

Kolmogorov Spectra of Turbulence I

Wave Turbulence

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1. Equations of Motion and the Hamiltonian Formalism

1.1 The Hamiltonian Formalism for Waves in Continuous Media

Equations describing waves in different media and written in natural variables are diverse. For example, the Bloch equations defining the motion of a magnetic moment are totally different from the Maxwell equations for nonlinear dielectrics. The latter radically differ from the Euler equations for compressible fluids. However all of them as well as many other equations describing nondissipative media, possess an implicit or explicit Hamiltonian structure. This was established empirically and is reflected by the fact that all these models may be derived from initial microscopic Hamiltonian equations of motion.

The Hamiltonian method is applicable to a wide class of weakly dissipative wave systems; it clearly manifests general properties of small-amplitude waves. For example, spin, electromagnetic and sound waves are just waves, i.e., medium oscillations, transferred from one point to another. If we are interested only in small-amplitude wave propagation phenomena, such as diffraction, we do not really need to know what it is that oscillates: magnetic moment, electrical field or density. Their respective dispersion law $\omega(\mathbf{k})$ contains all the information about the medium properties that is necessary and sufficient for studying the propagation of noninteracting waves. As we shall see now, the $\omega(\mathbf{k})$ -function is a coefficient in the term of the Hamiltonian which is quadratic with respect to wave amplitudes, i.e., to complex normal variables. The actual Hamiltonian is a power series in these variables that contains all the information about nonlinear wave interactions. Let us consider the transition to such variables using a simple yet very important example.

1.1.1 The Hamiltonian in Normal Variables

A continuous medium of dimensionality d may be defined in the simplest case by a pair of canonical variables $p(\mathbf{r}, t)$ and $q(\mathbf{r}, t)$. The canonical equations of motion are expressed as

$$\frac{\partial q(\mathbf{r}, t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta p(\mathbf{r}, t)}, \quad (1.1.1)$$

$$\frac{\partial p(\mathbf{r}, t)}{\partial t} = -\frac{\delta \mathcal{H}}{\delta q(\mathbf{r}, t)}. \quad (1.1.2)$$

The Hamiltonian \mathcal{H} depends on $p(\mathbf{r}, t)$ and $q(\mathbf{r}, t)$ as a functional. The symbols $\delta/\delta q$ and $\delta/\delta p$ designate variational derivatives which are extensions of partial derivatives for the continuous case (Sect. A.1). The formal advantage of the Hamiltonian method is that its equations are symmetric in coordinate q and momentum p . To illustrate this advantage, let us first go over to new canonical variables $Q = \lambda q$, $P = p/\lambda$, choosing the dimensional factor λ in such a way that P and Q have the same dimension. Then we introduce complex variables

$$a = (Q + iP)/\sqrt{2}, \quad (1.1.3)$$

$$a^* = (Q - iP)/\sqrt{2} \quad (1.1.4)$$

to obtain

$$\sqrt{2} \frac{\partial a}{\partial t} = \frac{\delta \mathcal{H}}{\delta P} - i \frac{\delta \mathcal{H}}{\delta Q}, \quad \sqrt{2} \frac{\partial a^*}{\partial t} = \frac{\delta \mathcal{H}}{\delta P} + i \frac{\delta \mathcal{H}}{\delta Q}. \quad (1.1.5)$$

Substituting for \mathcal{H} , P and Q , we obtain

$$i \frac{\partial a(\mathbf{r}, t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta a^*(\mathbf{r}, t)}, \quad i \frac{\partial a^*(\mathbf{r}, t)}{\partial t} = \frac{\delta \mathcal{H}}{\delta a(\mathbf{r}, t)}. \quad (1.1.6)$$

The second equation follows from the first by complex conjugation. Hence, we have obtained one complex equation instead of two real ones (1.1.1,2). In quantum mechanics such a substitution of variables corresponds to a transition from the coordinate-momentum representation to a representation using creation and annihilation Bose operators. Their classical analogues are complex canonical variables. Obviously, the canonical variables (1.1.3,4) are by no means the only possible variables to choose. A large choice of transformations from one set of variables to another $a, a^* \rightarrow b, b^*$ exists, such that the equations of motion retain their canonical form (1.1.6). For a given set of explicit variables the canonicity condition is expressed through the Poisson brackets of two functions

$$\{f(q), g(q')\} = \int d\mathbf{r}'' \left[\frac{\delta f(q)}{\delta a^*(\mathbf{r}'')} \frac{\delta g(q')}{\delta a(\mathbf{r}'')} - \frac{\delta f(q)}{\delta a(\mathbf{r}'')} \frac{\delta g(q')}{\delta a^*(\mathbf{r}'')} \right]$$

(see Sect. A.2) and has the simple form

$$\{b(q), b(q')\} = 0, \quad \{b(q), b^*(q')\} = \delta(q - q'). \quad (1.1.7)$$

To ensure that this is a one-to-one transformation the index q should cover a "complete set" of values, for example, the space R^d . It should be noted that this wide range of possibilities in selecting canonical variables is an important advantage of the Hamiltonian method. It ensures the choice of the most adequate variables for a specific problem. We shall define the canonical variables $a(p)$ in such a way that they determine wave amplitudes and become zero for vanishing waves.

