respective dimensions $[\varrho] = \mathrm{g} \cdot \mathrm{cm}^{-3}$ and $[\kappa] = \mathrm{g} \cdot \mathrm{cm}^{-1} \cdot \mathrm{s}^{-2}$. These values and the wave vector k combine to yield the dimension of a frequency $[\omega] = \mathrm{s}^{-1} = [\varrho^x \kappa^y k^z] = \mathrm{g}^{x+y} \cdot \mathrm{cm}^{-3x-y+z} \cdot \mathrm{s}^{-2y}$. Equating the exponents at g, cm and s, we have three equations $x + y = 0$, $3x + y + z = 0$, and $2y = 1$. Hence $x = -1/2$, $y = 1/2$, $z = 1$. Thus the dimensional analysis leads to a linear law for the wave dispersion

$$\omega_k = c_s k, \quad c_s = a(\kappa/\varrho)^{1/2}. \tag{1.1.36}$$

Here c_s is the sound velocity and a a dimensionless parameter of the order of unity. From the parameters of our problem, one can also obtain B_k with the dimension of the canonical variable b_k

$$B_k = (\varrho c_s/k)^{1/2} \tag{1.1.37}$$

and the interaction coefficient

$$V_{123} = \sqrt{\frac{k_1 k_2 k_3 c_s}{\varrho}} f\left(\frac{\boldsymbol{k}_1}{k_1}, \frac{\boldsymbol{k}_2}{k_1}, \frac{\boldsymbol{k}_3}{k_1}\right). \tag{1.1.38}$$

Here the dimensionless function f depends on eight dimensionless arguments: the two ratios k_2/k_1 and k_3/k_1, and six angle variables giving the directions of the three vectors. In fact, there are only three angle variables: $\cos\theta_{12}$, $\cos\theta_{13}$, and $\cos\theta_{23}$ [here $\cos\theta_{ij} = (\boldsymbol{k}_i \boldsymbol{k}_j)/k_i k_j$], as our system has no preferred direction.

In the Hamiltonian description, the wave amplitude is proportional to b_k. In the sound wave field, medium density $\varrho(r,t) = \varrho_0 + \varrho_1(r,t)$ and velocity oscillate:

$$\begin{aligned} \varrho_1(r,t) &= \mathrm{Re}[\varrho_k \exp(ikr - i\omega_k t)], \\ v(r,t) &= \mathrm{Re}[v_k \exp(ikr - i\omega_k t)] , \end{aligned} \tag{1.1.39}$$

where ϱ_k, v_k are the respective wave amplitudes in natural variables. In the linear approximation the relationship between natural and normal canonical variables is easily established from dimensional considerations

$$\varrho_k \propto (k\varrho_0/c_s)^{1/2} b_k, \quad v_k \propto (kc_s/\varrho_0)^{1/2} b_k. \tag{1.1.40}$$

The symbol "$\propto$" designates proportionality. In terms of canonical variables the condition of weak nonlinearity is written as

$$\xi_k \simeq \varrho_k/\varrho_0 \simeq v_k/c_s \ll 1. \tag{1.1.41}$$

The symbol "$\simeq$" denotes an estimate to an accuracy of a dimensionless factor of order unity.

Gravitational Waves on a Fluid Surface. These are relatively long waves for which surface tension is insignificant and the force tending to restore the equilibrium state of the surface is the gravitational force. Apart from fluid density ϱ the significant parameters should evidently include the gravitational acceleration g, $[g] = \mathrm{cm} \cdot \mathrm{s}^{-2}$. Following the scheme given in the preceding example and bearing in mind that this is a 2-dimensional problem ($d = 2$), we have:

$$\omega_k = \sqrt{gk}, \quad B_k = \left(\varrho^2 g k^{-5}\right)^{1/4}. \tag{1.1.42}$$

As we see, the dispersion law is of the nondecay type, $\omega_k \propto k^\alpha$, $\alpha = 1/2 < 1$. Therefore the principal interaction is four-wave, with the interaction coefficient

$$T_{k123} = \frac{k^3}{\varrho} f[(k_1/k), (k_2/k), (k_3/k), \cos\theta_{k1}, \cos\theta_{k2}, \cos\theta_{k3}]. \tag{1.1.43}$$

A natural variable describing water waves is $\eta(r)$, the deviation of the fluid surface from the unperturbed state and the dimensionless wave amplitude is $\xi_k = k\eta_k = b_k/B_k$. Whence, we obtain

$$\eta_k = (k/\varrho^2 g)^{1/4} b_k. \tag{1.1.44}$$

Capillary Waves. For sufficiently short waves the restoring force should be entirely determined by surface tension. The significant parameters should in this case instead of g include the surface tension coefficient σ having the dimension of a surface energy $[\sigma] = g \cdot s^{-2}$. Thus,

$$\omega_k = \sqrt{\frac{\sigma k^3}{\varrho}}, \quad B_k = \left(\varrho\sigma/k^3\right)^{1/4}, \quad \eta_k = (\varrho\sigma k)^{-1/4} b_k. \tag{1.1.45}$$

The dispersion law of capillary waves is of the decay type: $\alpha = 3/2 > 1$. Therefore the three-wave interaction remains as the most essential one

$$V_{k12} = \left(\frac{\sigma k^9}{\varrho^3}\right)^{1/4} f(k_1/k, k_2/k, \cos\theta_{k1} \cos\theta_{k2}) . \tag{1.1.46}$$

Comparing the dispersion laws of capillary (1.1.45) and gravitational waves (1.1.42), it is easy to find the boundary value of the wave vector at which these frequencies coincide:

$$k_* = \sqrt{\frac{\varrho g}{\sigma}} . \tag{1.1.47}$$

At $k \ll k_*$, the gravitational energy of a wave is larger than the surface tension energy and the latter may be neglected. Thus long waves on a fluid surface will be gravitational. Accordingly, at $k \gg k_*$ the surface waves will be capillary with the dispersion law (1.1.45). We shall show in Sect. 1.2 below that at arbitrary k's the dispersion law for waves on the surface of a deep fluid is expressed as

$$\omega_k = (gk + \sigma k^3/\varrho)^{1/2}. \quad (1.1.48)$$

Though dimensional estimates usually give answers to the accuracy of a dimensionless factor of the order of unity, the dispersion laws (1.1.42,45) are accurate in the limits of large and small wavelength, respectively.

For water waves at room temperature, $k_* \simeq 4\,\text{cm}$ ($\varrho = 1\text{g/cm}^3$, $\sigma = 70\,\text{g/s}^2$). This corresponds to a wavelength of $\lambda = 2\pi/k_* \simeq 1.6$ cm and frequency $f_* = \omega_*/2\pi \simeq 0.2$ Hz.

Vortex Motions of Incompressible Fluids. From the viewpoint of dimensional analysis, this problem radically differs from the preceding problems in that it has only one significant parameter, the fluid density $[\varrho_0] = \text{g/cm}^3$. Still, this allows to determine the Hamiltonian structure. In particular, since it is impossible to build from ϱ_0 and k a combination with the dimension of a frequency, it should not contain an $\mathcal{H}_2 = \int \omega_k a_k^* a_k \, dk$ term, i.e., $\mathcal{H}_2 = 0$. Using (1.1.31–32), one can easily see that among all factors of the $\mathcal{H}$-expansion into a power series, only the coefficient at a^4 has a dimension containing no time. Therefore, only this factor may be derived from ϱ_0 and k:

$$T_{k123} \simeq \frac{k^2}{\varrho_0}. \quad (1.1.49)$$

It immediately follows that the nonlinearity of incompressible fluid motion is extremely strong, $\xi = \mathcal{H}_4/\mathcal{H}_2 \to \infty$. Another important consequence of dimensional analysis is the nonlinear relationship between fluid velocity v_k and canonical variables a_k. If we formally represent v_k as a power series of a_k:

$$v_k = \sum_{i=1}^{\infty} \phi_i(k) a_k^i \, ,$$

the dimension of a single coefficient ϕ_2 will contain no time. From this, $\phi_1 = \phi_3 = \ldots = 0$, $\phi_2 = k/\varrho_0$. Therefore

$$v_k \simeq k a_k^2/\varrho_0 \, . \quad (1.1.50)$$

It is now clear why the Hamiltonian of the problem $\mathcal{H} = (1/2) \int |v(r)|^2 \, dr$ is proportional to the fourth power of the canonical variables a_k, a_k^*.

This section has given the general structure of canonical equations of motion for weakly nonlinear waves. The remaining sections of this chapter deal with various specific systems, the introduction of canonical variables and calculation of Hamiltonian coefficients. Readers who are not interested in the character of waves in different media and the technique for deriving the canonical equations may go over directly to Chap. 2 where the kinetic wave equation is obtained from the dynamic equations given in Sect. 1.1. The paragraphs left out in the first reading, may then be referred to when evaluating the coefficients of the Hamiltonian.

1.2 The Hamiltonian Formalism in Hydrodynamics

The ideal incompressible fluid is the simplest and most important representative of a wide class of dynamic systems of the hydrodynamic type and is thus widely used in physical problems. For zero dissipation all these systems possess an implicit Hamiltonian structure. The description of such structures and the related group-theoretical formulations constitute a formidable mathematical problem extending far beyond the scope of this book (those interested in it are referred to [1.3]). For our purposes it will be sufficient to discuss the introduction of canonical variables only for several cases that are most important for the turbulence theory. Appropriate canonical variables for an imcompressible fluid were first presented by *Clebsh* (see [1.4]) in the last century. Independently *Bateman* [1.5] and later on *Davydov* [1.6] gave the canonical variables for barotropic flows in incompressible fluids with single-valued functions for the pressure. These results will be discussed in Sect. 1.2.1. Further on we shall obtain the Hamiltonians for vortex motion $\mathcal{H}_V$ (Sect. 1.2.2), for small-amplitude (potential) motion of sound $\mathcal{H}_S$ (Sect. 1.2.3) and for sound-vortex interactions $\mathcal{H}_{SV}$(Sect. 1.2.4). Fluid stratification gives rise to new types of motions localized in the regions of maximal inhomogenities. In the extremely nonuniform case of the free surface of a fluid, these are the known surface waves. The canonical variables for them were obtained by *Zakharov* [1.7]. For the general case with arbitrary wavelength and fluid depth the Hamiltonian description of this type of motion will be given in Sects. 1.2.5,6.

1.2.1 Clebsh Variables for Ideal Hydrodynamics

Consider the Euler equations for compressible fluids:

$$\partial\varrho/\partial t + \operatorname{div} \varrho \boldsymbol{v} = 0 \, , \quad (1.2.1a)$$

$$\partial \boldsymbol{v}/\partial t + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\nabla p(\varrho)/\varrho \, . \quad (1.2.1b)$$

Here $\boldsymbol{v}(r,t)$ is the Eulerian fluid velocity (in the point $\boldsymbol{r}$ at the moment of time t); $\varrho(\boldsymbol{r},t)$ the density and $p(\boldsymbol{r},t)$ is the pressure which, in the general case, is a function of fluid density and specific entropy s, i.e., $p = p(\varrho, s)$. In ideal fluids where there is neither viscosity nor heat exchange, the entropy per unit volume is carried by the fluid, i.e., it obeys $\partial s/\partial t + (\boldsymbol{v}\nabla)s = 0$. A fluid in which the specific entropy is constant throughout the volume is called *barotropic*. In such a fluid the pressure is a single-valued function of the density $p = p(\varrho)$. In this case, $\nabla p/\varrho$ may be expressed via the gradient of the specific enthalpy of the unit mass $w = E + PV$ and $dw = VdP = dP/\varrho$. Thus, $\nabla p/\varrho = \nabla w$. The enthalpy in turn equals the derivative of the internal energy of the unit volume $\varepsilon(\varrho) = E\varrho$ with respect to the fluid density

$$w = \frac{\delta\varepsilon}{\delta\varrho} . \tag{1.2.1c}$$

Direct differentiation with respect to time shows that (1.2.1) conserves the full energy of the fluid

$$\mathcal{H} = \int [\varrho v^2/2 + \varepsilon(\varrho)]\, d\boldsymbol{r} . \tag{1.2.2}$$

In line with Thomson's theorem, these equations also conserve the velocity circulation around a "fluid" path. This means that there exists a scalar function $\mu(\boldsymbol{r}, t)$ which moves together with the fluid:

$$d\mu/dt = [\partial/\partial t + (\boldsymbol{v}\nabla)]\mu = 0 . \tag{1.2.3}$$

In our search for the canonical variables for the Euler equations (1.2.1), we shall use the Lagrangian approach. For that purpose, we shall consider the known expression for the Lagrangian of a mechanical system (kinetic minus potential energy), generalized for the continuous case and use as external constraints the continuity equation (1.2.1a) and Thomson's theorem (1.2.3):

$$\mathcal{L}(t) = \int \left[\varrho\frac{v^2}{2} - \varepsilon(\varrho) + \Phi\left(\frac{\partial\varrho}{\partial t} + \operatorname{div}\varrho\boldsymbol{v}\right) - \lambda\left(\frac{\partial\mu}{\partial t} + (\boldsymbol{v}\nabla)\mu\right)\right] d\boldsymbol{r} . \tag{1.2.4}$$

Φ and λ are the undetermined Lagrange multipliers. Variation with respect to them leads to (1.2.1a) and (1.2.3). A single integration by parts allows to rewrite the Lagrangian (1.2.4) as

$$\mathcal{L} = \int \Big\{\Phi\frac{\partial\varrho}{\partial t} - \lambda\frac{\partial\mu}{\partial t} + \varrho\frac{v^2}{2} - [\boldsymbol{v}(\varrho\nabla\Phi + \lambda\nabla\mu)] - \varepsilon(\varrho)\Big\}\, d\boldsymbol{r} . \tag{1.2.5}$$

Consider the action $\mathcal{S} = \int \mathcal{L}\, dt$. Due to its extremality, the condition $\delta\mathcal{S}/\delta\boldsymbol{v} = 0$ should be satisfied, which is equivalent to the condition $\delta\mathcal{L}/\delta\boldsymbol{v} = 0$. From (1.2.5), we have

$$\boldsymbol{v} = \lambda\frac{\nabla\mu}{\varrho} + \nabla\Phi . \tag{1.2.6}$$

As the Lagrangian (1.2.5) and (1.2.6) do not contain time derivatives we can substitute (1.2.6) into (1.2.5) to arrive at the Lagrangian of the Hamiltonian system,

$$\mathcal{L} = \int (\Phi\partial\varrho/\partial t - \lambda\partial\mu/\partial t)\, d\boldsymbol{r} - \mathcal{H} . \tag{1.2.7}$$

$$\partial\varrho/\partial t = \delta\mathcal{H}/\delta\Phi , \tag{1.2.8}$$

$$\partial\Phi/\partial t = -\delta\mathcal{H}/\delta\varrho , \tag{1.2.9}$$

$$\partial\lambda/\partial t = \delta\mathcal{H}/\delta\mu , \tag{1.2.10}$$

$$\partial\mu/\partial t = -\delta\mathcal{H}/\delta\lambda , \tag{1.2.11}$$

for which the pairs (ϱ, Φ) and (λ, μ) are pairs of canonically-conjugate variables. Using (1.2.2) and (1.2.6) it is easy to compute the variational derivatives with respect to ϱ and λ,

$$\frac{\delta\mathcal{H}}{\delta\varrho} = \frac{(\nabla\Phi)^2}{2} - \frac{(\lambda\nabla\mu)^2}{2\varrho^2} + \frac{\delta\varepsilon}{\delta\varrho} = (\boldsymbol{v}\nabla)\Phi - v^2/2 + w ,$$

$$\frac{\delta\mathcal{H}}{\delta\lambda} = \varrho[\boldsymbol{v}, \partial\boldsymbol{v}/\partial\lambda] = (\boldsymbol{v}\nabla)\mu ,$$

$[\boldsymbol{a}, \boldsymbol{b}] = \boldsymbol{a} \times \boldsymbol{b}$ denotes the vector product. In calculating the derivatives with respect to Φ and μ, one has to integrate by parts.

$$\delta\mathcal{H}/\delta\Phi = -\operatorname{div}\varrho\boldsymbol{v}, \quad \delta\mathcal{H}/\delta\mu = -\operatorname{div}\lambda\boldsymbol{v} .$$

Thus (1.2.9) and (1.2.10) have the forms

$$\partial\Phi/\partial t + (\boldsymbol{v}\nabla\Phi)/2 - v^2/2 + w = 0 ,$$

$$\partial\lambda/\partial t + \operatorname{div}\lambda\boldsymbol{v} = 0 .$$

As to be expected (1.2.8) and (1.2.11) coincide with the continuity equations (1.2.1a) and (1.2.3), respectively.