Let us expand the Hamiltonian \mathcal{H} in a power series of variables $a(\mathbf{r})$ and $a^*(\mathbf{r})$ assuming these to be small. The zeroth order term is of no interest for us, since it does not occur in the equation of motion. There are no first-order terms as the medium is assumed to be in equilibrium if the amplitudes of the waves were zero, and, consequently, the Hamiltonian to be minimal at $a = a^* = 0$. Thus the \mathcal{H} expansion starts from the second-order terms:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\text{int}}. \quad (1.1.8)$$

The most general form of \mathcal{H}_2 is:

$$\mathcal{H}_2 = \int d\mathbf{r} d\mathbf{r}' \{A(\mathbf{r}, \mathbf{r}') a(\mathbf{r}) a^*(\mathbf{r}') + (1/2)[B^*(\mathbf{r}, \mathbf{r}') a(\mathbf{r}) a(\mathbf{r}') + \text{c.c.}]\}. \quad (1.1.9)$$

Here "c.c." means the complex conjugate of the preceding term.

The \mathcal{H}_2 value is real (the Hamiltonian is Hermitian). Therefore

$$A(\mathbf{r}, \mathbf{r}') = A^*(\mathbf{r}', \mathbf{r}), \quad B(\mathbf{r}, \mathbf{r}') = B(\mathbf{r}', \mathbf{r}). \quad (1.1.10)$$

Below we shall consider the medium to be spatially homogeneous. This very important assumption is the basis of all following discussions. Because of space homogeneity, the functions $A(\mathbf{r}, \mathbf{r}')$ and $B(\mathbf{r}, \mathbf{r}')$ depend only on the difference of the arguments $\mathbf{R} = \mathbf{r} - \mathbf{r}'$. Now

$$A(\mathbf{R}) = A^*(-\mathbf{R}), \quad B(\mathbf{R}) = B(-\mathbf{R}). \quad (1.1.11)$$

The Hamiltonian can be significantly simplified using the Fourier transform:

$$a(\mathbf{k}) = a_{\mathbf{k}} = (1/V) \int a(\mathbf{r}) \exp(-i\mathbf{k}\mathbf{r}) d\mathbf{r}, \quad (1.1.12)$$

$$a(\mathbf{r}) = \sum_{\mathbf{k}} a_{\mathbf{k}} \exp(i\mathbf{k}\mathbf{r}).$$

Here V is the volume of a sample (the propagation medium). We consider the wave vector \mathbf{k} to be a discrete variable. This corresponds to imposing periodic space boundary conditions on the wave field $a(\mathbf{r}+L) = a(\mathbf{r})$, $V = L^d$. If required, one can pass in any conventional manner from the summation over \mathbf{k} to the integration

$$(2\pi)^d \sum_{\mathbf{k}} = V \int d\mathbf{k}. \quad (1.1.13)$$

The Fourier transform (1.1.12) is canonical but not unimodal. This means that the canonical equation of motion (1.1.6) retains its canonical form, but the new Hamiltonian differs from the old one by a factor, which is the inverse sample volume

$$V\mathcal{H}(a_{\mathbf{k}}, a_{\mathbf{k}}^*) = \mathcal{H}\{a(\mathbf{r}), a^*(\mathbf{r})\}, \quad (1.1.14)$$

(1.1.20), a change of sign in the even part of $\omega(\mathbf{k})$ means that there is a surface on which $A_1(\mathbf{k}) = 0$. In the general position on this surface $B(\mathbf{k}) \neq 0$. This implies that at least in the vicinity of the zero surface of $A_1(\mathbf{k})$ the square of the frequency is negative and the medium is unstable. Thus, if the even part of the $\omega(\mathbf{k})$ -function is sign-alternating and the $B(\mathbf{k})$ -function does not identically vanish wherever $A_1(\mathbf{k}) = 0$, the k -space contains a field of linear instability. Given this instability, it is impossible to transform the Hamiltonian into the form (1.1.17). In this case, however, the Hamiltonian (1.1.15) reduces to the simple form

$$\mathcal{H}_2 = \int C(\mathbf{k})[b(\mathbf{k}, t)b(-\mathbf{k}, t) + b^*(\mathbf{k}, t)b^*(-\mathbf{k}, t)] d\mathbf{k} ,$$

$$C(\mathbf{k}) = C(-\mathbf{k}) = C^*(\mathbf{k}) .$$

From the analogy to quantum-mechanics it is evident that such a Hamiltonian describes the creation of a pair of quasi-particles from the vacuum (and the reverse process), thus representing just such an unstable medium. Summarizing, the canonical transformations (1.1.16) allow us to eliminate the term with the least factor (in magnitude) in the Hamiltonian (1.1.15).

Up to now, we have considered the case of a medium containing only one type of waves described by a single dispersion law $\omega(\mathbf{k})$. We can examine, without any essential complications, a more general case with a medium having several types of waves. In this case, the medium is described by a set of equations

$$\frac{\partial q_j}{\partial t} = \frac{\delta \mathcal{H}}{\delta p_j}, \quad \frac{\partial p_j}{\partial t} = -\frac{\delta \mathcal{H}}{\delta q_j}, \quad j = 1, \dots, n .$$

Going over to complex variables $a_j = (1/\sqrt{2})(q_j + ip_j)$, we have for the quadratic part of the Hamiltonian

$$\mathcal{H}_2 = \sum_{i,j} \int d\mathbf{r} [A_{ij}(\mathbf{r} - \mathbf{r}_1)a_i(\mathbf{r}, t)a_j^*(\mathbf{r}_1, t) + (1/2)(B_{ij}(\mathbf{r} - \mathbf{r}_1)a_i(\mathbf{r}, t)a_j(\mathbf{r}_1, t) + \text{c.c.})] . \quad (1.1.21)$$

Now

$$A_{ij}(\mathbf{R}) = A_{ji}^*(-\mathbf{R}), \quad B_{ij}(\mathbf{R}) = B_{ji}(-\mathbf{R}) .$$

Normal variables are introduced by diagonalizing the Hamiltonian (1.1.21), which results in the reduction of the Hamiltonian for a stable medium to the form

$$\mathcal{H} = \sum_j \int \omega_j(\mathbf{k})b_j(\mathbf{k}, t)b_j^*(\mathbf{k}, t) d\mathbf{k} . \quad (1.1.22)$$

Diagonalization may be accomplished if all $\omega_j(\mathbf{k})$ have the same sign and do not identically coincide.

1.1.2 Interaction Hamiltonian for Weak Nonlinearity

In various problems of nonlinear wave dynamics, the wave amplitude may be defined by a natural dimensionless parameter ξ . For sound waves, this parameter is represented by the ratio of the density variation in the sound wave to the average density of the medium; for fluid surface waves it is the ratio of the wave height to wavelength. For spin waves, ξ is the precession angle of the magnetic moment. For ξ of the order of unity, phenomena specific for each of the above media arise: sound turns into shock waves, fluid surface waves form whitecaps and in ferromagnets an inversion of magnetization occurs, that is, a movable domain wall. Obviously, consideration of all these phenomena from a general viewpoint is not always constructive. If, however, the wave's nonlinearity parameter ξ is small, the characteristic features of the medium are negligible, and the wave dynamics may be described in general terms by expanding the Hamiltonian in terms of canonical variables. Let us now look in greater detail at the expansion we started to analyze in the preceding subsection. Suppose we have only a single wave type in a stable medium. Then the first term of the Hamiltonian expansion has the form (1.1.17), and the corresponding equation of motion may be written as

$$\frac{\partial b(\mathbf{k}, t)}{\partial t} + i\omega(\mathbf{k})b(\mathbf{k}, t) = 0, \quad b(\mathbf{k}, t) = b(\mathbf{k}) \exp[-i\omega(\mathbf{k})t] .$$

At this level of sophistication, waves in different media are only distinguished by their dispersion laws $\omega(\mathbf{k})$. All information about the wave interaction is contained in the higher coefficients of the expansion of \mathcal{H} in a power series of b :

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_{\text{int}}, \quad \mathcal{H}_{\text{int}} = \mathcal{H}_3 + \mathcal{H}_4 + \dots . \quad (1.1.23)$$

The physical meaning of \mathcal{H}_3 and \mathcal{H}_4 is easy to understand by analogy with quantum mechanics. The Hamiltonian \mathcal{H}_3 describes three-wave processes:

$$\mathcal{H}_3 = \frac{1}{2} \int (V_q b_1 b_2 b_3 + \text{c.c.}) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 + \frac{1}{6} \int (U_q b_1^* b_2^* b_3^* + \text{c.c.}) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 . \quad (1.1.24a)$$

Here and below a shorthand notation is to be used: b_1, b_2 are $b(\mathbf{k}_1, t), b(\mathbf{k}_2, t)$; $q = (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$ and thus $V_q = V_{123} = V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3)$. The first term in (1.1.24) defines the decay processes $1 \rightarrow 2$ and the reverse confluence processes $2 \rightarrow 1$. The second term describes mutual annihilation of three waves or their creation from vacuum.