Let us consider now to what extent the system of equations (1.2.8–11) is equivalent to the initial hydrodynamic system for the three components of velocity and density. The solvability of (1.2.8–11) should imply that (1.2.1) are satisfied for the velocity given by (1.2.6). This may be verified by direct calculation of the $\partial\boldsymbol{v}/\partial t$ derivative. The question of reverse correspondence reduces to the following one: can we always represent the velocity field $\boldsymbol{v}(\boldsymbol{r}, t)$ in the form of (1.2.6)? To answer this question, we calculate the vorticity. From (1.2.6), we have

$$\operatorname{rot}\boldsymbol{v} = [\nabla(\lambda/\varrho), \nabla\mu] = [\nabla\vartheta, \nabla\mu], \quad \vartheta = \lambda/\varrho . \tag{1.2.12}$$

Evidently, $\operatorname{div}[\nabla\vartheta, \nabla\mu] = 0$. Now we introduce the $q_s = (\boldsymbol{v}\ \operatorname{rot}\boldsymbol{v})$ value, the helicity density of the velocity field. From (1.2.6) and (1.2.16), we have

$$q_s = \big(\boldsymbol{v}[\nabla\vartheta, \nabla\mu]\big) = \big(\nabla\Phi[\nabla\vartheta, \nabla\mu]\big) = \operatorname{div}\Phi[\nabla\vartheta, \nabla\mu] . \tag{1.2.13}$$

The $Q_s = \int q_s \, d\boldsymbol{r}$ value is called the helicity of the velocity field and represents an integral of motion of the hydrodynamic equations. From (1.2.13) it is seen that the possibility of representing the velocity as (1.2.6) means that $Q_s = 0$. As a matter of fact, it is easy to construct examples of the velocity fields for which $Q_s \neq 0$. Let, e.g., $\boldsymbol{v}$ satisfy of the system of equations (α =const)

$$\operatorname{rot} \boldsymbol{v} = \alpha \boldsymbol{v}, \quad \operatorname{div} \boldsymbol{v} = 0 \qquad \text{(Beltramy flow)} .$$

It should be noted that the fields obeying these equations are the stationary solutions of the hydrodynamic equations with the density ϱ =const. It is easily seen that for such flows $Q_s = \alpha \int v^2 \, d\boldsymbol{r} \neq 0$. The condition allowing to represent the velocity field as (1.2.6) may be interpreted geometrically. According to the logic of our construction, $\mu(\boldsymbol{r})$ and $\vartheta(\boldsymbol{r})$ are single-valued functions of the coordinates. It follows from (1.2.12) that the vector rot $\boldsymbol{v}$ is directed along the intersection line of the level surfaces of these functions. Not every closed line may be represented as an intersection line of level surfaces of single-valued functions. This is, for example, not feasible if the line is knotted, i.e., if it represents the circle image unhomotopic to it. Let such a line be specified by the equation $r = l(y)$ where y is a parameter on the line, and the vorticity field is expressed as (the vortex line)

$$\operatorname{rot} \boldsymbol{v} = \kappa \int \boldsymbol{n}(y) \delta[\boldsymbol{r} - \boldsymbol{l}(y)] dy, \quad |\boldsymbol{n}|^2 = 1 .$$

Topology manuals (see also [1.8]) prove that in this case $Q_s = m\kappa^2$ where m is an integer defining the winding number (knotticity) of the line. A similar formula holds if the line represents a pair of linked circles with $m = 1$.

We shall refer to the variables ϱ, Φ, λ and μ as the Clebsch variables. Evidently, a global definition of the Clebsch variables is not always possible, as it requires zero knotticity of the vortex lines. At least in the vicinity of a regular point of the velocity field, the local introduction of the Clebsch variables is always possible, but attempts to expand the range of validity of their functions may lead to a loss of the single-valuedness of λ and ϑ. Nevertheless, it is possible to introduce several pairs of Clebsch variables (λ_i, μ_i), $i = 1, \dots N$:

$$\partial \mu_i / \partial t + (\boldsymbol{v} \nabla) \mu_i = 0, \qquad \partial \lambda_i / \partial t + \operatorname{div} (\lambda \boldsymbol{v}) = 0 ,$$

$$\boldsymbol{v} = \left(\frac{1}{\varrho} \sum_{i=1}^{N} \lambda_i \nabla \mu_i + \nabla \Phi \right) .$$

Probably, one can prove that $N = 2$ is sufficient to establish a one-to-one equivalence between the initial hydrodynamic system and a canonical one for arbitrary flows.

1.2.2 Vortex Motion in Incompressible Fluids

In the case of an incompressible fluid with $\partial \varrho / \partial t = 0$, we can set $\varrho = 1$ (to simplify the notation). The velocity may now be written as $\boldsymbol{v} = \lambda \nabla \mu + \nabla \Phi$. The condition div $\boldsymbol{v} = 0$ yields Φ : $\Phi = -\Delta^{-1} \operatorname{div} \lambda \nabla \mu$. The formula for the velocity may be rewritten as

$$\boldsymbol{v} = -\Delta^{-1} \operatorname{rot}[\nabla \lambda, \nabla \mu] . \tag{1.2.14}$$

Here Δ^{-1} is the inverse operator to the Laplacian. Hence, we have now only one pair of the Clebsch variables λ, μ. Fourier transformation and the transition to complex variables

$$\mu(\boldsymbol{k}) = [a(\boldsymbol{k}) + a^*(-\boldsymbol{k})]/\sqrt{2}, \quad \lambda(\boldsymbol{k}) = [a(\boldsymbol{k}) - a^*(-\boldsymbol{k})]/i\sqrt{2}$$

allow to cast the canonical equations (1.2.10–11) into the standard form (1.1.6):

$$\partial a(\boldsymbol{k}, t)/\partial t = \delta \mathcal{H}_V / \delta a^*(\boldsymbol{k}, t).$$

The Hamiltonian $\mathcal{H}_V$ is obtained by substituting the formula for the velocity into the expression for the kinetic energy (1.2.2)

$$\boldsymbol{v}(\boldsymbol{k}) = \int \varphi_{12} a_1^* a_2 \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2) \, d\boldsymbol{k}_1 d\boldsymbol{k}_2, \quad a_j = a(\boldsymbol{k}_j, t) . \tag{1.2.15a}$$

Here

$$\varphi_{12} = \varphi(\boldsymbol{k}_1, \boldsymbol{k}_2) = \frac{1}{2\varrho_0 (2\pi)^{3/2}} \left[\boldsymbol{k}_1 + \boldsymbol{k}_2 - (\boldsymbol{k}_1 - \boldsymbol{k}_2) \frac{k_1^2 - k_2^2}{|\boldsymbol{k}_1 - \boldsymbol{k}_2|^2} \right] \tag{1.2.15b}$$

which follows from (1.2.14). As a result, we have

$$\mathcal{H}_V = \frac{1}{4} \int T_{12,34} a_1^* a_2^* a_3 a_4 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4) \, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 , \tag{1.2.16a}$$

where the interaction coefficient is

$$T(\boldsymbol{k}_1 \boldsymbol{k}_2, \boldsymbol{k}_3 \boldsymbol{k}_4) = \varrho_0 [(\varphi_{13} \varphi_{24}) + (\varphi_{14} \varphi_{23})] . \tag{1.2.16b}$$

The expressions (1.2.15, 16) agree with the dimensional estimates (1.1.49–50) obtained in Sect. 1.1.5.

1.2.3 Sound in Continuous Media

As seen from (1.2.6), the case with $\lambda = 0$ or μ =const corresponds to potential fluid motion which is according to (1.2.8–9) defined by a pair of variables (ϱ, Φ). Following the standard scheme presented in Sect. 1.1, we go in the $\boldsymbol{k}$-representation from the real canonical variables $\Phi(\boldsymbol{k}), \varrho(\boldsymbol{k})$ over to the complex $b(\boldsymbol{k}), b^*(\boldsymbol{k})$:

$$\Phi(\boldsymbol{k}) = -(i/k)(\omega_k / 2\varrho_0)^{1/2} [b(\boldsymbol{k}) - b^*(-\boldsymbol{k})] ,$$

$$\delta\varrho(\boldsymbol{k}) = k(\varrho_0/2\omega_k)^{1/2}[b(\boldsymbol{k}) + b^*(-\boldsymbol{k})] , \tag{1.2.17a}$$

$$\omega(\boldsymbol{k}) = kc_s, \quad c_s^2 = (\partial p/\partial\varrho) . \tag{1.2.17b}$$

Here $\delta\varrho = \varrho - \varrho_0$ is density deviation from the steady state and c_s the sound velocity. The derivative $(\partial p/\partial\varrho)$ is calculated with the entropy s treated as a constant, which corresponds to assuming a barotropic motion of the fluid (without heat exchange). Equations (1.2.17) coincide to an accuracy of a dimensionless multiplier of the order of unity with the dimensional estimates (1.1.36) and (1.1.40). In order to obtain the sound Hamiltonian $\mathcal{H}_S$ one should expand the expression for energy (1.2.2) in terms of $\delta\varrho$ and $\boldsymbol{v} = \nabla\Phi$

$$\mathcal{H} = \mathcal{H}_S = \mathcal{H}_{2S} + \mathcal{H}_{SS} , \tag{1.2.18a}$$

$$\mathcal{H}_{2S} = \frac{1}{2}\int \left[\varrho_0|\nabla\Phi|^2 + c_S^2(\delta\varrho)^2/\varrho_0\right] d\boldsymbol{r} , \tag{1.2.18b}$$

$$\mathcal{H}_{SS} = \frac{1}{2}\int \left[\delta\varrho|\nabla\Phi|^2 + gc_S^2(\delta\varrho)^3\right] d\boldsymbol{r} , \tag{1.2.18c}$$

and substitute Φ and $\delta\varrho$ from (1.2.17) into these expressions. As a result, we see that the quadratic part of the Hamiltonian is diagonal in the variables b_k, b_k^*.

$$\mathcal{H}_{2S} = \int \omega(\boldsymbol{k})b^*(\boldsymbol{k})b(\boldsymbol{k})\, d\boldsymbol{k} . \tag{1.2.19}$$

The Hamiltonian of the sound-sound interaction $\mathcal{H}_{SS}$ has the form (1.1.24a) with the interaction coefficients:

$$V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = U(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$$
$$= \left(\frac{c_s k k_1 k_2}{4\pi^3\varrho_0}\right)^{1/2} (3g + \cos\theta_{k1} + \cos\theta_{k2} + \cos\theta_{12}) . \tag{1.2.20}$$

As expected, this expression is consistent with the result (1.1.38) obtained from a dimensional analysis by specifying the type of angular dependence f.

The supposition that the density of the internal energy $\varepsilon(\boldsymbol{r})$ depends only on $\varrho(\boldsymbol{r})$ is true only in the range of small inhomogeneities. In the general case, the internal energy E_{in} is a density functional which may be represented as a power series in $\nabla\varrho$:

$$E_{\text{in}} = \int \left[\varepsilon(\varrho) + \beta|\nabla\varrho|^2/2 + \ldots\right] d\boldsymbol{r} . \tag{1.2.21}$$

Whence, the expression for the frequency $\omega(k)$ will change from (1.2.17b) to

$$\omega^2(k) = c_s^2k^2 + \beta k^4, \quad \omega(k) = c_s k\left[1 + \frac{\beta k^2}{2c_s^2} + \ldots\right] . \tag{1.2.22}$$

It should be noted, that β may be either positive or negative [see, e.g., (1.2.39, 41) and (1.3.10)]. The expression (1.2.22) is true if the dispersion of sound is small: $\beta k^2 \ll 2c_s^2$. Otherwise, one should take into account the next $\nabla\varrho$-terms.

1.2.4 Interaction of Vortex and Potential Motions in Compressible Fluids

As shown above, in incompressible fluids, the variables λ and μ define vortex motion. A purely potential motion is described by the pair ϱ and Φ. However, in the general case it is wrong to assert that this pair ϱ and Φ describes potential motion and λ and μ vortex motion. Indeed, dividing $\boldsymbol{v}$ into two parts

$$\boldsymbol{v} = \boldsymbol{v}_1 + \boldsymbol{v}_2, \quad \operatorname{rot}\boldsymbol{v}_1 = 0, \quad \operatorname{div}\boldsymbol{v}_2 = 0 ,$$

we see that

$$\boldsymbol{v}_1 = \nabla\tilde{\Phi}, \qquad \tilde{\Phi} = \Delta^{-1}\operatorname{div}[(\lambda\nabla\mu)/\varrho] + \Phi . \tag{1.2.23}$$

Therefore the initial *Clebsh* variables are inconvenient for describing the turbulence of compressible fluids: the fields (ϱ, Φ) and (λ, μ) are strongly coupled; even for small fluctuation velocity (with the Mach number M)

$$\langle v^2\rangle/c_s^2 = M^2 \ll 1 . \tag{1.2.24}$$

Formally this manifests itself in the fact that the coefficient of interaction between these fields increases with the sound velocity $\propto \sqrt{c_s}$.

Assuming the Mach number M to be small, *L'vov* and *Mikhailov* obtain a canonical transformation separating potential and vortex motions in the new variables (q, p) and (Q, P) [1.9]. In doing so we shall try to determine the vortex velocity v_2 and the potential velocity v_1 by equations close to (1.2.12) and (1.2.23), respectively. We choose the desired canonical transformation using the generating functional F depending on the new coordinates q, Q and the old momenta Φ, μ (see (A.2.12) in Sect. A.2)

$$\lambda = \delta F/\delta\mu, \quad P = \delta F/\delta Q, \quad \varrho = \delta F/\delta\Phi, \quad p = \delta F/\delta q . \tag{1.2.25a}$$

We write the generating functional as

$$F = F_0 + F_1, \quad F_0 = \int (\tilde{\Phi}q + Q\mu)\, d\boldsymbol{r} \tag{1.2.25b}$$

where F_0 is a functional of the identity transformation chosen in such a way that the pair of canonical variables responsible for potential motion is $q = \varrho$ and $p = \tilde{\Phi}$. The functional F_1 is independent of $\tilde{\Phi}$, bilinear in μ and Q and represents a power series of the variable part of density $\delta\varrho = \varrho(r, t) - \varrho_0$:

$$\varrho_0 F_1 = \int [Q\nabla\mu, \nabla\Delta^{-1}\varrho]\, d\boldsymbol{r} . \tag{1.2.25c}$$

The expansion parameter is

$$\xi = \frac{k_V\delta\varrho}{k_S\varrho_0} \simeq \frac{\lambda_S}{L}\sqrt{\frac{E_S}{\varrho_0 c_s^2}} , \tag{1.2.25d}$$

where $k_S \simeq 1/\lambda_S$ and $k_V = 1/L$ are the characteristic wave vectors of sound waves and vortices, respectively; E_S is the energy of sound motions.

Substituting (1.2.25c) and (1.2.25d) in (1.2.25a) and solving the resulting equations by iterations with regard to the small parameter $\xi \ll 1$, we obtain

$$q = \varrho, \quad p = \tilde{\Phi} = \Phi + \Delta^{-1}\operatorname{div}[(\lambda\nabla\mu)/\varrho] , \tag{1.2.26a}$$

$$\begin{aligned} \lambda &= Q + (\nabla, \Delta^{-1}\nabla\delta\varrho)Q/\varrho_0 + O(\xi^2) , \\ \mu &= P + (\nabla P, \Delta^{-1}\nabla\delta\varrho)/\varrho_0 + O(\xi^2) . \end{aligned} \tag{1.2.26b}$$

In the new variables

$$\boldsymbol{v}_1 = \nabla p, \tag{1.2.27a}$$

$$\begin{aligned} \boldsymbol{v}_2 = {}& \Delta^{-1}[\nabla,[\nabla Q, \nabla P]]/\varrho_0 \\ & - [\nabla,[\nabla,[\Delta^{-1}\nabla\varrho,[\nabla P, \nabla Q]]]] + O(\xi^2) . \end{aligned} \tag{1.2.27b}$$

Thus we achieved the desired result: the potential motions are defined by the pair (q,p) only; the main contribution to vortex motion is made by the pair Q, P. The last term in (1.2.27b) for v_2 describes the effect of compressibility on the vortex motion.