The Hamiltonian \mathcal{H}_4 describes processes involving four waves:

$$\begin{aligned}
\mathcal{H}_4 &= (1/4) \int W_p b_1^* b_2^* b_3 b_4 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \\
&+ \int (G_p b_1 b_2^* b_3^* b_4^* + \text{c.c.}) \delta(\mathbf{k}_1 - \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4 \\
&+ \int (R_p^* b_1 b_2 b_3 b_4 + \text{c.c.}) \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4, \\
&\text{with } p = (\mathbf{k}_1 \mathbf{k}_2 \mathbf{k}_3 \mathbf{k}_4). \tag{1.1.24b}
\end{aligned}$$

The coefficients of the interaction Hamiltonian have the following obvious properties

$$\begin{aligned}
V(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= V(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2), \\
U(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3) &= U(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2) = U(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_3), \\
G(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) &= G(\mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_4) = G(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_3), \\
W(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4) &= W(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_4) \\
&= W(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_3) = W^*(\mathbf{k}_3, \mathbf{k}_4, \mathbf{k}_1, \mathbf{k}_2), \\
R(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_3) &= R(\mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_3, \mathbf{k}_4) = R(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_4, \mathbf{k}_3) \\
&= R(\mathbf{k}_3, \mathbf{k}_2, \mathbf{k}_1, \mathbf{k}_4) = R(\mathbf{k}_4, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_1). \tag{1.1.25}
\end{aligned}$$

The last equation for $W(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4)$ follows since the Hamiltonian is real.

But to which order in b, b^* should we expand the Hamiltonian \mathcal{H} ? The answer turns out to be as general as the question: “ \mathcal{H}_5 and higher-order terms should normally not be taken into account.” This can be explained as follows: Since the expansion uses a small parameter, every subsequent term is smaller than the preceding one, and the dynamics of the wave system is determined by the very first term of \mathcal{H}_{int} , i.e., normally \mathcal{H}_3 . However, three-wave processes may turn out to be “nonresonant” which means that the spatio-temporal synchronization condition (or, in terms of quasi-particles, the momentum-energy conservation law)

$$\omega(\mathbf{k} + \mathbf{k}_1) = \omega(\mathbf{k}) + \omega(\mathbf{k}_1) \tag{1.1.26}$$

cannot be satisfied. Let d be the dimensionality of the medium and \mathbf{k} the vector in d -meric space ($d > 1$). Equation (1.1.26) specifies the hypersurface of dimension $2d - 1$ in the $2d$ -meric space of vectors \mathbf{k}, \mathbf{k}_1 . If this surface does in fact exist [i.e., $\omega(\mathbf{k})$ is real], the dispersion law $\omega(\mathbf{k})$ is of the decay type. If (1.1.26) has no real solutions, the dispersion law is of the nondecay type.

In isotropic media, $\omega(\mathbf{k})$ is a function of k only. Let $\omega(0) = 0$, $\omega' = \partial\omega(\mathbf{k})/\partial k > 0$. In this important case a simple criterion for the decay may be formulated: the dispersion law is of the decay type if $\omega'' = \partial^2\omega(\mathbf{k})/\partial k^2 > 0$ and of the nondecay type if $\omega'' < 0$. This criterion has a clear geometric meaning. Consider, for example, the case with $d = 2$. The dispersion law $\omega(\mathbf{k})$ then specifies the surface of rotation in a three-dimensional space $\omega, \mathbf{k}_x, \mathbf{k}_y$. In Fig.1.1 this is the surface S for $\omega(\mathbf{k})$ and S_1 for $\omega(\mathbf{k}_1)$. It is seen that (1.1.26) is satisfied if these surfaces intersect, then all the three points (ω, \mathbf{k}) , (ω_1, \mathbf{k}_1) and

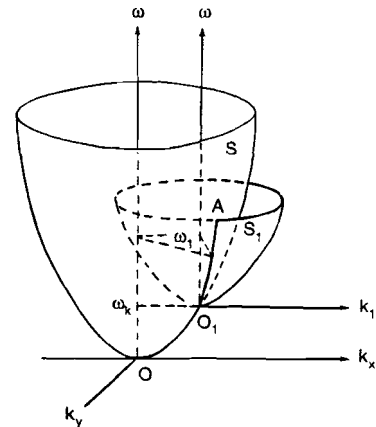


Fig. 1.1. Intersection of the frequency surfaces of the convex function $\omega(\mathbf{k})$

$(\omega(\mathbf{k} + \mathbf{k}_1), \mathbf{k} + \mathbf{k}_1)$ lie on the intersection line O_1A . Surface section requires a convex function, i.e., $\omega'' > 0$. For power functions $\omega(\mathbf{k}) \propto k^\alpha$, the dispersion law is of the decay type if $\alpha > 1$. In the limiting case at $\alpha = 1$, decays are only allowed for collinear vectors.

Four-wave processes are always permitted. This is evident from the conservation laws for scattering processes

$$\omega(\mathbf{k}) + \omega(\mathbf{k}') = \omega(\mathbf{k} + \boldsymbol{\kappa}) + \omega(\mathbf{k}' - \boldsymbol{\kappa}), \tag{1.1.27}$$

which are allowed at $\boldsymbol{\kappa} \rightarrow 0$ for any dispersion laws. Consequently, it is \mathcal{H}_4 that will govern the dynamics of wave systems with a nondecay dispersion law. Subsequent expansion terms \mathcal{H}_5, \dots will define small and, generally, insignificant corrections. However, nature is certainly more interesting than this formal scheme. Some problems may involve an additional small parameter (apart from the nonlinearity level). For example, for spin waves in magnets (Sect.1.4), not only \mathcal{H}_3 but also \mathcal{H}_4 should be taken into account, even in the decay region of the spectrum. The Hamiltonian \mathcal{H}_3 arises only because of the magnetic dipole-dipole interaction and has relativistic smallness as compared with the Hamiltonian \mathcal{H}_4 which results from the exchange interaction. The authors are aware of problems where one has to take into consideration the fifth- and even sixth-order terms (for example, while considering an approximate conservation of the wave action integral, see below). These problems are, however, rather specialized and go beyond the frame of this book. Hence, it is usually sufficient to expand the Hamiltonian in the decay region of the spectrum up to third-order, and in the nondecay region up to fourth-order terms.

1.1.3 Dynamic Perturbation Theory.

Elimination of Nonresonant Terms

It is intuitively clear that in the case of a nondecay dispersion law, the Hamiltonian \mathcal{H}_3 describing three-wave processes may turn out to be irrelevant in some respect. We shall show now that in this case one can go over to new canonical variables c_k, c_k^* , such that $\mathcal{H}_3\{c_k, c_k^*\} = 0$. This is possible because the dynamic system under consideration, the weakly nonlinear wave field, is close to a completely integrable dynamic system (a set of noninteracting oscillators). Traditionally classical perturbation theory is employed to handle systems close to completely integrable ones. In this procedure a canonical transformation is derived by sequentially excluding the nonintegrable terms from the Hamiltonian. It is known that the procedure may encounter the problem of "small resonance denominators"; then the only terms to be excluded from the Hamiltonian are those for which the resonance condition is not satisfied. As shown by Zakharov [1.1], we can in this case to some extent apply classical perturbation theory.

Let us demonstrate such a procedure using a simple example. Consider the expansion of the one-wave Hamiltonian:

$$\mathcal{H} = \mathcal{H}_2 + \mathcal{H}_3 + \mathcal{H}_4 = \omega bb^* + \frac{V}{2}(b^2 b^* + b^{*2} b) + \frac{U}{6}(b^3 + b^{*3}) + \frac{W}{4}b^2 b^{*2} + G(b^3 b^* + bb^{*3}) + R(b^4 + b^{*4}).$$

We assume every subsequent term to be smaller than the preceding one, i.e. $\mathcal{H}_2 \gg \mathcal{H}_3 \gg \mathcal{H}_4$. Since one needs to eliminate the \mathcal{H}_3 term without changing \mathcal{H}_2 , the transformation must be close to the unity transformation. Thus it is reasonable to search for the transformation in the form of an expansion, which starts from a linear term:

$$b = c + A_1 c^2 + A_2 c c^* + A_3 c^{*2} + B_1 c^3 + B_2 c^* c^2 + B_3 c c^{*2} + B_4 c^{*3} + \dots \quad (1.1.28a)$$

Why do we take the following (c^3 -order) terms into account? Taking only the linear and quadratic terms in (1.1.28a) into account is indeed sufficient to eliminate the \mathcal{H}_3 term. In that case the fourth-order terms govern the nonlinear interaction. Due to the transformation they will acquire additional terms. To derive these new terms c^3 -order terms in (1.1.28a) have to be taken into account. Moreover, we will use them to exclude the last two terms describing the $1 \rightarrow 3$ and $0 \rightarrow 4$ processes in the Hamiltonian (1.1.24b).

Thus we look for seven coefficients: $A_1, A_2, A_3, B_1, \dots, B_4$. The canonicity condition (1.1.7) is expressed through the Poisson brackets and has the form

$$\{bb^*\} = \frac{\partial b}{\partial c} \frac{\partial b^*}{\partial c^*} - \frac{\partial b^*}{\partial c} \frac{\partial b}{\partial c^*} = 1.$$