In the $\boldsymbol{k}$-representation, we go over to the complex variables $b(\boldsymbol{k})$, $b^*(\boldsymbol{k})$, and $a(\boldsymbol{k})$, $a^*(\boldsymbol{k})$ following formulas similar to (1.2.17) and (1.1.3–4). In these variables the hydrodynamic equations have the canonical form

$$i\frac{\partial a(\boldsymbol{k},t)}{\partial t} = \frac{\delta\mathcal{H}}{\delta a^*(\boldsymbol{k},t)} , \tag{1.2.28a}$$

$$i\frac{\partial b(\boldsymbol{k},t)}{\partial t} = \frac{\delta\mathcal{H}}{\delta b^*(\boldsymbol{k},t)} , \tag{1.2.28b}$$

with the Hamiltonian

$$\mathcal{H} = \mathcal{H}_S + \mathcal{H}_V + \mathcal{H}_{SV} . \tag{1.2.29a}$$

The sound Hamiltonian has the form (1.2.18), the Hamiltonian of vortex motions is specified by (1.2.16), and the Hamiltonian of sound-vortex interaction has two terms:

$$\mathcal{H}_{SV} = \mathcal{H}_{SV1} + \mathcal{H}_{SV2} , \tag{1.2.29b}$$

$$\mathcal{H}_{SV1} = \int S_{12,34} a_1^* a_2 b_3^* b_4 \delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 , \tag{1.2.29c}$$

$$S_{12,34} = \left(k_3 k_4/32\pi^3\right)^{1/2}[(\boldsymbol{n}_3\varphi_{12}) + (\boldsymbol{n}_4\varphi_{12})] , \tag{1.2.29d}$$

$$\begin{aligned} \mathcal{H}_{SV2} = {}& \frac{1}{4}\int W_{1234k} a_1^* a_2^* a_3 a_4 \left[b(\boldsymbol{k}) + b^*(-\boldsymbol{k})\right] \\ & \times \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4 - \boldsymbol{k})\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 d\boldsymbol{k} , \end{aligned} \tag{1.2.29e}$$

$$W_{1234k} = \sqrt{\frac{\varrho_0 k}{4c_s\pi^3}}\,[(\varphi_{13}\boldsymbol{n}_k)(\varphi_{24}\boldsymbol{n}_k) + (\varphi_{23}\boldsymbol{n}_k + \varphi_{14}\boldsymbol{n}_k)] . \tag{1.2.29f}$$

Here $\boldsymbol{n}_j = \boldsymbol{k}_j/k$, and φ_{ij} are determined by (1.2.15b). The term $\mathcal{H}_{SV1}$ describes sound scattering processes and the $\mathcal{H}_{SV2}$-term describes generation and absorption of sound by turbulence. In (1.2.29a) we have not written out the terms $S^{(n)}a_1^* a_2 b^{n+2}$, $W^{(n)}a^2 a^{*2} b^{n+1}$, $n \geq 1$, which are small in the ξ^n parameter and insignificant for our future considerations.

1.2.5 Waves on Fluid Surfaces

Let us consider potential motion of incompressible fluids with a free surface in a homogeneous gravitational fluid [1.7]. In the quiescent state the fluid surface is a plane $z = 0$ with the bottom at $z = -h$. We describe the surface form by $\eta = \eta(\boldsymbol{r},t)$ where $\boldsymbol{r} = (x,y)$ is the coordinate in the transverse plane. The full energy of the fluid $\mathcal{H} = T + \Pi$ is a sum of the kinetic energy

$$T = \frac{1}{2}\int d\boldsymbol{r}\int_{-h}^{\eta} v^2\, dz \tag{1.2.30a}$$

and the potential energy

$$\Pi = \frac{1}{2}g\int \eta^2\, d\boldsymbol{r} + \sigma\int\left[\sqrt{1 + |\nabla\eta|^2} - 1\right] d\boldsymbol{r} . \tag{1.2.30b}$$

Here g is the gravitational acceleration; σ the surface tension coefficient and the free surface element is expressed by $ds = dr\sqrt{1 + |\nabla\eta|^2}$.

In constructing the canonical variables we shall proceed from the Hamiltonian (1.2.7) where we set

$$\varrho = \Theta\big(\eta(\boldsymbol{r}) - z\big) . \tag{1.2.31}$$

Here $\Theta(\xi) = 1$ at $\xi > 0$, $\Theta(\xi) = 0$ at $\xi < 0$. Further on we shall consider only irrotational fluid flows implying that may use $\lambda = 0$. Substituting (1.2.31) into (1.2.7) and taking advantage of the fact that $\partial\eta(\boldsymbol{r},z,t)/\partial t = \delta[\eta(\boldsymbol{r},t) - z]\partial\eta(\boldsymbol{r},t)/\partial t$, the Lagrangian reads

$$\mathcal{L} = -\int \Psi(\boldsymbol{r},t)\frac{\partial\eta(\boldsymbol{r},t)}{\partial t}\, d\boldsymbol{r} - \mathcal{H} . \tag{1.2.32}$$

Here $\Psi(\boldsymbol{r},t) = \Phi(\boldsymbol{r},z,t)$ at $z = \eta(\boldsymbol{r},t)$.

The Lagrangian (1.2.32) yields the canonical equations

$$\frac{\partial\eta(\boldsymbol{r},t)}{\partial t} = -\frac{\delta\mathcal{H}}{\delta\Psi(\boldsymbol{r},t)} , \tag{1.2.33a}$$

$$\frac{\partial\Psi(\boldsymbol{r},t)}{\partial t} = \frac{\delta\mathcal{H}}{\delta\eta(\boldsymbol{r},t)} . \tag{1.2.33b}$$

Thus the canonical pair of variables is now given by $\eta(\boldsymbol{r},t)$, $\Psi(\boldsymbol{r},t)$. Having specified them, we have now to solve the boundary value problem for the Laplace equation

$$\Delta\Phi(\boldsymbol{r},z,t)=0, \quad \Phi[\boldsymbol{r},\eta(\boldsymbol{r},t),t]=\Psi(\boldsymbol{r},t), \Phi(\boldsymbol{r},-h,t)=0 \tag{1.2.34}$$

in order to determine the fluid's velocity field. Now $\boldsymbol{v}=\nabla\Phi$ and we get

$$T=\frac{1}{2}\int d\boldsymbol{r}\int_{-h}^{\eta} dz|\nabla\Phi|^2$$

for the kinetic energy. The variation $\delta\mathcal{H}/\delta\Psi=\delta T/\delta\Psi$ may be carried out explicitly, but the result is known in advance. To obtain it, we substitute $\varrho=\Theta(\eta-z)$ into the continuity equation (1.2.1) to get

$$\frac{\partial\eta(\boldsymbol{r},t)}{\partial t}+v_\perp\nabla\eta=v \tag{1.2.35}$$

as the kinematic condition on the fluid surface which should coincide with (1.2.33a). The physical meaning of this condition is rather simple: the velocity of fluid height-variations should be the same as the velocity of the fluid itself in the given point at the surface.

The explicit solution of the boundary value problem (1.2.34) is not possible but it may be solved in the small nonlinearity limit by expanding the Hamiltonian in a power series with regard to its canonical variables. In coordinate representation, every term in this series is a nonlocal functional of η and Ψ. This is due to the above-mentioned necessity to solve the Laplace equation at every iteration step. Going over to a Fourier representation we obtain

$$\begin{aligned}
\mathcal{H}_2 &= \frac{1}{2}\int\left[(\varrho g+\sigma k^2)|\eta(\boldsymbol{k},t|^2+\varrho k\tanh(kh)|\Psi(\boldsymbol{k},t)|^2\right]d\boldsymbol{k}\ ,\\
\mathcal{H}_3 &= \frac{1}{4\pi}\int L_{123}\Psi(\boldsymbol{k}_1)\Psi(\boldsymbol{k}_2)\eta(\boldsymbol{k}_3)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\,d\boldsymbol{k}_1d\boldsymbol{k}_2d\boldsymbol{k}_3\ ,\\
L_{123} &= \frac{1}{2}(k_1^2+k_2^2-k_3^2)-k_1k_2\tanh(k_1h)\tanh(k_2h)\ ,\\
\mathcal{H}_4 &= \frac{1}{(2\pi)^2}\int M_{1234}\Psi(\boldsymbol{k}_1)\Psi(\boldsymbol{k}_2)\eta(\boldsymbol{k}_3)\eta(\boldsymbol{k}_4)\\
&\quad\times\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4)\,d\boldsymbol{k}_1d\boldsymbol{k}_2d\boldsymbol{k}_3d\boldsymbol{k}_4, \qquad \text{etc}\ .
\end{aligned} \tag{1.2.36}$$

In the case of gravitational waves on deep water the expression for M_{1234} is required, see (1.2.43). In the limit $\sqrt{\varrho g/\sigma}=k_*\gg k\gg h^{-1}$ we obtain

$$\begin{aligned}
M_{1234} &= \frac{1}{4}\sqrt{k_1k_2k_3k_4}[2(k_1+k_2)\\
&\quad-|\boldsymbol{k}_1+\boldsymbol{k}_3|-|\boldsymbol{k}_1+\boldsymbol{k}_4|-|\boldsymbol{k}_2+\boldsymbol{k}_3|-|\boldsymbol{k}_2+\boldsymbol{k}_4|]\ .
\end{aligned}$$

The transition to normal complex variables is given by

$$\begin{aligned}
\eta(\boldsymbol{k}) &= \sqrt{\frac{\lambda_k}{2}}[a(\boldsymbol{k})+a^*(-\boldsymbol{k})]\ ,\\
\Psi(\boldsymbol{k}) &= -i\sqrt{\frac{1}{2\lambda_k}}[a(\boldsymbol{k})-a^*(-\boldsymbol{k})],\\
\lambda_k &= \frac{\omega(k)}{g+\sigma k^2/\varrho}\ ,
\end{aligned} \tag{1.2.37}$$

where $\omega(k)$ is the dispersion law of waves on a fluid surface with depth h:

$$\omega^2(k)=k[g+\sigma k^2/\varrho]\tanh(kh)\ . \tag{1.2.38}$$

Written in these variables the canonical equations of motions have the normal form (1.1.14) and the quadratic part of the Hamiltonian is diagonal with respect to $a(\boldsymbol{k})$ and $a^*(\boldsymbol{k})$. We give the coefficient of the interaction only for the limiting cases in which the problem becomes scale-invariant. As seen from (1.2.38), there are two characteristic scales in k-space: $1/h$ and $k_*=\sqrt{\varrho_0 g/\sigma}$ [see also (1.1.47)]. If the scales strongly differ, the k-space contains the regions of scale-invariant behavior. For example, at $1\gg k_*h$ we have

1. $k\ll k_*$ are the *shallow-water gravitational-capillary waves*. Their dispersion law is close to that of sound

$$\omega(k)=\sqrt{gh}\,k[1+k^2/2k_*^2]\ . \tag{1.2.39}$$

The velocity of such a wave is determined by gravity only, while the dispersion is determined by surface tension as well. The positive addition to the linear dispersion law makes three-wave processes possible, the interaction coefficients coincide with (1.2.20) where $c_s=\sqrt{gh}$ should be assumed. It should be noted that, despite the absence of complete self-similarity (there is a parameter with the dimension of a length h), the Hamiltonian coefficients are scale invariant. Such cases are usually referred to as incomplete, or second order, self-similarity.

2. $k_*\ll k\ll h^{-1}$ are the *shallow water capillary waves* [1.10]. In this case

$$\omega(k)=\sqrt{\frac{\sigma h}{\varrho}}\,k^2\ . \tag{1.2.40a}$$

The dispersion law is of the decay type. It is sufficient to consider only three-wave processes. Written in normal variables the Hamiltonian $\mathcal{H}_{\text{int}}$ has the standard form (1.1.23) with rather simple interaction coefficients

$$V_{k12}=U_{k12}=(k^2/8\pi)(\sigma/4\varrho h)^{1/4}\ . \tag{1.2.40b}$$

3. $h^{-1}\ll k$ are the *deep-water capillary waves* [1.7,11]. In this limit

$$\omega(k)=\sqrt{\frac{\sigma}{\varrho}}\,k^{3/2}\ , \tag{1.2.41a}$$

$$V(\mathbf{k},\mathbf{12}) = \frac{1}{8\pi}\left(\frac{\sigma}{4\varrho^3}\right)^{1/4}\left[(\mathbf{k}_1\mathbf{k}_2 + k_1k_2)\left(\frac{k_1k_2}{k}\right)^{1/4} + (\mathbf{k}_1\mathbf{k} - k_1k)\left(\frac{k_1k}{k_2}\right)^{1/4} + (\mathbf{k}\mathbf{k}_2 - kk_2)\left(\frac{kk_2}{k_1}\right)^{1/4}\right] . \tag{1.2.41b}$$

The reader should compare these expressions with (1.1.45) and (1.1.46)

However, for $k_*h \simeq 1$ there are only two scale-invariant regions, namely at very short and very long waves, respectively. At $k \to \infty$ we have the case (1.2.41) considered above. At $k \to 0$, we have a dispersion law close to the acoustic one:

$$\omega(k) = \sqrt{ghk}\left(1 + [(1/2k_*) - (h^3/3)]k^2\right) . \tag{1.2.42}$$

In the case of $k_*h > 3/2$, the law (1.2.42) is the nondecay type, and the principal role is played by the four-wave interaction with an interaction coefficient of the form (1.1.30) where $V(k,12)$ is given by (1.2.20). For $k_*h \gg 1$ the dispersion law (1.2.42) is determined by gravity. These waves are called *shallow water gravitational waves*.

At $k_*h \gg 1$ in the intermediate region $k_* \gg k \gg h^{-1}$ we have *deep-water gravitational waves* with the nondecay dispersion law $\omega(k) = \sqrt{gk}$ (1.1.42) and the interaction coefficient [1.7]

$$U_{k,12} = V_{-k12} = \frac{1}{8\pi}\left(\frac{g}{4\varrho^2}\right)^{1/4}\left[(\mathbf{k}_1\mathbf{k}_2 + k_1k_2)\left(\frac{k}{k_1k_2}\right)^{1/4} + (\mathbf{k}\mathbf{k}_1 + kk_1)\left(\frac{k_2}{kk_1}\right)^{1/4} + (\mathbf{k}\mathbf{k}_2 + kk_2)\left(\frac{k_1}{kk_2}\right)^{1/4}\right] , \tag{1.2.43a}$$

$$W(\mathbf{k1},\mathbf{23}) = \frac{(kk_1k_2k_3)^{1/2}}{64\varrho\pi^2}[R(\mathbf{k123}) + R(\mathbf{k123}) - R(\mathbf{k213}) - R(\mathbf{k312}) - R(\mathbf{12k3}) - R(\mathbf{13k2})] , \tag{1.2.43b}$$

$$R(\mathbf{k123}) = \left(\frac{kk_1}{k_2k_3}\right)^{1/4}[2(k + k_1) - |\mathbf{k} - \mathbf{k}_2| - |\mathbf{k} - \mathbf{k}_3| - |\mathbf{k}_1 - \mathbf{k}_2| - |\mathbf{k}_1 - \mathbf{k}_3|] . \tag{1.2.43c}$$

The resulting coefficient of the four-wave interaction has after elimination of three-wave processes the form (1.1.29b) and possesses the same homogeneity properties as $W(\mathbf{k1},\mathbf{23})$ [see also (1.1.43)].

Thus surface waves obey in different limiting cases either decay or nondecay power dispersion laws [$\omega(k) \propto k^\alpha$, $\alpha = 1/2, 1, 3/2, 2$] and have scale-invariant interaction coefficients.

1.3 Hydrodynamic-Type Systems

1.3.1 Langmuir and Ion-Sound Waves in Plasma

In some situations, a plasma may be regarded as a set of two fluids: electronic and ionic ones, each defined by a system of hydrodynamic equations. If there are no external magnetic fields, this is possible if the wavelengths induced in the plasma are large compared with the Debye length. The simplest model of such a plasma does not take into account the generation of the magnetic field by currents and the electric field in such a plasma is potential.

The equations of motion of the electronic fluid have in this model the form of (1.2.1)

$$\frac{\partial \mathbf{v}_e(\mathbf{r},t)}{\partial t} + (\mathbf{v}_e\nabla)\mathbf{v}_e + \nabla\left(\frac{\delta\varepsilon}{\delta\varrho_e}\right) = 0, \qquad \frac{\partial \varrho_e(\mathbf{r},t)}{\partial t} + \operatorname{div}(\varrho\mathbf{v}_e) = 0 \tag{1.3.1}$$

with the internal energy being the sum of electrostatic and gas kinetic terms

$$\varepsilon = \frac{e^2}{2m^2}\int \frac{\delta\varrho_e(\mathbf{r},t)\delta\varrho_e(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}\,d\mathbf{r}\,d\mathbf{r}' + \frac{3T_e}{2m\varrho_0}\int[\delta\varrho_e(\mathbf{r},t)]^2\,d\mathbf{r} . \tag{1.3.2}$$

Here $\delta\varrho_e = \varrho_e - \varrho_0$ is the electronic density variation; e, m and T_e are the charge, mass and temperature of the electrons. In the second term defining the kinetic pressure of the gas the coefficient 3 emphasizes the phenomenological character of the model: it has been chosen to obtain the correct dispersion law of Langmuir waves

$$\omega^2(k) = \omega_p^2(1 + 3k^2r_D^2) \tag{1.3.3}$$

arising from the precise kinetic description (see e.g. [1.12]). Here ω_p and r_D are, respectively, plasma frequency and Debye length:

$$\omega_p^2 = \frac{4\pi\varrho_0}{m^2}, \qquad r_D^2 = \frac{T_e m}{4\pi\varrho_0 e^2} . \tag{1.3.4}$$

The Langmuir waves depict a type of plasma motions possessing a potential. In such cases one can introduce in a conventional way the velocity potential $\nabla\Phi = \mathbf{v}$. The normal canonical variables are introduced in the same way as for the potential motions of an ordinary fluid, using the formulas (1.2.17a) where $\omega(k)$ should be given by (1.3.3). The coefficients of three-wave interactions are calculated similarly to (1.2.20) and have the form [1.13]

$$U_{k12} = V_{k12} = \frac{1}{8\sqrt{2\pi^3}\varrho_0}\left[\left(\frac{\omega_1\omega_2}{2\omega_k}\right)^{1/2}k\cos\theta_{12} + \left(\frac{\omega_k\omega_1}{2\omega_2}\right)^{1/2}k_2\cos\theta_1 + \left(\frac{\omega_k\omega_2}{\omega_1}\right)^{1/2}k_1\cos\theta_2\right] .$$

However, the dispersion law (1.3.3) is valid only in the long wave range $kr_D \ll 1$ and is of the nondecay type. Using the transformation (1.1.28), one can obtain the effective Hamiltonian (1.1.29). For $kr_D \ll 1$ the interaction coefficients U and V become scale-invariant with the scaling index unity and the effective four-wave interaction coefficient (1.1.29b) has the scaling index two since $\omega(k) \approx \omega_p$:

$$T_{k123} = \frac{1}{\omega_p}\left[V_{k+1,k1}V_{2+3,23} - V_{-k-1,k1}V_{-2-3,23} - V_{k,2k-2}V_{3,13-1} - V_{k,3k-3}V_{2,12-1} - V_{2,k2-k}V_{131-3} - V_{3,k3-k}V_{1,22-1}\right] . \tag{1.3.5}$$

That expression satisfies symmetry properties and is a homogeneous function of the second degree.

In describing the electronic oscillations we have assumed the ions to be at rest. The slow motion of the ionic fluid will be considered in nonisothermal plasma where the electron temperature is a lot larger than the one of the ion $T_e \gg T_i$. We shall consider the phase velocities of the waves to be much higher than the thermal velocities of the ions but much smaller than the thermal velocities of the electrons. Then, at each moment of time the electrons may be taken to have a Boltzmann distribution $n_e = n_0 \exp(e\phi/T_e)$. The electric field potential $\varphi(\boldsymbol{r},t)$ satisfies the Poisson equation

$$\Delta\varphi = -4\pi e[n - n_0\exp(e\phi/T_e)] , \tag{1.3.6}$$

where $n(\boldsymbol{r},t)$ is the ion concentration. Neglecting the ion thermal pressure, we obtain a system of ion hydrodynamic equations:

$$\begin{aligned} &\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{e\nabla\varphi}{M} , \\ &\frac{\partial \varrho(\boldsymbol{r},t)}{\partial t} + \operatorname{div}(\varrho\boldsymbol{v}) = 0 . \end{aligned} \tag{1.3.7}$$

Here v and M are ion velocity and mass and $\varrho = Mn$. As (1.2.1) and (1.3.1) the system of equations (1.3.6, 7) may be written in the form of Hamilton equations with the Hamiltonian

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\int \varrho v^2\, d\boldsymbol{r} + E_{\text{in}} , \\ E_{\text{in}} &= \frac{1}{8\pi}\int |\nabla\varphi|^2\, d\boldsymbol{r} + T_e n_0 \int \left[\left(\frac{e\varphi}{T_e} - 1\right)\exp\left(\frac{e\varphi}{T_e}\right) + 1\right] d\boldsymbol{r} , \end{aligned} \tag{1.3.8}$$

i.e., to a sum over the ion kinetic, electrostatic field and thermal energies of the electron gas. The canonically conjugate variables are ϱ and velocity potential Φ: $\boldsymbol{v} = \nabla\Phi$; thus $\partial\varrho/\partial t = -\delta\mathcal{H}/\delta\Phi$ yields the first of the equations (1.3.7). To obtain the right-hand side of the first equation (1.3.7) one should calculate the internal-energy variational derivative which, by virtue of the Poisson equation (1.3.6), equals

$$\begin{aligned} \frac{\delta E_{\text{in}}}{\delta\varrho(\boldsymbol{r})} &= \int \varphi(\boldsymbol{r}') \left[-\frac{1}{4\pi}\Delta[\delta\varphi(\boldsymbol{r}')/\delta\varrho(\boldsymbol{r})] + \frac{\delta\varphi(\boldsymbol{r}')}{\delta\varrho(\boldsymbol{r})} e^2 n_0 \exp\left(\frac{e\varphi}{T_e}\right)\right] d\boldsymbol{r}' \\ &= \frac{e}{M}\delta(\boldsymbol{r} - \boldsymbol{r}') . \end{aligned}$$

Thus the second Hamilton equation $\partial\Phi/\partial t = \delta\mathcal{H}/\delta\varrho$ coincides with the second of relations (1.3.7). Assuming the unperturbed plasma to be quasi-neutral, $n(r,t) = n_0$ holds for small perturbations $n, \boldsymbol{v} \propto \exp[i(\boldsymbol{kr} - i\Omega(k)t]$ and the dispersion law reads

$$\Omega^2(k) = \frac{k^2 T_e}{M(1 + k^2 r^2)} . \tag{1.3.9}$$

In the long-wave range $kr_D \ll 1$, the dispersion law (1.3.9) is almost linear

$$\Omega(k) \approx kc_s(1 - k^2 r_D^2/2) . \tag{1.3.10a}$$

Such oscillations are called *ion sound*, they are only at $T_e \gg T_i$ well defined (i.e., they are weakly damped). The sound velocity $c_s^2 = T_e/M$ is determined by electron temperature and ion mass (inertia). A correction to the linear term in $\Omega(k)$ is negative, therefore the dispersion law (1.3.10) [as in (1.3.3)] is of the nondecay type, the resonance interaction of ion-sound waves with each other is specified by the four-wave Hamiltonian (1.1.29) where U and V are computed in a similar way as for ordinary sound and are given by (1.2.20) with $g = -1/3$:

$$T_{1234} = \frac{V_{1+212}V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}} - \frac{2V_{131-3}V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{2V_{232-3}V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} . \tag{1.3.10b}$$

Here we neglected the small terms which do not contain small denomitators $\propto r_D^2$. Thus three-wave processes are forbidden for systems containing either Langmuir or ion-sound waves. However, there should be an interaction between Langmuir and ion sound waves. The physical reason for it is in the joint action of two mechanisms: the slow ion-sound density variations alter the plasmon frequency, and the high-frequency field creates a mean ponderomotive force (proportional to the gradient of the square of the field) which affects ions. Such phenomena can be described in the framework of the so called Zakharov equations

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} - c_s^2\Delta\right)\delta n &= \frac{1}{16\pi M}\Delta|E|^2 \\ \Delta\left(2i\frac{\partial\varphi}{\partial t} + 3r_D^2\Delta\varphi\omega_p\right) &= \frac{\omega_p}{n_0}\operatorname{div}(\delta n\nabla\varphi) . \end{aligned} \tag{1.3.11}$$

Here

$$E = \nabla\varphi, \quad c_s = \sqrt{(T_e + 5T_i/2)/M} \ .$$

These equations correspond to the two fluid plasma model [1.13]. The canonical variables are introduced similarly to the above cases. So we obtain plasmons with the dispersion law (1.3.3), ion sound with that of (1.3.10a) and the interaction Hamiltonian

$$\mathcal{H}_{\rm int} = \int (V_{123} b_1 a_2 a_3^* + \text{c.c.})\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3) \ , \qquad (1.3.12)$$

where b and a are amplitudes of ion-sound and Langmuir waves, respectively, and the interaction coeffecient is equal to [1.13]

$$V_{k12} = \frac{\omega_p \sqrt{k}}{4\pi\sqrt{4\pi M n_0 c_s}} \frac{(\boldsymbol{k}_1 \boldsymbol{k}_2)}{k_1 k_2} \ . \qquad (1.3.13)$$

This Hamiltonian describes plasmon decay with sound wave emission

$$\omega(\boldsymbol{k}) = \omega(\boldsymbol{k} - \boldsymbol{k}_1) + \Omega(\boldsymbol{k}_1) \ .$$

a process which is sometimes called Cherenkov emission, as it is analogous to wave emission by a particle moving in a medium with a velocity exceeding the phase velocity of waves. Similarly, the process (1.3.12) is allowed if the group velocity of the plasmons is larger than the sound velocity.

The ion interaction also contributes to the four-plasmon interaction. For example, in an isothermal plasma, the interaction coefficient of Langmuir waves with virtual ion-sound waves is [1.14]

$$T_{k123} = \frac{\omega_p(\cos\theta_{k2}\cos\theta_{13} + \cos\theta_{k3}\cos\theta_{12})}{8\pi^3 n_0 T} \ . \qquad (1.3.14)$$

Its scaling index is zero. Curiously enough T_{k123} vanishes for one-dimensional motion if $\boldsymbol{k} + \boldsymbol{k}_1 = \boldsymbol{k}_2 + \boldsymbol{k}_3$.

In a constant external magnetic field H, the Hamiltonian coefficients depend on the angles in $\boldsymbol{k}$-space. In particular, the dispersion laws of both Langmuir and ion-sound waves in strong enough fields are of the decay type. Scale-invariance of $\omega(\boldsymbol{k})$ and $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$ is observed separately for the two components of the wave vector: one parallel to the field k_z and the other perpendicular $k_\perp$.

Let us consider ion sound in a magnetized plasma [1.15]. The presence of magnetic field will give rise to a Lorentz force in the Euler equation, and instead of (1.3.7), we have

$$\frac{\partial \boldsymbol{v}(\boldsymbol{r},t)}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{e\nabla\varphi}{M} + \frac{e[\boldsymbol{v}, \boldsymbol{H}]}{Mc} \ ,$$
$$\frac{\partial \varrho(\boldsymbol{r},t)}{\partial t} + \text{div}\,(\varrho\boldsymbol{v}) = 0 \ . \qquad (1.3.15)$$

Assuming the motion to be quasi-neutral (the criterion for this will be obtained below), we shall consider the ions, like the electrons, to obey the Boltzmann distribution law

$$\varrho = \varrho_0 \exp\left(\frac{e\varphi}{T}\right) \ . \qquad (1.3.16)$$

Equations (1.3.15, 16) form a closed system. Having obtained φ from (1.3.16), we can rewrite the term $e\nabla\varphi/M$ in (1.3.15) in the standard form $e\nabla\varphi/M = \nabla w$, where w is the enthalpy

$$w = (T/M)\ln(\varrho/\varrho_0) = c_s^2 \ln(\varrho/\varrho_0) \ .$$

The Hamiltonian has the form known from hydrodynamic-type systems (1.2.2) where the internal energy $\varepsilon(\varrho)$ is related to the enthalpy: $w = \delta\varepsilon/\delta\varrho$. As usual, the canonical variables (λ, μ) and (ϱ, Φ) are introduced and similarly as in (1.2.5a) and (1.2.23) the velocity $\boldsymbol{v}$ reads then

$$\boldsymbol{v} = (\lambda\nabla\mu - \mu\nabla\lambda)/2\varrho + \nabla\Phi - e\boldsymbol{A}/Mc$$

(considering that in the magnitic field the generalized momentum is renormalized to vector potential $\boldsymbol{p} \to \boldsymbol{p} - e\boldsymbol{A}/c$).

The vector potential of the constant field is $2\boldsymbol{A} = [\boldsymbol{H}\boldsymbol{r}]$, $\boldsymbol{r} = (x, y)$. In the new variables the equations have the form of the Hamilton equations (1.2.6, 7) containing the coordinate r in an explicit form. In order to eliminate $\boldsymbol{r}$ from $\boldsymbol{v}$, we perform the canonical transformation

$$\lambda \to \lambda + \sqrt{\varrho\Omega_H}\, x, \quad \mu \to \mu - \sqrt{\varrho\Omega_H}\, y \ ,$$

$$\Phi \to \Phi - \sqrt{\varrho\Omega_H}(\lambda y + \mu x) \ .$$

Here $\Omega_H = eH/Mc$ is the Larmor frequency of ion rotation in the magnetic field H. Going then over to normal variables and expanding the Hamiltonian, we obtain $\omega(\boldsymbol{k})$ and $V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2)$.

In this case it will be more convenient to derive first a truncated equation describing the waves in the range involved and to go then over to normal variables. Supposing the magnetic energy to be much greater than the thermal one ($8\pi nT \ll H^2$) and considering low-frequency waves $(\omega(\boldsymbol{k}) \ll \Omega(\boldsymbol{k}))$ with weak dispersion ($k_\perp c_s \ll \Omega_H$), we get by virtue of (1.3.15, 16) the dispersion law for magnetized ion sound

$$\omega(k) = c_s k_z (1 - k_\perp^2 c_s^2 / 2\Omega_H^2) \ . \qquad (1.3.17)$$

Since the deviations from quasi-neutrality lead, as seen from (1.3.10), to the correction $k^2 r_D^2/2$, (1.3.7) holds for not too small $k_\perp$'s, when $k_\perp/k > r_D^2\Omega_H/c_s$. The dispersion law (1.3.17) is of the decay type. It should be noted that the group velocity of such waves is directed along the magnetic field. If we restrict our consideration to unidirectional waves and the quadratic nonlinearity, then for the value $u = \partial\Phi/\partial z$ from (1.3.15, 16), we obtain

$$\frac{\partial u(\boldsymbol{r},t)}{\partial t} - c_s \frac{\partial}{\partial z}\left[u(\boldsymbol{r},t) + \frac{c_s^2}{2\Omega_H^2}\Delta_\perp u(\boldsymbol{r},t) - \frac{u^2(\boldsymbol{r},t)}{2c_s}\right] = 0 \ , \qquad (1.3.18)$$

whose linear part corresponds to (1.3.17). Going over to a reference system moving along the z-axis with speed c_s yields

$$\frac{\partial u(\boldsymbol{r},t)}{\partial t} = c_s \frac{\partial}{\partial z}\left[\frac{c_s^2}{2\omega_H^2}\Delta_\perp u(\boldsymbol{r},t) - \frac{u^2(\boldsymbol{r},t)}{2c_s^2}\right] . \tag{1.3.19}$$

After transition to normal variables according to

$$u_k = c_s\sqrt{k_z/2}[a(k_z,k_\perp) + a^*(-k_z,-k_\perp)] ,$$

(1.3.19) corresponds to the standard Hamiltonian $\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3$ [see (1.1.17, 23)] with the coefficients

$$\begin{aligned} \omega(k_z,k_\perp) &= \frac{c_s^3 k_z k_\perp^2}{2\Omega_H^2} , \\ V(k,k_1,k_2) &= \frac{1}{4\pi}\sqrt{\frac{c_s k_z k_{1z} k_{2z}}{2\varrho_0}} . \end{aligned} \tag{1.3.20}$$

In conclusion, we give, without calculations the formulas defining the dispersion law and the matrix elements of the decay interaction for Langmuir waves in magnetized plasma [1.13]. The system of equations differs from (1.3.1) in the Lorentz force substituted for the gas kinetic term on the right-hand-side of the Euler equation. The resulting dispersion law has two branches

$$\omega_\pm^2(\boldsymbol{k}) = \frac{1}{2}\left(\omega_p^2 + \omega_H^2\right) \pm \frac{1}{2}\sqrt{\omega_p^4 + \omega_H^4 - 2\omega_p^2\omega_H^2\cos 2\theta_k} , \tag{1.3.21}$$

$\omega_H = eH/mc$ is the electronic Larmor frequency and θ_k the angle between the wave vector $\boldsymbol{k}$ and the constant magnetic field. We shall be concerned with the lower branch in the two limiting cases when the problem becomes scale-invariant:

1. $\omega_H \gg \omega_p$ is *the strong field case*. Considering waves propagating almost perpendicular to the field, one can get

$$\omega_k = \omega_p |k_z/k_\perp| . \tag{1.3.22a}$$

For the angular range $\cos\theta_k \ll \omega_p/\omega_H$ we obtain

$$\begin{aligned} V_{k12} &= i\frac{\omega_p}{\omega_H}\frac{(\boldsymbol{k}_{1\perp}[\boldsymbol{h},\boldsymbol{k}_{2\perp}])}{\sqrt{32\varrho_0}}\sqrt{\frac{\omega_1\omega_2}{\omega_k}}\operatorname{sign} k_z \\ &\times \left[\frac{\operatorname{sign} k_z}{k_\perp}\left(\frac{k_{1\perp}}{k_{2\perp}} - \frac{k_{2\perp}}{k_{1\perp}}\right) + \frac{\operatorname{sign} k_{1z}}{k_{2\perp}} + \frac{\operatorname{sign} k_{2z}}{k_{1\perp}}\right] . \end{aligned} \tag{1.3.22b}$$

Here $\boldsymbol{h} = \boldsymbol{H}/H$.