Computing the Poisson bracket to an accuracy of c^3 -order terms, we obtain three equations

$$A_2 = -2A_1, \quad B_2 = A_3^2 - A_1^2, \quad B_3 + 3B_1 = 2A_2(A_3 - A_1).$$

Substituting (1.1.28a) into the Hamiltonian and demanding that all nonlinear terms (except $c^2 c^{*2}$) vanish, we have four equations

$$\begin{aligned} 2\omega(A_1 + A_2) + V &= 0, & 6\omega A_3 + U &= 0, \\ \omega B_4 + \omega A_1 A_3 + \frac{1}{2}V A_3 + \frac{1}{2}U A_1 + R &= 0, \\ \omega(A_1 A_2 + A_2 A_3 + B_1 + B_3) + V A_3 + \frac{1}{2}U A_2 + G &= 0. \end{aligned}$$

From these it is easy to find the transformation coefficients

$$\begin{aligned} A_1 &= \frac{V}{2\omega}, & A_2 &= -\frac{V}{\omega}, & A_3 &= -\frac{U}{6\omega}, \\ B_1 &= \frac{V^2}{4\omega^2} + \frac{VU}{6\omega^2} + \frac{G}{2\omega}, & B_2 &= \frac{U^2}{36\omega^2} - \frac{V^2}{4\omega^2}, \\ B_3 &= \frac{V^2}{4\omega^2} + \frac{7UV}{12\omega^2} - \frac{3G}{2\omega}, & B_4 &= -\frac{UV}{12\omega^2} - \frac{R}{\omega}. \end{aligned} \quad (1.1.28b)$$

In the new variables the Hamiltonian has the simple form

$$\mathcal{H} = \omega c c^* + \frac{1}{4}T c^2 c^{*2}, \quad T = W - \frac{3V^2}{\omega} - \frac{U^2}{3\omega}.$$

It is easily seen that neglect of the cubic terms in (1.1.28a), would have given wrong values of the additional interaction coefficients supplementing W .

Following the same pattern, let us return to the general case and use a transformation in the form of a power series to eliminate cubic and nonresonant fourth-order terms. In the new variables the Hamiltonian of interaction describes $2 \rightarrow 2$ processes (for details see Sect. A.3):

$$\mathcal{H}_4 = \frac{1}{4} \int T_p c_1^* c_2^* c_3 c_4 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3 d\mathbf{k}_4, \quad (1.1.29a)$$

$$p = (\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}_4), \quad T_p = W_p + T'_p.$$

$$\begin{aligned} T'_p &= -\frac{U_{-1-212} U_{-3-434}}{\omega_3 + \omega_4 + \omega_{3+4}} + \frac{V_{1+212}^* V_{3+434}}{\omega_1 + \omega_2 - \omega_{1+2}} \\ &\quad - \frac{V_{131-3}^* V_{424-2}}{\omega_{4-2} + \omega_2 - \omega_4} - \frac{V_{242-4}^* V_{313-1}}{\omega_{3-1} + \omega_1 - \omega_3} \\ &\quad - \frac{V_{232-3}^* V_{414-1}}{\omega_{4-1} + \omega_1 - \omega_4} - \frac{V_{141-4}^* V_{323-2}}{\omega_{3-2} + \omega_2 - \omega_3}. \end{aligned} \quad (1.1.29b)$$

Here $(j \pm i) = k_j \pm k_i$.

Note that (1.1.29b) is true on the resonant surface

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) = \omega(\mathbf{k}_3) + \omega(\mathbf{k}_4), \quad \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}_4$$

only, where the coefficient T'_p has the same properties (1.1.25) as W_p . The necessity of taking cubic terms into account for the transformation to yield the correct value of the four-wave interaction coefficient was first pointed out by Krasitskii [1.2].

Let us discuss the singularities of (1.1.29). The denominators become zero on the resonance surfaces of three-wave processes:

$$\mathbf{k} + \mathbf{k}_1 + \mathbf{k}_2 = 0, \quad \omega_k + \omega_1 + \omega_2 = 0 \quad (1.1.30a)$$

and

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2, \quad \omega_k = \omega_1 + \omega_2. \quad (1.1.30b)$$

The conditions (1.1.30a) can only be satisfied if the medium allows for negative-energy waves. Without such waves, denominators of the type (1.1.30a) do not vanish and the corresponding terms in the three-wave Hamiltonian may be eliminated. The condition of nonzero denominators of the type (1.1.30b) coincides with the nondecay condition for the dispersion law $\omega(k)$. In the nondecay case, the cubic terms in the Hamiltonian may thus be completely excluded. The same holds for the terms in the fourth-order Hamiltonian (1.1.24b) differing in their form from (1.1.29). Prohibition of the $1 \rightarrow 2$ and $2 \rightarrow 1$ processes implies in general that the $1 \rightarrow 3$ and $3 \rightarrow 1$ processes are not feasible. One can definitely state that the interaction Hamiltonian of type (1.1.29) is a fundamental model for considering nonlinear processes in media that obey a nondecay dispersion law. Additional terms in it may be interpreted as scattering processes that arise in the second order perturbation theory for three-wave processes. In that case, a virtual forced wave appears at an intermediate stage for which the resonance condition is not satisfied. In this interpretation, every term in (1.1.29b) may be juxtaposed with a picture (see Fig. 1.2) to illustrate which particular process is meant.