At larger angles $\omega_p/\omega_H \ll \cos\theta_k \ll 1$ we have

$$\begin{aligned} V_{k12} &= \operatorname{sign}(k_z k_{1z} k_{2z})\sqrt{\frac{\omega_k\omega_1\omega_2}{32\varrho_0\omega_p^2}} \\ &\times (k_\perp \operatorname{sign} k_z + k_{1\perp}\operatorname{sign} k_{1z} k_{2\perp} \operatorname{sign} k_{2z}) . \end{aligned} \tag{1.3.22c}$$

2. $\omega_H \ll \omega_p$ is *the weak field case* ($\cos\theta_k \ll 1$):

$$\begin{aligned} \omega_k &= \omega_H \frac{k_z}{k_\perp} , \\ V_{k12} &= \sqrt{\frac{\omega_1\omega_2}{32\varrho_0\omega_k}}\frac{(\boldsymbol{k}_{1\perp}[\boldsymbol{h},\boldsymbol{k}_{2\perp}])}{k_{1\perp}k_{2\perp}} \\ &\times \left(k_\perp + k_{2\perp}\operatorname{sign}(k_z k_{2z}) + k_{1\perp}\operatorname{sign}(k_z k_{1z})\right) . \end{aligned} \tag{1.3.23}$$

The scaling indices of $\omega(\boldsymbol{k})$ and $V(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2)$ are the same as for (1.3.22a,b).

1.3.2 Atmospheric Rossby Waves and Drift Waves in Inhomogeneous Magnetized Plasmas

Drift Rossby waves propagate in atmospheres of planets and in oceans. Their frequencies are small as compared to the frequency of global rotation of planets Ω_0, and the lengths are rather large compared to the extensions of the medium L (depth of the ocean or height/thickness of the atmosphere). At large amplitudes these waves become planetary vortices. The largest among them is the Big Red Spot of Jupiter. The planetary waves (vortices) are named after the Swedish geophysicist Rossby who revealed, in the 1930–40, their important role in the processes of global circulation of the atmosphere [1.16], although theoretically they had been known since the end of the last century [1.17]. Those waves are successfully simulated in laboratories [1.18–19], observed in the atmosphere of the Earth and in oceans [1.20]. Rossby waves are analogues to the drift waves in inhomogeneous magnetized plasmas [1.21–22]. They may have some relation to the generation of magnetic fields in nature [1.23–24].

We distinguish barotropic and baroclinic Rossby waves. The former allow to treat phenomena observed in nature (atmosphere or ocean) as occuring in a quasi-two-dimensional medium where the Rossby waves have a wave length λ which is much larger than the vertical extension L. In describing the baroclinic waves, one should take into account the vertical inhomogeneity of the density of an ocean (which is due to the vertical temperature profile and salt concentration) or atmosphere. This inhomogeneity gives rise to vertical oscillations of the fluid which is stable against convection. In barotropic Rossby waves there are no oscillations and the medium may be regarded as two-dimensional. We shall deal with this simple case in more detail.

We suppose the planet to rotate with angular velocity Ω_0. Proceeding from the Euler equation complemented by the Coriolis force one can obtain the equation which describes the atmospheric Rossby waves:

$$\begin{aligned} \frac{\partial}{\partial t}(\Delta\psi - k_0^2\psi) + \beta\frac{\partial\psi}{\partial x} &= \frac{\partial\Omega}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\Omega}{\partial y}\frac{\partial\psi}{\partial x} \\ \Omega &= \Delta\psi - k_0^2\psi . \end{aligned} \tag{1.3.24}$$

This equation and its derivation are described in details in various monographs (see, e.g.[1.25–28]). So we shall not dwell upon that matter. We shall only explain the notation used and give the applicability criterion for (1.3.24). The stream function $\psi(x,y,t)$ is via

$$v_y = -\frac{\partial\psi}{\partial y}, \quad v_x = \frac{\partial\psi}{\partial x}.$$

related to the velocity $v = (v_x, v_y)$. The parameter β describes the dependence of the Coriolis force $f = 2\Omega_0 \cos\alpha$ on the latitude

$$\beta = \frac{\partial f}{\partial y} = -R^{-1}\frac{\partial f}{\partial \alpha}. \tag{1.3.25}$$

R is the radius of the planet's curvature, $\alpha = \pi/2 - \phi$ where ϕ is the geographical latitude, $k_0 = 1/r_R$ and the Rossby-Obukhov radius equals

$$r_R = \frac{\sqrt{gL}}{f}. \tag{1.3.26}$$

In the literature two names are assigned to (1.3.24): in hydrodynamics it is called the Charney-Obukhov equation [1.26] and in the plasma physics the Hasegawa-Mima equation [1.27]. It describes not only Rossby waves but also some other phenomena.

Some of them are:

1. Drift waves in inhomogeneous magnetized plasmas [1.21, 27, 28]. In this case $\psi = e\phi/\Omega_H M$ holds where e is the electron charge; ϕ is the potential of the electric field. The Rossby-Obukhov radius is $r_R = \sqrt{T_e/M}/\Omega_H$, the Larmor radius for ions calculated by the electron temperature. As elsewhere in this section, M, Ω_H are the mass and the Larmor frequency of ion rotation, respectively. The parameter β is the plasma inhomogeneity $\beta = \Omega_H \partial[\ln(n_0/\Omega_H)]/\partial y$ and is calculated at $y = 0$. Here n_0 is the equilibrium concentration of the plasma.
2. Low-hybrid drift waves in plasmas of compact toruses, pinches with reverse field and ionospheric F-stratum [1.29]. For these waves, $\psi = e\phi/\omega_H m$, $r_R = T_i/(m\omega_H^2)$, $\beta = \omega_H \partial[\ln(n_0/\omega_H)]/\partial y$. Here m and ω_H are mass and Larmor frequency of electron rotation, respectively.
3. Electromagnetic oscillations of the electronic component in inhomogeneous magnetized plasmas occur in z-pinches and other pulsed high-current discharges [1.30]. Here $\psi = \phi cm/H_0 m$, $r_R = c/\omega_p$, $\beta = \omega_H \partial[\ln(n_0/\omega_H)]/\partial y$, where c is the velocity of light; H_0 is the equilibrium magnetic field and ω_p the plasma frequency.
4. Density waves in rotating gas disks of galaxies [1.31]. In this case, ψ is the gravitational potential, $\beta = 2\Omega\partial[\ln(\varrho/\Omega)]/\partial r$, where Ω is the frequency of gas disk rotation; ϱ is the unperturbed density of galactic gas as a function of the radial coordinate r.

Equation (1.3.24) is applicable for intermediate wavelengths λ larger than the depth L (shallow water approximation) but less than R so we may regard the medium under consideration to be plane (not spheric). To provide a feel for the orders of magnitude of the characteristic lengths Table 1.1 gives the values of R, r_R and the mean height of atmosphere L for the Earth, Jupiter and Saturn.

Table 1.1. Characteristic lengths determined the Rossby waves: planet's radius R, Rossby-Obukhov radius $r_R = \sqrt{gL}/f$, and mean height of atmosphere L

Planet	R in km	r_R in km	L in km
Earth	6400	3000	8
Jupiter	71000	6000	25
Saturn	$\simeq$70000	6000	80

We see that there is a large interval of wave lengths λ where, on the one hand, $\lambda \gg L$ and the motion may be considered to be two-dimensional, and on the other hand, $R \gg \lambda$, so that one can neglect the planet's curvature. Thus we arrive at the "β-plane approximation" used, in effect, in deriving (1.3.24). This approximation considers waves to be on a "β-plane" tangent to the planet's surface rather than on its spheric surface. The dependence of the coefficient β (1.3.25) on y (or α) is not taken into account.

From (1.1.24) follows the dispersion law of the Rossby waves:

$$\omega(k) = -\frac{\beta k_x}{k^2 + k_0^2}, \quad k^2 = k_x^2 + k_y^2. \tag{1.3.27}$$

The phase velocity of Rossby waves is directed westwards, against the global rotation of the planet. The phase velocity decreases with increasing k, its maximal value

$$v_R = \beta k_0^{-2} = \beta r_R^2 \tag{1.3.28}$$

is called the Rossby velocity. The wave frequency (1.3.27) $\omega(k) \to 0$ at $k \to 0, \infty$. It reaches its maximal value

$$\omega_R = \frac{\beta}{2k_0} \tag{1.3.29}$$

at $k \parallel k_x$, $k = k_0$. It is interesting to compare the Rossby velocity v_R with the linear velocity of the planet's surface motion $\Omega_0 R$, and the maximal frequency of Rossby waves ω_R with the planet's rotation frequency Ω_0. From (1.3.26–29), we get the estimates

$$\frac{v_R}{\Omega_0 R} \simeq \left(\frac{r_R}{R}\right)^2, \qquad \frac{\omega_R}{\Omega_0} \simeq \frac{r_R}{R},$$

whence follows

$$\frac{r_R}{R} \simeq \frac{\sqrt{gL}}{\Omega_0 R}.$$

For the atmosphere, $\sqrt{gL} \simeq c_s$ is the velocity of sound near the planet's surface.

Baroclinic Rossby Waves. As mentioned earlier, in view of the vertical inhomogeneity of the medium one should not only consider horizontal wave motion (as in the case of barotropic waves), but also vertical wave motion. This complicates the equation of motion which will be given here without derivation (see, e.g., [1.26]). The function ψ now depends also on the vertical coordinate z: $\psi(x,y,z)$, and the desired equation is obtained by substituting into (1.3.24) an expression for the vortex density Ω different from the previous one

$$\Omega = \Delta\psi + \frac{\partial}{\partial z}\frac{f^2}{N^2}\frac{\partial\psi}{\partial z} . \tag{1.3.30a}$$

The equation for $\Omega(x,y,z,t)$ can thus be written

$$\frac{\partial\Omega}{\partial t} + \beta\frac{\partial\psi}{\partial x} = \frac{\partial(\Omega,\psi)}{\partial(x,y)} = \frac{\partial\Omega}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial\Omega}{\partial y}\frac{\partial\psi}{\partial x} . \tag{1.3.30b}$$

Here N is the frequency of vertical oscillations of a convection-stable inhomogeneous fluid. The fluid density is assumed to decrease in the vertical direction. In an incompressible medium

$$N = -\frac{g}{\varrho}\frac{\partial\varrho}{\partial z} , \tag{1.3.31}$$

whereas for a compressible fluid, the term g^2/c_s^2 should be added to the right-hand-side of (1.3.31), where c_s is the sound velocity in the medium.

It is seen from (1.3.30) that the dispersion relation for the baroclinic waves has the same form as for the barotropic ones (1.3.27). But instead of the Rossby-Obukhov radius (1.3.26) (it would be natural to call it barotropic) as a characteristic dispersion scale, it contains the so-called baroclinic Rossby radius r_l:

$$k_0 \to k_l = r_l^{-1} = \frac{\pi l f}{NL} . \tag{1.3.32}$$

Here l is the number of the vertical mode: $\psi \propto \exp(i\pi l z/L)$. In the Earth' ocean at $l = 1$ $r_1 \simeq 50\,\mathrm{km}$, which is much less then the barotropic radius $r_R \simeq 3000\,\mathrm{km}$.

Hamiltonian Description of Barotropic Rossby Waves. It has been shown independently by *Weinstein* [1.32] and *Zakharov* and *Kuznetsov* [1.3] that (1.3.24) is a Hamiltonian system and may be represented as:

$$\frac{\partial\Omega}{\partial t} = \{\Omega, \mathcal{H}\}. \tag{1.3.33a}$$

Here $\mathcal{H}$ is the energy of the system

$$\mathcal{H} = \frac{1}{2}\int\left[(\nabla\psi)^2 + k_0^2\psi^2\right] dxdy . \tag{1.3.33b}$$

which is the Hamiltonian, and the symbol $\{F,G\}$ denotes the Poisson bracket determined on functionals of $\Omega(x,y)$ by

$$\{F,G\}_\Omega = \int(\Omega + \beta y)\frac{\partial(\delta F/\delta\Omega, \delta G/\delta\Omega)}{\partial(x,y)}\, dxdy . \tag{1.3.33c}$$

The equivalence of (1.3.24) and (1.3.33) may be checked by the direct calculation.

To introduce the canonical variables for the system (1.3.33) means to diagonalize the Poisson bracket, i.e., to represent it in a form involving constant coefficients. This problem has been solved by *Zakharov and Piterbarg* [1.33]. They introduce a function $\xi(x,y)$ related to $\Omega(x,y)$ by two equivalent equations

$$\Omega(x,y) = \xi\left(x, y + \beta^{-1}\Omega(x,y)\right), \tag{1.3.34a}$$

$$\xi(x,y) = \Omega\left(x, y - \beta^{-1}\xi(x,y)\right) . \tag{1.3.34b}$$

Then they prove that

$$\{F,G\}_\Omega = \beta\int\frac{\delta F}{\delta\xi}\frac{\partial}{\partial x}\frac{\delta G}{\delta\xi}\, dxdy, \tag{1.3.35}$$

i.e., in the variables $\xi(x,y)$ the bracket (1.3.33c) becomes a bracket with constant coefficients. Here we repeat the proof:
Equation (1.3.34a) is represented as

$$\Omega(x,w) = \int\xi(x,z)\delta\left(z - w - \beta^{-1}\Omega(x,w)\right) dz ,$$

hence

$$\frac{\delta\Omega(x,w)}{\delta\xi(x,y)} = \frac{\beta}{\beta-\xi_y}\delta\left(y - w - \beta^{-1}\Omega(x,w)\right) = \delta\left(w - y + \beta^{-1}\xi(x,y)\right)$$

and, consequently,

$$\frac{\delta F}{\delta\xi(x,y)} = \int\frac{\delta}{\delta\Omega(x,w)}\frac{\delta\Omega(x,w)}{\delta\xi(x,y)}\, dw = \frac{\delta F}{\delta\Omega(x,w)} .$$

Here

$$w = y - \beta^{-1}\xi(x,y), \qquad dy = \frac{\beta\, dw}{\beta - \xi_y} . \tag{1.3.36}$$

Now we calculate

$$\begin{aligned}\{F,G\}_\xi &= \beta\int\frac{\delta F}{\delta\xi(x,y)}\frac{\partial}{\partial x}\frac{\delta G}{\delta\xi(x,y)}\, dx\, dy \\ &= \beta\int\frac{1}{1-\beta^{-1}\xi_y}\frac{\delta G}{\delta\Omega(x,w)} \\ &\quad\times\left[\frac{\partial}{\partial x}\frac{\delta F}{\delta\Omega(x,w)} - \frac{\xi_x}{\beta}\frac{\partial}{\partial w}\frac{\delta F}{\delta\Omega(x,w)}\right] dx\, dw .\end{aligned}$$