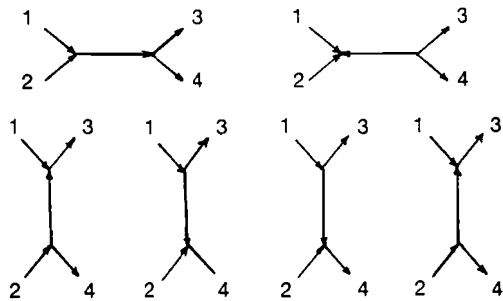


Fig. 1.2. The processes corresponding to different terms in (1.1.29b)

On the resonance manifold (1.1.27) an attempt to exclude the Hamiltonian term (1.1.29) by using a canonical transformation will lead to small denominators describing wave-wave scattering. These processes are allowed irrespective of the type of the dispersion law; hence, it is impossible to exclude this term from the Hamiltonian. The Hamiltonian (1.1.29) may be said to describe wave scattering.

Scattering processes possess an important feature: they do not change the total number of waves. Therefore the equations of motion corresponding to Hamiltonian (1.1.29) preserve one more integral besides energy, namely

$$N = \int c_k^* c_k d\mathbf{k},$$

which gives the total number of quasi-particles. This is known as the wave action integral. A complete system that has to include minor effects associated with higher-order processes generally preserves the N value only approximately.

One can similarly exclude the cubic terms corresponding to nondecay three-wave processes in the Hamiltonian specifying wave interactions of several types.

1.1.4 Dimensional Analysis of the Hamiltonian Coefficients

As stated above, at weak nonlinearity the Hamiltonian of a system of interacting waves has the standard form (1.1.17, 24, 25). Is it possible to evaluate the coefficients ω_k , V_{123} , T_{1234} without considering the specific nature of every particular problem and to understand, e.g., how these values depend on the wave vectors? The answer is positive if the parameters specifying waves of this type do not yield a quantity with the dimension of a length. In this case the problem is said to possess complete self-similarity (first-order self-similarity). The Hamiltonian coefficients are estimated by dimensional considerations.

Let us first obtain the dimensions of the canonical variables b_k and the coefficients of the interaction Hamiltonian V_{123} and T_{1234} . Bearing in mind that \mathcal{H} has the dimension of an energy density and ω_k the one of a frequency, the dimension of b_k is found by (1.1.18).

$$[\mathcal{H}] = g \cdot \text{cm}^{2-d} \cdot \text{s}^{-2}, \quad [\omega] = \text{s}^{-1}, \quad [b_k] = g^{1/2} \cdot \text{cm} \cdot \text{s}^{-1/2}. \quad (1.1.31)$$

Here d is the dimensionality of the medium, cm = centimeter, g = gramm, s = second. In view of the fact that $[\omega] = [V_{123}b] = [T_{1234}b^2]$, it is easy to establish

$$[V_{123}] = g^{-1/2} \cdot \text{cm}^{d/2-1} \cdot \text{s}^{-1/2}, \quad [T_{1234}] = g^{-1} \cdot \text{cm}^{d-2}. \quad (1.1.32)$$

As to be expected, the dimension of Vb^2 (here V is the volume of the system!) coincides with the one of Planck's constant \hbar . Naturally, our classical approach is true when the quantum-mechanical occupation numbers $N(k) = Vb^2/\hbar$ are large compared with unity. On the other hand, wave amplitudes b_k should not be too large for the interaction Hamiltonian \mathcal{H}_{int} to remain small compared with \mathcal{H}_2 . This gives an upper estimate for b_k . It may be schematically written as

$$\omega_k \gg V_{kkk} \sum_{k'} b_{k'}. \quad (1.1.33)$$

If we introduce the dimensionless wave amplitude

$$\xi_k = b_k/B_k, \quad B_k = |\omega_k/V_{kkk}|, \quad (1.1.34)$$

the weak nonlinearity condition may be written as

$$\xi_k \ll 1. \quad (1.1.35)$$

Now we can discuss some particular examples.

Sound in Continuous Media. As parameters the equations of motion for this problem may include only the medium's density ρ and the elasticity coefficient κ with the respective dimensions $[\rho] = \text{g} \cdot \text{cm}^{-3}$ and $[\kappa] = \text{g} \cdot \text{cm}^{-1} \cdot \text{s}^{-2}$. These values and the wave vector k combine to yield the dimension of a frequency $[\omega] = \text{s}^{-1} = [\rho^x \kappa^y k^z] = \text{g}^{x+y} \cdot \text{cm}^{-3x-y+z} \cdot \text{s}^{-2y}$. Equating the exponents at g, cm and s, we have three equations $x + y = 0$, $3x + y + z = 0$, and $2y = 1$. Hence $x = -1/2$, $y = 1/2$, $z = 1$. Thus the dimensional analysis leads to a linear law for the wave dispersion