Differentiating this equation with respect to x and y, we find [the w and y points are related by (1.3.36)]

$$\frac{\beta\xi_y}{\beta-\xi_y} = \Omega_w, \quad \frac{\beta\xi_x}{\beta-\xi_y} = \Omega_x . \tag{1.3.37}$$

Substituting (1.3.37) into the previous equation, we get

$$\begin{aligned}\{F,G\}_\xi &= \beta\int \frac{\delta G}{\delta\Omega}\frac{\partial}{\partial x}\frac{\delta F}{\delta\Omega}\,dx \\ &+\int\left(\Omega_x\frac{\partial}{\partial w}\frac{\delta F}{\delta\Omega} - \Omega_w\frac{\partial}{\partial x}\frac{\delta F}{\delta\Omega}\right)\frac{\delta G}{\delta\Omega}\,dx\,dw \\ &= \{F,G\}_\Omega .\end{aligned}$$

We used in the proof the local reversibility of (1.3.36) and the inverse equation

$$y = w + \beta^{-1}\Omega(x,w)\ . \tag{1.3.38}$$

Reversibility of (1.3.38) has a clear topological meaning, indicating the in-closed character of isocurlus lines of a full planetary vortex $\omega + \beta y$. It is known that this condition precludes the existence of solitons or localized vortices in the solution of (1.3.24) and is satisfied for the waves of sufficiently small amplitude. At small ξ, Ω equations (1.3.34) may be expanded in a series

$$\Omega(x,y) = \sum_{n=0}^{\infty}\frac{(-1)^n\beta^n}{(n+1)!}\frac{\partial^n}{\partial y^n}\xi^{n+1}\ , \tag{1.3.39a}$$

$$\xi(x,y) = \sum_{n=0}^{\infty}\frac{\beta^n}{(n+1)!}\frac{\partial^n}{\partial y^n}\Omega^{n+1}\ . \tag{1.3.39b}$$

The statement we have proved shows that after the transition to the ξ-variable, (1.3.33a) reduces to

$$\frac{\partial\xi}{\partial t} = \beta\frac{\partial}{\partial x}\frac{\delta\mathcal{H}}{\delta\xi}\ . \tag{1.3.40}$$

It is now easy to see that use of the representation

$$\xi(x,y) = \frac{\sqrt{\beta/2}}{\pi}\int_{p>0}\sqrt{p}\left[a(p,q)e^{i(px+qy)} + a^*(p,q)e^{-i(px+qy)}\right]dpdq\ , \tag{1.3.41}$$

converts (1.3.40) into the Hamiltonian form

$$\frac{\partial a}{\partial t} = -i\frac{\delta\mathcal{H}}{\delta a^*}\ . \tag{1.3.42}$$

The Hamiltonian $\mathcal{H}$ is an infinite power series of the variables $a(\boldsymbol{k})$, $a^*(\boldsymbol{k})$. The first terms of this series have the form

$$\begin{aligned}\mathcal{H} &= 2\int\omega(k)|a(\boldsymbol{k})|^2\,d\boldsymbol{k} + \int V_{k12}\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) \\ &\times\left[a^*(\boldsymbol{k})a(\boldsymbol{k}_1)a(\boldsymbol{k}_2) - a(\boldsymbol{k})a^*(\boldsymbol{k}_1)a^*(\boldsymbol{k}_2)\right]d\boldsymbol{k}d\boldsymbol{k}_1d\boldsymbol{k}_2\ .\end{aligned} \tag{1.3.43}$$

Here $\boldsymbol{k}_i = (p_i, q_i)$, the integration domain is $p_i > 0$ and $\omega(\boldsymbol{k}) = \beta p(k^2+k_0^2)^{-1}$ is the dispersion law of Rossby waves. The interaction coefficient has the simple form

$$V_{k12} = \frac{i}{\pi}\sqrt{\frac{\beta pp_1p_2}{2}}\left(\frac{q}{k^2+k_0^2} - \frac{q_1}{k_1^2+k_0^2} - \frac{q_2}{k_2^2+k_0^2}\right)\ . \tag{1.3.44}$$

These expressions may be considered from two different viewpoints. On the one hand, if (1.3.24) is a postulate, then the corresponding canonical equation is (1.3.42) with the Hamiltonian (1.3.43–44). But in deriving (1.3.24) we can consider the limit $k > k_0$. Strictly speaking, it is only in this region that we can handle the three-wave matrix element for Rossby waves

$$V_{k12} \to \frac{i}{\pi}\sqrt{\frac{\beta pp_1p_2}{2}}\left(\frac{q}{k^2} - \frac{q_1}{k_1^2} - \frac{q_2}{k_2^2}\right)\ . \tag{1.3.45}$$

Hamiltonian Description of Baroclinic Rossby Waves. As shown by *Zakharov et al.* [1.34], the canonical variables for baroclinic Rossby waves are introduced similarly to the barotropic case. At first one can carry out a direct calculation to see that the equations of motion (1.3.30) reduce to a Hamiltonian form similar to (1.3.33a):

$$\frac{\partial\Omega}{\partial t} = \{\Omega,\mathcal{H}\}_\Omega\ , \tag{1.3.46a}$$

where Ω is given by (1.3.30a). The Hamiltonian

$$\mathcal{H} = \frac{1}{2}\int\left[(\nabla\psi)^2 + \frac{f^2}{N^2}\left(\frac{\partial\psi}{\partial z}\right)^2\right]d\boldsymbol{r}dz\ , \tag{1.3.46b}$$

differs from (1.3.33b) and the Poisson bracket between two functionals F and G,

$$\{F,G\}_\Omega = \int(\Omega+\beta y)\frac{\partial(\delta F/\delta\Omega, \delta G/\delta\Omega)}{\partial(x,y)}\,dxdydz\ . \tag{1.3.46c}$$

has a form which deviates from (1.15c) by an integration over z. Similarly to the barotropic case, one can prove [1.34] that in the ξ-variables determined like (1.3.39),

$$\begin{aligned}\Omega(x,y,z) &= \xi\left(x, y, +\beta^{-1}\Omega(x,y,z), z\right)\ , \\ \xi(x,y,z) &= \Omega\left(x, y-\beta^{-1}\xi(x,y,z), z\right)\ ,\end{aligned} \tag{1.3.47}$$

(1.3.46) reduces to (1.3.40). For functions of x, y, z we shall use representations of the form

$$\xi(x,y,z) = \frac{1}{2\pi}\sum_m\varphi_m(z)\int_{k_x>0}\left[\xi_m(k)e^{i(kx)} + \xi_m^*(k)e^{-i(kx)}\right]dk\ . \tag{1.3.48}$$

where $\boldsymbol{x} = (x,y)$, $\{\varphi_m(z)\}$ is an orthonormal system of eigenfunctions of the operator $\hat{L}$:

$$\hat{L} = \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\right)\frac{\partial}{\partial z} ,$$
$$\hat{L}\varphi_m = -k_m^2\varphi_m , \quad \varphi_m(0) = \varphi_m(L) = 0, \tag{1.3.49}$$
$$\frac{1}{L}\int\limits_0^L \varphi_n\varphi_m\left(\frac{f_0^2}{N^2}\right)dz = \delta_{nm} .$$

In such terms the Hamiltonian (1.3.46) reduces to a form

$$\mathcal{H} = \frac{1}{2}\sum_m (k_m^2 + k^2)|\varphi_m(k)|^2\,dk = \frac{1}{2}\sum_m \int \frac{|\Omega_m(k)|^2\,dk}{k_m^2 + k^2} . \tag{1.3.50}$$

Now we go over to normal canonical variables,

$$a_m(k) = (-2\beta k_x)^{1/2}\xi_m(k) ,$$

in which the Hamilton equation (1.3.40) takes the canonical form (1.3.42), and all that is left to do is to express the Hamiltonian (1.3.50) in terms of the normal variables. In view of the fact that for the Rossby waves, three-wave resonance interactions are possible, in calculating $\mathcal{H}$ we shall restrict ourselves to those terms that are quadratic and cubic in $a_m(k)$. This allows us to use, instead of (1.3.47), an approximate relationship between Ω and ξ which holds to an accuracy of the order of β^{-2}:

$$\Omega(x,y,z) = \xi(x,y,z) + \frac{1}{2\beta}\frac{\partial}{\partial y}\xi^2(x,y,z) , \tag{1.3.51a}$$

$$\begin{aligned}\Omega_m(\boldsymbol{k}) &= \xi_m(\boldsymbol{k}) + \frac{1}{4\pi\beta}\int \exp[-i(\boldsymbol{k}\boldsymbol{x})]\varphi_m(z)\left(\frac{f_0^2}{N^2}\right)\frac{\partial\xi^2}{\partial y}\,dx dy dz \\ &= \xi_m(\boldsymbol{k}) + \frac{ik_y}{4\pi\beta}\sum_{m_1,m_2} B(m,m_1,m_2)\xi_{m_1}(\boldsymbol{k}_1)\xi_{m_2}(\boldsymbol{k}_2) \\ &\quad\times \delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2)\,d\boldsymbol{k}_1 d\boldsymbol{k}_2 ,\end{aligned} \tag{1.3.51b}$$

$$B(m,m_1,m_2) = L^{-1}\int_0^L \varphi_m(z)\varphi_{m_1}(z)\varphi_{m_2}(z)\left(f_0^2/N^2\right)dz . \tag{1.3.51c}$$

Substitution of (1.3.51b) into (1.3.50) gives

$$\begin{aligned}\mathcal{H} &= 2\sum_m \int \sigma_m(\boldsymbol{k})|a_m(\boldsymbol{k})|^2\,d\boldsymbol{k} \\ &+ 2\sum_{mm_1m_2}\int [V_{k12}\left(a^* a_1 a_2 - \text{c.c.}\right)\delta(\boldsymbol{k} - \boldsymbol{k}_1 - \boldsymbol{k}_2) \\ &+ U_{k12}\left(a^* a_1^* a_2^* - \text{c.c}\right)\delta(\boldsymbol{k} + \boldsymbol{k}_1 + \boldsymbol{k}_2)]\,d\boldsymbol{k}d\boldsymbol{k}_1 d\boldsymbol{k}_2 ,\end{aligned}$$

$$\begin{aligned}V_{k12} &= \frac{i}{4\pi}\sqrt{-2\beta k_x k_{1x} k_{2x}}[S(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) \\ &\quad - S(\boldsymbol{k}_2,-\boldsymbol{k}_1,\boldsymbol{k}) - S(\boldsymbol{k}_1,\boldsymbol{k},-\boldsymbol{k}_2)] ,\end{aligned}$$

$$U_{k12} = -\frac{i}{4\pi}\sqrt{-2\beta k_x k_{1x} k_{2x}}\,S(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) ,$$

$$S(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = S(m,m_1,m_2,\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = \frac{B(m,m_1,m_2)k_y}{k_m^2 + k^2} ,$$

$$\sigma_m(\boldsymbol{k}) = -\frac{\beta k_x}{k_m^2 + k^2} .$$

If the vertical inhomogeneity is linear, then according to (1.3.31) N =const holds and

$$k_m = \frac{\pi m f}{NL} , \qquad \varphi_m(z) = \sin(\pi m z/L) .$$

and for the coefficients (1.3.51c) we obtain

$$B(m,m_1,m_2) = \frac{f^2}{N^2} \times \frac{\left[(-1)^{m+m_1+m_2} - 1\right]mm_1m_2}{(m+m_1+m_2)(m-m_1+m_2)(-m-m_1+m_2)(-m+m_1+m_2)} .$$

It should be noted that in the case of the barotropic waves, this expression is replaced by unity and k_m by the constant k_0.

1.4 Spin Waves

1.4.1 Magnetic Order, Energy and Equations of Motion

To date, a large variety of magnetically-ordered substances are known: dielectrics, semiconductors and metals, both crystalline and amorphous [1.35]. Their structure includes paramagnetic atoms (ions) with uncompensated spin $\boldsymbol{S}$, thus having magnetic moment $\mu\boldsymbol{S}$ (μ is the Bohr magneton here). Such atoms give rise to the exchange interaction which is of electrostatic nature and is to be associated with the Pauli principle prohibiting more than one electron to be in a given quantum-mechanical state [1.36]. At low temperatures, this interaction leads to magnetic ordering orienting the magnetic moments of the atoms in a definite manner. The simplest type of magnetic ordering is the ferromagnetic state in which the magnetic moments of all atoms are parallel. This results in a macroscopic magnetic moment with density $\boldsymbol{M}$. In contrast to ferromagnets, the total magnetic moment of antiferromagnets is zero. In the simplest case, an elementary cell of crystalline antiferromagnet has two magnetic atoms whose moments are antiparallel and equal in magnitude. In describing antiferromagnets one uses the notion of a magnetic sublattice, which contains the translational-invariant magnetic atoms, i.e., the positions of them differ by an integer number of elementary translations of the crystalline lattice. In the simplest antiferromagnet, there are two magnetic sublattices with the moments $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$, with $\boldsymbol{M} = \boldsymbol{M}_1 + \boldsymbol{M}_2 = 0$.

At low temperatures the long-wave magnetic excitations may be described classically, using the functions $\boldsymbol{M}_j(\boldsymbol{r},t)$. These excitations are spin waves or precession waves of the magnetic moment. The equation of motion for $\boldsymbol{M}(\boldsymbol{r},t)$ (the Bloch equation) describes the precession of a vector with a fixed length $|\,\boldsymbol{M}(\boldsymbol{r},t)\,|^2 = \boldsymbol{M}^2(T)$ =const in an effective magnetic field $\boldsymbol{H}_{\text{eff}}(\boldsymbol{r},t)$ (see, e.g., [1.37]):

$$\partial\boldsymbol{M}(\boldsymbol{r},t)/\partial t = g_m[\boldsymbol{H}_{\text{eff}}(\boldsymbol{r},t), \boldsymbol{M}(\boldsymbol{r},t)] , \tag{1.4.1a}$$

$$\boldsymbol{H}_{\text{eff}}(\boldsymbol{r},t) = -\delta W/\delta\boldsymbol{M}(\boldsymbol{r},t) . \tag{1.4.1b}$$

Here $g_m = \mu/\hbar$ is the ratio of magnetic to mechanical moment of the electrons, and W, the energy of the system. The g_m value is approximately equal to $2\pi 2.8$ MHz/Ö. The energy W is a functional of $\boldsymbol{M}(\boldsymbol{r},t)$. In ferromagnets it includes W_0, the interaction energy of a spin subsystem with an external field $\boldsymbol{H}_0$, the exchange energy W_{ex} and a number of the terms of relativistic origin. The main contributions stem from the energy of the magnetic dipole-dipole interaction W_{dd} and the energy of the crystalline anisotropy W_a:

$$W = W_0 + W_{ex} + W_{dd} + W_a \ , \tag{1.4.2a}$$

$$W_0 = -g_m \int (\boldsymbol{H}\boldsymbol{M})\, d\boldsymbol{r} \ , \tag{1.4.2b}$$

$$W_{ex} = \frac{1}{2}\kappa_{ik} \int \frac{\partial M_j}{\partial x_i}\frac{\partial M_j}{\partial x_k}\, d\boldsymbol{r} \ , \tag{1.4.2c}$$

$$W_a = K \int M_z^2\, d\boldsymbol{r} \ , \tag{1.4.2d}$$

$$W_{dd} = -\frac{1}{2}\int (\boldsymbol{H}_m\boldsymbol{M})\, d\boldsymbol{r} \ . \tag{1.4.2e}$$

Here κ_{ik} and K are material constants; $\boldsymbol{H}_m$ is the static magnetic field created by the magnetic moment distribution. The phenomenological expression (1.4.2) for the energy of a ferromagnet is discussed in detail in [1.37–38]. Here we shall only comment on it in brief. The physical meaning of W_0 is obvious, it is the magnetic dipole energy in the external field $\boldsymbol{H}$; the integral in the expression for W_{dd} defines the dipole energy in the self-magnetic field $\boldsymbol{H}_m$, with the interaction energy of each pair taken into account twice. The factor $\frac{1}{2}$ in (1.4.2e) compensates for this double counting. The expression (1.4.2c) for W_{ex} is general enough. Indeed, if we suggest that (i) W_{ex} is independent of the magnetization relative to the crystal axes; (ii) the crystal has an inverted symmetry element; (iii) the energy dependence on the magnetization is quadratic. For the derivation of (1.4.2c) see, e.g., [1.37]. Item (i) follows from the nature of the exchange interaction which is invariant relative to the total rotation of all spins. (ii) is satisfied since most magnets are just of this kind. (iii) holds strictly speaking only for magnetic atoms with the spin $S = 1/2$. At $S > 1/2$ this suggestion is invalid, but experiment shows that corrections to (1.4.2c) are small and this has been theoretically substantiated [1.39]. The term (1.4.2d) for the crystalline anisotropy energy appears in second order perturbation theory with regard to the spin-orbit interaction as a weak one. The constant K is nonzero for uniaxial crystals. At $K < 0$, the energetically favorable orientation of magnetization is that along the crystal's symmetry axis (z axis). At $K > 0$ the orientation in the plane perpendicular to the z axis is favored. In the former case, the anisotropy is referred to as being of the *easy axis* type, and in the latter case, it is said to be of the *easy plane* type. Finally, in crystals with a cubic symmetry, the expression for κ_{ik} is simplified compared with that for the isotropic continuous medium $\kappa_{ik} = \kappa\delta_{ik}$.

1.4.2 Canonical Variables

In (1.4.1) we go over to circular variables

$$M_\pm = M_x + iM_y, \quad \frac{\partial M_\pm}{\partial t} = 2ig_m M_z, \quad M_z^2 = M_0^2 - M_+M_- \ . \tag{1.4.3}$$

We choose the z axis along the equilibrium direction of magnetization. Then at small oscillation amplitudes of the magnetic moment the $M_\pm$ values will be small, and M_z will be close to the length of $\boldsymbol{M}$, i.e., M_0. Comparing (1.4.3) and (1.1.6), we see that these equations have in the $M_\pm$-linear approximation the form of Hamilton equations if we take as the canonical variables

$$a(\boldsymbol{r},t) = \frac{M_+}{\sqrt{2g_m M_0}}, \quad a^*(\boldsymbol{r},t) = \frac{M_-}{\sqrt{2g_m M_0}} \ .$$

Therefore it is reasonable to write these canonical variables as

$$\begin{aligned} M_+ &= af(a^*a)\sqrt{2g_m M_0} \ , \\ M_- &= a^*f(a^*a)\sqrt{2g_m M_0}, \quad f(0) = 1 \ . \end{aligned} \tag{1.4.4}$$

Substituting (1.4.4) into (1.4.3), we obtain an equation for $\partial a(\boldsymbol{r},t)/\partial t$:

$$\frac{\partial a(\boldsymbol{r},t)}{\partial t} = -\frac{iM_z(a^*a)}{(f^2 + 2aa^*ff')M_0}\frac{\delta W}{\delta a^*}, \tag{1.4.5a}$$

Here

$$M_z(a^*a) = M_0\sqrt{1 - 2f^2 g_m a^*a/M_0}. \tag{1.4.5b}$$

Demanding these equations to coincide with the canonical equations (1.1.6), we obtain the differential equation for the function $f(x)$

$$f^2 + 2xff' = 2g\sqrt{M_0^2 - xf^2}, \tag{1.4.6a}$$

of which the only solution satisfying the condition $f(0) = 1$ is

$$f(x) = \sqrt{1 - g_m a^*a/2M_0} \tag{1.4.6b}$$

Thus we have expressed the natural variables of the ferromagnet's spin subsystem $M_z, M_\pm$ through the canonical ones:

$$\begin{aligned} M_+ &= a\sqrt{2g_m M_0[1 - g_m a^*a/2M_0]}, \quad M_- = M_+^*, \\ M_z &= M_0 - g_m a^*a \ . \end{aligned} \tag{1.4.7}$$

This equation is nonlinear and valid if $g_m a^*a < 2M_0$. The ferromagnet's energy W expressed via the canonical variables becomes the Hamiltonian $\mathcal{H}(a^*, a)$. In quantum mechanics, the Holstein-Primakoff representation has long been known [1.37–38]. It gives the spin operators in terms of Bose operators. The formulas (1.4.7) are the classical analogue of this representation. They were first used by

Schlömann for the analysis of nonlinear processes in a spin wave system [1.40]. The choice of canonical variables is certainly not unique. For (1.4.1) one can introduce other canonical variables to express the vector M as follows

$$M_z + iM_x = M_0\sqrt{1+\frac{g_m \mid b^*-b \mid}{2M_0}}\ \exp\left[i(b^*+b)\sqrt{g_m/2M_0}\right], \qquad (1.4.8)$$
$$M_y = i\sqrt{g_m M_0/2}(b^*-b) .$$

These formulas are the classical analogue of the spin operator representation via Bose operators as suggested by *Baryakhtar* and *Yablonsky* [1.41].

Comparing (1.4.7) and (1.4.8), one can see that the Holstein-Primakoff a^*, a and Baryakhtar-Yablonsky b^*, b variables coincide in the linear approximation. The specific character of the problem under consideration determines the type of representation to be preferred.

1.4.3 The Hamiltonian of a Heisenberg Ferromagnet

The procedure for calculating the Hamiltonian coefficients is standard. The magnetization (1.4.8) is expanded into a series of canonical variables, the result is substituted into (1.4.2) for the energy to go then over to the k-representation. Neglecting the relativistic interaction W_{dd} and W_a, one thus finds that the quadratic part of the Hamiltonian is diagonal in $a^*(\boldsymbol{k},t)$, $a(\boldsymbol{k},t)$, $\mathcal{H}_3 = 0$, and that out of the fourth-order terms, only those terms in (1.1.24) that are proportional to W and describe $2 \to 2$ scattering are nonzero. In the isotropic case,

$$\omega(k) = \omega_0 + \beta k^2, \quad \omega_0 = gH, \quad \beta = 2\kappa g_m M_0 , \qquad (1.4.9a)$$

$$W_{12,34} \equiv W_{ex}(\boldsymbol{k}_1, \boldsymbol{k}_2; \boldsymbol{k}_3, \boldsymbol{k}_4) = -\kappa g_m[(\boldsymbol{k}_1\boldsymbol{k}_2) + (\boldsymbol{k}_3\boldsymbol{k}_4)]. \qquad (1.4.9b)$$

As mentioned in Sect. 1.1, the scattering processes do not change the total number of waves, therefore the equations of motion corresponding to such a Hamiltonian preserve an additional integral of motion

$$N = \int a^*(\boldsymbol{k},t)a(\boldsymbol{k},t)\, d\boldsymbol{k}. \qquad (1.4.10)$$

As seen from (1.4.7), conservation of N means the constancy of the z-projection of the magnetization $M_z = M_0 - gN$. This is a consequence of the fact that the operator $\hat{M}_z$ commutes with the Heisenberg Hamiltonian of the ferromagnet [1.37]. Inclusion of relativistic interactions violates this relation, leading to the terms in $\mathcal{H}_{\text{int}}$ describing processes that do not conserve the number of waves: those of type $1 \to 2$, $2 \to 1$, $1 \to 3$, etc. In cubic ferromagnets, which were the object of extensive experimental research on nonlinear spin wave dynamics [1.38], the dominant relativistic interaction is of the magnetic dipole-dipole type (1.4.2c). The consequence of such interaction is the fact that the "circular" (circularly polarized) canonical variables $a^*(\boldsymbol{k},t)$, $a(\boldsymbol{k},t)$ are no longer normal for the quadratic Hamiltonian $\mathcal{H}_2$. Diagonalization of $\mathcal{H}_2$ with the help of the

(u,v)-transformation (1.1.16) leads to the "elliptical" (elliptically polarized) variables $b^*(\boldsymbol{k},t)$, $b(\boldsymbol{k},t)$ and the expression for the frequency (1.4.9a) is replaced by

$$\omega^2(k) = \left[\omega_0 - \omega_M N_z + \beta k^2 + \frac{1}{2}\omega \sin^2\theta\right]^2 - \frac{1}{4}\omega_M^2 \sin^4\theta . \qquad (1.4.11a)$$

Here $\omega_M = 4\pi g_m M_0$, θ is the angle between vectors M and $\boldsymbol{k}$; N_z is the demagnetization factor (equal to $1/3$ for a spherical sample, to 0 for a longitudinally magnetized cylinder and 1 for a tangentially magnetized disk). Computing the contribution of the dipole-dipole interaction to the three- and four-wave Hamiltonian does not present any special difficulties, but the computation procedure and the result are rather cumbersome. Monograph [1.38] gives them in full. We restrict our considerations to the case of a relatively small dipole-dipole interaction where one can do without the (u,v)-transformation in the expression for $\mathcal{H}_3$ and $\mathcal{H}_4$:

$$V_{1,23} = \frac{1}{2}(V_2 + V_3), \quad V_k = -\omega_M\sqrt{\frac{g}{2M_0}}\sin\theta\cos\theta\exp(i\varphi_k) . \qquad (1.4.12)$$

$$\begin{aligned} W_{12,34} &= W_{ex}(12,34) + \frac{1}{4}(C_{14}+C_{13}+C_{23}+C_{24}) \\ &\quad - \frac{1}{4}(D_1+D_2+D_3+D_4) , \\ C_{ij} &\equiv C(\boldsymbol{k}_i,\boldsymbol{k}_j) = C(\boldsymbol{k}_i - \boldsymbol{k}_j) , \\ C(\boldsymbol{k}) &= 4\pi g_m^2\cos^2\theta, \quad C(0) = 4\pi g_m^2 N_z , \\ D_i &= D(\boldsymbol{k}_i) = 4\pi g_m^2 \sin^2\theta\exp(2i\varphi_k) . \end{aligned} \qquad (1.4.13)$$

We see that the problem of spin waves in ferromagnets has no complete self-similarity. This is due to the presence of two interactions, the exchange and dipole-dipole interactions characterized by frequencies βk^2 and ω_M with different dependences on the wave vector. Nevertheless, one can single out regions in k-space where the Hamiltonian coefficients are scale-invariant. At $\beta k^2 \gg \omega_M$, the dispersion law is quadratic with a gap (1.4.9a), and the coefficient of the four-wave interaction (1.4.9b) has the homogeneity degree two. At $\omega_0 - \omega_M N_z + \beta k^2 \ll \omega_M \sin^2\theta \ll \omega_M$ for the waves propagating at small angles with the z axis, like in the case of magnetized plasma, a separate self-similarity in $k_z, k_\perp$ results:

$$\omega(\boldsymbol{k}) = \sqrt{\omega_M(\omega_0 - \omega_M N_z)}\ \mid k_z/k_\perp \mid . \qquad (1.4.14a)$$

It is of the decay type, like (1.3.22) so that the interaction coefficient (1.4.12) becomes

$$V_{1,2,3} = \frac{1}{2}\omega_M\sqrt{g_m/2M_0}[\exp(i\varphi_1)k_{1\perp}/k_{1z} + \exp(i\varphi_2)k_{2\perp}/k_{2z}]. \qquad (1.4.14b)$$

Other cases of self-similarity are given in [1.42].

1.4.4 The Hamiltonian of Antiferromagnets

The simplest antiferromagnets have two magnetic sublattices and, accordingly, two spin wave branches. The quadratic part of the Hamiltonian has the standard form (1.1.17)

$$\mathcal{H}_2 = \int \left[\omega(\boldsymbol{k}) a_k a_k^* + \Omega(\boldsymbol{k}) b_k b_k^*\right] d\boldsymbol{k}$$

We give this expression here to introduce the notations for the spin wave frequencies in the two branches $\omega(\boldsymbol{k})$, $\Omega(\boldsymbol{k})$ and the normal canonical variables $a_k = a(\boldsymbol{k}, t)$, $a_k^* = a^*(\boldsymbol{k}, t)$. In the uniaxial ferromagnets with an "easy axis"-type anisotropy, the (crystalline) anisotropy field $\boldsymbol{H}_a$ tends to keep the magnetization parallel to that axis (usually called the z axis).

By analogy with ferromagnets, spin wave frequencies with $k \to 0$ would be expected to correspond to sublattice magnetization precession in the field $\boldsymbol{H}_a$, i.e.

$$\omega_0 = \Omega_0 = \omega_a ,$$

where

$$\omega_a = gH_a . \qquad (1.4.15)$$

However this is not so. In fact, the magnetization of the sublattice $\boldsymbol{M}_1$ oriented upwards is affected by the anisotropy field $\boldsymbol{H}_{a1}$ which is also oriented upwards: $\boldsymbol{H}_{a1} = \boldsymbol{H}_a$. The second sublattice $\boldsymbol{M}_2 = -\boldsymbol{M}_1$ is affected by another field $\boldsymbol{H}_{a2} = -\boldsymbol{H}_{a1}$. As a result, the sublattices tend to precess in opposite directions. In this case, the antiparallel arrangement of $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ will inevitably be broken up, which is prevented by the strong exchange interaction between sublattices. As a result we have [1.37]

$$\omega_0^2 = 2\omega_{ex}\omega_a - \omega_H^2, \quad \Omega_0^2 = 2\omega_{ex}\omega_a + \omega_H^2, \quad \omega_H = gH . \qquad (1.4.16)$$

Here $\omega_{ex} = g_m BM$ characterizes the antiferromagnetic exchange between sublattices. The order of magnitude of the dimensionless exchange constant is $B \simeq 10^3$.

In uniaxial antiferromagnets with "easy plane"-type anisotropy, the anisotropic field rotates the sublattice moments into the plane perpendicular to the z axis. The possibility of almost free oscillations of moments in this plane leads to the fact that one of the spin wave frequencies at $k = 0$ turns out to be small, i.e., $\omega_0 \simeq \omega_H$. The upper branch gap lies much higher:

$$\Omega_0^2 = 2\omega_{ex}\omega_a + \omega_0^2 . \qquad (1.4.17)$$

This situation is of special interest for experiments on the nonlinear properties of spin waves: the investigated frequency range below 50 GHz includes the lower branch ω_k. Thus for "easy-plane" antiferromagnets we write down the results for the Hamiltonian coefficients in question [1.38]:

$$\omega_k^2 = \omega_0^2 + (vk)^2, \quad \Omega_k^2 = 2\omega_{ex}\omega_a + \omega_k^2 . \qquad (1.4.18)$$

$$\begin{aligned}
\mathcal{H}_3 &= \frac{1}{2}\int \big[\left(V_1 b_1 a_2^* a_3^* + 2V_2 a_1 b_2^* a_3^* + \text{c.c}\right)\delta(\boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) \\
&\quad + \left(U b_1^* a_2^* a_3^* + \text{c.c}\right)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\big]\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 , \\
\mathcal{H}_4 &= \frac{1}{4}\int W a_1^* a_2^* a_3 a_4 \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_3 - \boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 , \\
V_1 &= -\frac{1}{4}\sqrt{2g_m\omega_{ex}/M\omega_2\omega_3\Omega_1}\,(\Omega_1 + \omega_2 + \omega_3)\omega_H , \\
V_2 &= -\frac{1}{4}\sqrt{2g_m\omega_{ex}/M\omega_1\omega_3\Omega_2}\,(\omega_1 - \Omega_2 + \omega_3)\omega_H , \\
U &= -\frac{1}{4}\sqrt{2g_m\omega_{ex}/M\omega_2\omega_3\Omega_1}\,(\Omega_1 - \omega_2 - \omega_3)\omega_H , \\
W &= 9\omega_{ex}\frac{\left[\omega_1^2 + \omega_2^2 + \omega_3^2 + \omega_4^2 - \omega_{(1-3)}^2 - \omega_{(2-3)}^2 - \omega_{(1+2)}^2\right]}{4M\sqrt{\omega_1\omega_2\omega_3\omega_4}} .
\end{aligned} \qquad (1.4.19)$$

We used the shorthand notation

$$\Omega_i = \Omega(k_i), \ \omega_i = \omega(k_i), \ \omega_{i-j} = \omega(k_i - k_j), \quad \text{etc} .$$

It should be borne in mind that a scattering process of type $2 \to 2$ does not just require to take into account the Hamiltonian $\mathcal{H}_4$ but also $\mathcal{H}_3$ in second order perturbation theory. It may be shown that the dominant contribution to the ω_k/ω_{ex} parameter will be due to processes involving the upper-branch virtual wave, as explicitly given in (1.4.19). Using (1.1.29), we can derive an expression for the effective coefficient of the lower branch four-wave processes T [1.38]. The general expression for T is rather cumbersome. For $k_1 = k_3$, $k_2 = k_4$, $\omega_1 = \omega_2 = \omega_3 = \omega_4$ it reads

$$T = -\frac{g^2\omega_{ex}}{8\omega_k^2}\left[\omega_0^2 + \omega_H^2(3\Omega_0^2 - 4\omega_k^2)/(\omega_0^2 - 4\omega_k^2)\right] . \qquad (1.4.20)$$