$$\omega_k = c_s k, \quad c_s = a(\kappa/\rho)^{1/2}. \quad (1.1.36)$$

Here c_s is the sound velocity and a a dimensionless parameter of the order of unity. From the parameters of our problem, one can also obtain B_k with the dimension of the canonical variable b_k

$$B_k = (\rho c_s / k)^{1/2} \quad (1.1.37)$$

and the interaction coefficient

$$V_{123} = \sqrt{\frac{k_1 k_2 k_3 c_s}{\rho}} f\left(\frac{k_1}{k_1}, \frac{k_2}{k_1}, \frac{k_3}{k_1}\right). \quad (1.1.38)$$

Here the dimensionless function f depends on eight dimensionless arguments: the two ratios k_2/k_1 and k_3/k_1 , and six angle variables giving the directions of the three vectors. In fact, there are only three angle variables: $\cos \theta_{12}$, $\cos \theta_{13}$, and $\cos \theta_{23}$ [here $\cos \theta_{ij} = (\mathbf{k}_i \mathbf{k}_j) / k_i k_j$], as our system has no preferred direction.

In the Hamiltonian description, the wave amplitude is proportional to b_k . In the sound wave field, medium density $\rho(r, t) = \rho_0 + \rho_1(r, t)$ and velocity oscillate:

$$\begin{aligned} \rho_1(r, t) &= \text{Re}[\rho_k \exp(ikr - i\omega_k t)], \\ v(r, t) &= \text{Re}[v_k \exp(ikr - i\omega_k t)], \end{aligned} \quad (1.1.39)$$

where ρ_k , v_k are the respective wave amplitudes in natural variables. In the linear approximation the relationship between natural and normal canonical variables is easily established from dimensional considerations

$$\rho_k \propto (k \rho_0 / c_s)^{1/2} b_k, \quad v_k \propto (k c_s / \rho_0)^{1/2} b_k. \quad (1.1.40)$$

The symbol " \propto " designates proportionality. In terms of canonical variables the condition of weak nonlinearity is written as

$$\xi_k \simeq \rho_k / \rho_0 \simeq v_k / c_s \ll 1. \quad (1.1.41)$$

The symbol " \simeq " denotes an estimate to an accuracy of a dimensionless factor of order unity.

Gravitational Waves on a Fluid Surface. These are relatively long waves for which surface tension is insignificant and the force tending to restore the equilibrium state of the surface is the gravitational force. Apart from fluid density ρ the significant parameters should evidently include the gravitational acceleration g , $[g] = \text{cm} \cdot \text{s}^{-2}$. Following the scheme given in the preceding example and bearing in mind that this is a 2-dimensional problem ($d = 2$), we have:

$$\omega_k = \sqrt{gk}, \quad B_k = (\rho^2 g k^{-5})^{1/4}. \quad (1.1.42)$$

As we see, the dispersion law is of the nondecay type, $\omega_k \propto k^\alpha$, $\alpha = 1/2 < 1$. Therefore the principal interaction is four-wave, with the interaction coefficient

$$T_{k123} = \frac{k^3}{\rho} f[(k_1/k), (k_2/k), (k_3/k), \cos \theta_{k1}, \cos \theta_{k2}, \cos \theta_{k3}]. \quad (1.1.43)$$

A natural variable describing water waves is $\eta(r)$, the deviation of the fluid surface from the unperturbed state and the dimensionless wave amplitude is $\xi_k = k\eta_k = b_k/B_k$. Whence, we obtain

$$\eta_k = (k/\rho^2 g)^{1/4} b_k. \quad (1.1.44)$$

Capillary Waves. For sufficiently short waves the restoring force should be entirely determined by surface tension. The significant parameters should in this case instead of g include the surface tension coefficient σ having the dimension of a surface energy $[\sigma] = \text{g} \cdot \text{s}^{-2}$. Thus,

$$\omega_k = \sqrt{\frac{\sigma k^3}{\rho}}, \quad B_k = (\rho \sigma / k^3)^{1/4}, \quad \eta_k = (\rho \sigma k)^{-1/4} b_k. \quad (1.1.45)$$

The dispersion law of capillary waves is of the decay type: $\alpha = 3/2 > 1$. Therefore the three-wave interaction remains as the most essential one

$$V_{k12} = \left(\frac{\sigma k^9}{\rho^3}\right)^{1/4} f(k_1/k, k_2/k, \cos \theta_{k1} \cos \theta_{k2}). \quad (1.1.46)$$

Comparing the dispersion laws of capillary (1.1.45) and gravitational waves (1.1.42), it is easy to find the boundary value of the wave vector at which these frequencies coincide:

$$k_* = \sqrt{\frac{\rho g}{\sigma}}. \quad (1.1.47)$$