It is evident from these formulas that the problem of spin waves in antiferromagnets does not possess complete self-similarity, like in ferromagnets. However, a second-order self-similarity does exist. Thus in the dispersion law (1.4.18), three regions may be singled out. At $k \to 0$, the ω_k and Ω_k branches have a gap with a quadratic addition. In the region of large k, the functions ω_k and Ω_k follow a linear law. Since usually $\omega_a\omega_{ex} \gg \omega_H^2$, there is an intermediate range of the k values where the frequency ω_k has already become asymptotically linear and $\Omega_k = \Omega_0 + \beta k^2$. We shall not go through the simple but cumbersome analysis of the asymptotics of the interaction coefficient (1.4.16, 17). We shall only note that at $k \to 0$, the coefficient of the four-wave interaction becomes constant. In this case the Hamiltonian of the problem for low-frequency waves has an especially simple form:

$$\mathcal{H} = \int \omega(k) b^*(\boldsymbol{k}) b(\boldsymbol{k})\, d\boldsymbol{k} + \frac{T_0}{4}\int b^*(\boldsymbol{k}_1) b^*(\boldsymbol{k}_2) b(\boldsymbol{k}_3) b(\boldsymbol{k}_4)\, d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \,, \quad \omega_k = \omega_0 + \beta k^2 \,. \tag{1.4.21}$$

Why have we taken into account the term $\propto k^2$ in the expression for ω_k but neglected it in the expression for $T = T_0 + O(k^2)$ although it has in fact the same relative order of magnitude? The answer is: the constant ω_0 is not involved in the problem and is eliminated by the time-dependent canonical transformation

$$c(\boldsymbol{k}, t) = b(\boldsymbol{k}, t) \exp(i\omega_0 t/2) \,. \tag{1.4.22}$$

In the variables $c(\boldsymbol{k}, t)$, the Hamiltonian (1.4.21) has the form:

$$\mathcal{H} = \beta \int k^2 c^*(\boldsymbol{k}) c(\boldsymbol{k}) d\boldsymbol{k} + \frac{T_0}{4}\int c^*(\boldsymbol{k}_1) c^*(\boldsymbol{k}_2) c(\boldsymbol{k}_3) c(\boldsymbol{k}_4) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 d\boldsymbol{k}_4 \,. \tag{1.4.23}$$

The respective dynamic equation $i\partial c(\boldsymbol{k}, t)/\partial t = \delta\mathcal{H}/\delta c(\boldsymbol{k}, t)$ has after the inverse Fourier transform $c(\boldsymbol{k}, t) \to \psi(\boldsymbol{r}, t)$ the form of the nonlinear Schrödinger equation

$$i\partial\psi/\partial t + \beta \,\Delta\, \psi + \frac{T_0}{2} \mid \psi \mid^2 \psi = 0 \,. \tag{1.4.24}$$

Indeed, that is the quantum mechanical Schrödinger equation with $|\psi|^2$ as a potential.

1.5 Universal Models

The nonlinear Schrödinger equation which we encountered at the end of the previous section is a member of a small family of famous universal equations that arise in the nonlinear wave theory. Every such equation describes a large variety of physical systems belonging to different topics of physics. As we shall show just now, the nonlinear Schrödinger equation describes the behavior of a narrow wave packet envelope for most nonlinear wave systems, see Sect. 1.5.1. If we consider another physical situation, namely long-wave perturbations of the acoustic type (which can exist in most media), then we obtain the well-known Korteweg - de Vries (KdV) equation in the one-dimensinal case and the rather famous Kadomtsev-Petviashvili equation for weakly two-dimensional distributions, see Sect. 1.5.2. In Sect. 1.5.3 we shall consider a third rather universal physical situation: the interaction of three-wave packets in media with a decay dispersion law. All three of these models have extremely wide applications, ranging from solid state physics to hydrodynamics, plasma physics, etc.

1.5.1 Nonlinear Schrödinger Equation for Envelopes

This equation, like the Hamiltonian (1.4.23), has a wide range of applications. Basically, these applications are associated with the fact that (1.4.24) describes the behavior of an envelope of quasi-monochromatic waves in isotropic nonlinear media. Equation (1.4.24) should be regarded as written in a reference system moving with the group velocity of a wave packet.

Now let us show how the dynamical equation (1.4.24) can be obtained for a narrow wave packet. If the carrier wave vector is denoted by $\boldsymbol{k}_0$ and $|\boldsymbol{k} - \boldsymbol{k}_0| = |\boldsymbol{\kappa}| \ll k_0$, we assume

$$\omega(\boldsymbol{k}) \approx \omega(\boldsymbol{k}_0) + (\boldsymbol{\kappa v}) + \frac{1}{2}\kappa_i\kappa_j \left(\frac{\partial^2\omega}{\partial k_i \partial k_j}\right)_{k=k_0} \tag{1.5.1}$$

to hold. Here $\boldsymbol{v} = (\partial\omega/\partial\boldsymbol{k})_{k=k_0}$ is the group velocity. In isotropic media, with the frequency depending only on the modulus of the wave vector, the third term in (1.5.1) can be simplified

$$\kappa_i\kappa_j \left(\frac{\partial^2\omega}{\partial k_i \partial k_j}\right)_{k=k_0} = \frac{v}{k_0}\kappa_\perp^2 + \omega''\kappa_\parallel^2, \quad \kappa_\parallel = (\boldsymbol{\kappa k}_0)/k_0 \,.$$

In the propagation of a single narrow envelope the three-wave interaction is not important (but it must be taken into account in higher-order perturbation theory). Waves with approximately equal wave vectors are involved in the four-wave interaction. The dynamic equation has the form

$$\frac{\partial c(\boldsymbol{k}, t)}{\partial t} + i\omega(k) c(\boldsymbol{k}, t) = -i \int T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) \times c^*(\boldsymbol{k}_1, t) c(\boldsymbol{k}_2, t) c(\boldsymbol{k}_3, t) \delta(\boldsymbol{k} + \boldsymbol{k}_1 - \boldsymbol{k}_2 - \boldsymbol{k}_3) d\boldsymbol{k}_1 d\boldsymbol{k}_2 d\boldsymbol{k}_3 \,. \tag{1.5.2}$$

For a narrow packet it can be assumed that

$$T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3) \approx T(\boldsymbol{k}_0, \boldsymbol{k}_0, \boldsymbol{k}_0, \boldsymbol{k}_0) \equiv T(2\pi)^{-3}.$$

Introducing the envelope of the quasi-monochromatic wave

$$\psi(\boldsymbol{r}, t) = (2\pi)^{-3/2} \exp[i(\boldsymbol{k}_0\boldsymbol{r}) - i\omega(k_0)t] \int c(\boldsymbol{k}_0 + \boldsymbol{\kappa}, t) \exp[i(\boldsymbol{\kappa r})] d\boldsymbol{\kappa}$$

and making the inverse Fourier transform in (1.5.2), we arrive at

$$i\frac{\partial\psi}{\partial t} + iv\frac{\partial\psi}{\partial z} + \frac{v}{2k_0}\Delta_\perp\psi + \frac{\omega''}{2}\frac{\partial^2\psi}{\partial z^2} - T|\psi|^2\psi = 0 \,. \tag{1.5.3}$$

The z-axis is chosen in the direction of the wave propagation. The second term in (1.5.3) can be eliminated by the transition to a reference system moving with the group velocity: $z \to z - vt$. Expanding the z-coordinate according to $z \to z(k_0\omega''/v)^{1/2}$ and using $\Delta = \Delta_\perp + \partial^2/\partial z^2$, one can reduce (1.5.3) to

(1.4.24). It should be noted that (1.5.3) can be considered both for three- and two-dimensional media. The latter applies mainly to water wave envelopes.

In deriving (1.5.3) we assume that the coefficient $T(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ is a continuous function when all the arguments tend to $\boldsymbol{k}_0$. This is not always true; in some cases, this limit depends on the direction of the vector $\boldsymbol{k}$ with respect to the direction of $\boldsymbol{k}_0$. In such cases (1.5.3) should be replaced by a more complicated equation.

However in most of the isotropic media the behavior of envelopes is governed by (1.5.3). In particular, this equation describes self-focussing of light in nonlinear dielectrics [1.43] and the quasi-classical limit of a weakly nonhomogeneous Bose gas [1.44–45]. It should also be noted that the coefficient T_0 may be either positive (corresponding to the effective attraction of quasi-particles) or negative (corresponding to their repulsion). Scale transformations of the coordinates and the field ψ may be used to obtain unity for the values of the parameters β and T_0. The turbulence of envelopes is sometimes called *optical turbulence* because nonlinear optics is a field with a multitude of applications for nonlinear wave theory.

1.5.2 Kadomtsev-Petviashvili Equation for Weakly Dispersive Waves

Let us consider a hydrodynamic type system with the usual Hamiltonian

$$\mathcal{H} = \int \frac{\varrho v^2}{2}\, d\boldsymbol{r} + E_{\text{in}} ,$$

where the internal energy E_{in} is connected with the density variation $\varrho = \varrho_0 + \delta\varrho$ and has the form (1.2.18,21)

$$E_{\text{in}} = \frac{1}{2} \int \left[c_s^2 (\delta\varrho)^2/\varrho_0 + g c_s^2 (\delta\varrho)^3 + \beta |\nabla\varrho|^2 \right] d\boldsymbol{r} .$$

Such a Hamiltonian expansion using the smallness of nonlinearity and dispersion is quite universal [for example, we obtain the equations of the ion sound in plasmas (1.3.10) with such an approximation]. Respective dynamical equations have the standard form (1.2.8–9):

$$\frac{\partial\delta\varrho}{\partial t} + \varrho_0 \operatorname{div} \boldsymbol{v} = -\operatorname{div}(\boldsymbol{v}\delta\varrho) ,$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \frac{c_s^2}{\varrho_0} \nabla\delta\varrho = -(\boldsymbol{v}\nabla)\boldsymbol{v} - \beta\nabla\Delta\delta\varrho .$$

The right-hand-sides of these equations contain the small effects of nonlinearity and dispersion while the left-hand-side describes the main phenomenon: the motion of a perturbation with the sound velocity ($\boldsymbol{v}_{tt} - c_s^2 \Delta \boldsymbol{v} = 0$). We can eliminate the latter by going over to a moving reference frame. Let the x-axis coincide with the direction of motion of the sound velocity. We suppose

$\boldsymbol{v} \approx v_x = v$, $\partial v/\partial x \gg \partial v/\partial y, \partial v/\partial z$ and the same inequality for $\delta\varrho$. Substituting into the terms on the right the zero-order relations $\delta\varrho_t \approx -\varrho_0 \operatorname{div} \boldsymbol{v} \approx \varrho_0 v_x$, $v_t \approx -c_s^2 \nabla\delta\varrho/\varrho_0$, we obtain

$$v_{tt} - c_s^2 v_{xx} = c_s^2 \Delta_\perp v + c_s \frac{\partial}{\partial x}(v v_x) + \beta\varrho_0 \Delta^2 v$$

Here the terms on the left-hand-side are much greater than those on the right-hand-side. We introduce instead of x, t the slow variables $\xi = x - vt$ and slow time τ, so we can substitute $v_{tt} - v_{xx} = (\frac{\partial}{\partial x} - c_s \frac{\partial}{\partial t})(\frac{\partial}{\partial x} + c_s \frac{\partial}{\partial t}) v = 2 \frac{\partial}{\partial \xi}\frac{\partial}{\partial \tau} v$ to obtain finally the Kadomtsev-Petviashvili equation [1.46]

$$\frac{\partial}{\partial\xi} \left(v_\tau + v v_\xi + \frac{\beta\varrho_0}{2c_s} v_{\xi\xi\xi} \right) = \frac{c_s}{2} \Delta_\perp v . \tag{1.5.4}$$

As it is clear from its derivation, this equation is valid in the case of weak nonlinearity ($v \ll c_s$), weak dispersion ($\beta v_{xx} \ll v$) and small deviations from one-dimensionality ($v_x \gg \nabla_\perp v$). In the truncated equation all terms are generally of the same order, since linear effects are excluded. Following (1.2.17), one can introduce for (1.5.4) normal canonical variables with $v = \nabla\Phi$. So we obtain

$$\omega(k_x, k_\perp) = \frac{\beta\varrho_0}{2c_s} k_x^3 + \frac{c_s}{2k_x} k_\perp^2 , \tag{1.5.5a}$$

$$V(\boldsymbol{k}, \boldsymbol{k}_1, \boldsymbol{k}_2) = 3\sqrt{\frac{c_s k_x k_{1x} k_{2x}}{4\pi^3 \varrho_0}}\, \Theta(k_x)\Theta(k_{1x})\Theta(k_{2x}) . \tag{1.5.5b}$$

For purely one-dimensional motions the Kadomtsev-Petviashvili equation returns to the famous KdV equation:

$$v_\tau + v v_\xi + v_{\xi\xi\xi} = 0 .$$

1.5.3 Interaction of Three Wave Packets

Let us discuss the interaction of three narrow wave packets with characteristic wave vectors $\boldsymbol{k}_1, \boldsymbol{k}_2$ and $\boldsymbol{k}_3$. For this interaction to be essential, it is necessary that these vectors should lie in the vicinity of the resonant surface $\omega(\boldsymbol{k}_1 + \boldsymbol{k}_2) = \omega(\boldsymbol{k}_1) + \omega(\boldsymbol{k}_2)$. Let us represent the wave amplitude $c(\boldsymbol{k}, t)$ as

$$c(\boldsymbol{k}, t) = c_1(\boldsymbol{k}_1 + \boldsymbol{\kappa}, t) + c_2(\boldsymbol{k}_2 + \boldsymbol{\kappa}, t) + c_3(\boldsymbol{k}_3 + \boldsymbol{\kappa}, t) \tag{1.5.6}$$

and, making use of the narrowness of the packets ($\kappa \ll k_j$), expand the function $\omega(\boldsymbol{k})$:

$$\omega(\boldsymbol{k}_j + \boldsymbol{\kappa}) = \omega_j + \boldsymbol{\kappa}\boldsymbol{v}_j, \quad \omega_j = \omega(\boldsymbol{k}_j), \quad \boldsymbol{v}_j = \frac{\partial\omega(\boldsymbol{k}_j)}{\partial\boldsymbol{k}_j}, \quad j = 1,2,3 . \tag{1.5.7}$$

The value $c(\boldsymbol{k}, t)$ obeys the usual dynamic equation for the three-wave case

$$i\frac{\partial c(\boldsymbol{k},t)}{\partial t} - \omega_k c(\boldsymbol{k},t) = \int \Big[\frac{1}{2}V_{k12}c_1c_2\delta(\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2) + V^*_{1k2}c_1c^*_2\delta(\boldsymbol{k}_1-\boldsymbol{k}-\boldsymbol{k}_2)\Big]\,d\boldsymbol{k}_1 d\boldsymbol{k}_2 \,. \tag{1.5.8}$$

Neglecting its dependence on κ we shall regard the interaction coefficient to be a constant. Then we shall use

$$(2\pi)^{3/2}c_j(\boldsymbol{r},t) = \exp(-i\omega_j t)\int c_j(\boldsymbol{k}_j+\boldsymbol{\kappa})\exp[-i(\boldsymbol{\kappa r})]\,d\boldsymbol{\kappa}\;,$$

which also includes a procedure for eliminating the fast time dependence to go over to the r-representation. As a result, we obtain the known equation for the three-wave resonant interaction

$$\begin{aligned}\left[\frac{\partial}{\partial t}+(\boldsymbol{v}_1\nabla)\right]c_1(\boldsymbol{r},t) &= -(2\pi)^{3/2}iV_{123}c_2c_3.\\ \left[\frac{\partial}{\partial t}+(\boldsymbol{v}_2\nabla)\right]c_2(\boldsymbol{r},t) &= (2\pi)^{3/2}iV^*_{123}c_2c^*_3.\\ \left[\frac{\partial}{\partial t}+(\boldsymbol{v}_3\nabla)\right]c_3(\boldsymbol{r},t) &= (2\pi)^{3/2}iV^*_{123}c_1c^*_2\;.\end{aligned} \tag{1.5.9}$$

These relations show that the wave packets move in the r-space with group velocities v_j, the characteristic time of their amplitude and slow-phase variation being

$$t_{\text{int}} \simeq \frac{1}{2\pi}\left|V_{123}c_{\max}\right|\;. \tag{1.5.10}$$